ASTR 422
Problem Set 2
Due Friday, September 28, 2007

1. Spherical geometry. Suppose that you have two points on a sphere of radius R. In spherical polar coordinates, their angular locations are (θ,ϕ) and $(\theta+\Delta\theta,\phi+\Delta\phi)$. If $\Delta\theta \ll 1$ and $\Delta\phi \ll 1$, demonstrate explicitly that the metric has only higher-order corrections to the flat space form. That is, show that you can define local distances $dx \propto \Delta\theta$ and $dy \propto \Delta\phi$ so that the distance between the points (measured, as always, along a great circle) is given by $ds^2 = dx^2 + dy^2 +$ higher order terms.

2. The versatile Dr. I. M. N. Sane has proposed a new component of the universe that he calls "elastic matter". The effective density of elastic matter is $\rho_{\rm elast} = C_S\left[(a_S/a)\sin(a/a_S)\right]^2$. Here C_S is the "Sane constant" and a_S is the "Sane scale factor", which he says is much less than the current scale factor of the universe. Dr. Sane asserts that elastic matter currently dominates the universe, i.e., $\rho_{\rm elast} \gg \rho_{\rm matter}$, and that the geometry of the universe is flat, $k = 0$. He has submitted a paper to this effect to the journal *Nature*. Dr. Leslie Sage, astronomy editor for Nature, has asked for your expert opinion.

Granting Dr. Sane his assumption that elastic matter is the only important component of the universe, demonstrate as simply as you can that his model is inconsistent with observations. Assume, as Dr. Sane does, that when $\dot{a}$ reaches zero during the evolution it then changes sign (i.e., if it had been positive and goes to zero, it becomes negative, and vice versa).

3. Escape speed and fine tuning

If the universe is close to flat now, it had to be amazingly close to flat when it was much younger. Let's pursue this with an escape speed calculation.

(a) Suppose that a rock is launched straight up from the surface of the Moon; once it leaves the surface, it coasts. At a distance of 10 Moon radii from the center of the Moon, you measure the speed of the rock and find that it is twice the escape speed from that point. How fast was it moving when it was launched? The Moon's radius is about $R = 2\times10^6$ m and its mass is about $M = 8\times10^{22}$ kg. Newton's constant is $G = 6.67\times10^{-11}$ kg^{-1} m^3 s^{-2}.

(b) How fast was the rock launched if the speed you measure at 10 Moon radii was half the escape speed there? For comparison, recall that the speeds are different by a factor of four at ten Moon radii.

(c) Now repeat parts (a) and (b) except assuming that you measure either twice or half the escape speed at 10^{30} times the Moon's radius. This was roughly the change in the scale factor of the universe from the end of inflation until now. What was the speed of launch in each case? Pretty close, aren't they!

4. Consider a static sphere of radius R. Define the "proper distance" D_M between two points to be the smaller of the two possible distances between them along a great circle. Supposing that light follows great circles as well, calculate the angular diameter distance D_A and luminosity distance D_L as a function of D_M. For D_A, this is simply $l/\Delta\phi$, where l is the actual transverse width of an object and $\Delta\phi$ is the angle it subtends. For D_L, note that light that travels isotropically from a point for a distance D_M along the surface will be on a circle; the circumference of the circle is $2\pi D_L$.