Special Relativity: Basics

High-energy astrophysics involves not only light, which is intrinsically relativistic, but also particles that are accelerated to nearly the speed of light. Newtonian mechanics therefore has to be supplanted by special relativity. In this lecture and the next, we will go over some of the principles and applications of the special theory. In later lectures, we will consider general relativity, which generalizes these principles to accelerated frames and turns out to be our best current theory of gravity. For basic physics such as this, by the way, my opinion is that any serious physicist should at some point read the Feynman Lectures on Physics. His clarity of thought was exceptional, and probably the best way to approach those volumes is to look at them after you have already had a course in a given subject; it allows you to appreciate his profound insights better.

Philosophy

First, let's start with a little philosophy. After the fact, it is easy to present physical principles as if they are self-evident and derivable from pure mathematics. This is not the case. We can marvel at the brilliance of Einstein and the other pioneers of relativity, and appreciate the philosophical way that they drew their conclusions, but to be scientific one must at some point have contact with experiments. Therefore, ultimately, we have to point to the universe as a whole (or at least, what we've probed observationally) to argue that the theory is correct.

A second philosophical point that many people mistakenly derive from relativity, probably because of the name of the theory, is that the essential point is "everything is relative". In fact, one of the postulates of relativity, and one of its deepest points, is that there are some quantities that are *invariant*, meaning that all observers will measure the same value for those quantities. We'll try to emphasize such invariants when we derive aspects of special relativity.

Galilean Relativity

We should also not get the idea that Einstein was the first one to suggest a principle of relativity. In fact, Galileo used thought experiments quite similar to Einstein's to show that something coasting along at a constant velocity should experience all the same local effects as something at rest. He asked his readers to consider experiments performed by someone in a ship's cabin if the ship is moving at a constant speed. He notes that a ball tossed straight up will appear to come straight down; a tank of water will remain level; and in general the experimenter will not be able to tell that the ship is moving. From our standpoint a more familiar and extreme example is traveling in a plane. We might be going 75% of the speed of sound relative to the ground, but we can still be served bad food without it ending up in our faces!

Put more formally, all local experiments we do in an inertial frame will turn out the same independent of our velocity relative to a given frame. However, note the restrictions to *local* experiments and *inertial* frames. If you somehow opened the window of your plane and stuck your head out, it would be the last thing you ever did; there is a quite clear difference in physical effects when you have contact with other frames! In addition, when the plane accelerates (e.g., by hitting turbulence) it is sickeningly clear that you are not at rest. In more benign situations, such as experiments on a rotating Earth, the non-inertial nature of the frame leads one to introduce fictitious forces such as the Coriolis force.

How, then, would we phrase Galilean relativity mathematically? A useful way to do this is to consider two observers moving at a constant velocity v relative to each other. Let us set up Cartesian coordinate systems for both: for one frame the coordinates are (t, x, y, z) and for the other are (t', x', y', z'). We will refer to these as, respectively, the unprimed and primed frames. Here t means time, and we will make our lives easier by ensuring that the x axis is parallel to the x' axis, and similarly for y and z.

Suppose that, as seen in the unprimed system, the primed system is moving in the $+x$ direction with speed v. Note that we can always rotate our coordinate axes so that the x axis lines up with this speed; if you prefer making your algebra messier you can always do it more generally, but we won't bother. If we set up our initial conditions so that at time $t = t' = 0$ we have $x' = 0$ (i.e., the origins of the two systems are coincident), this implies that at time t, the origin of the primed system is at $x = vt$ as measured in the unprimed system. Of course, in the primed system, the origin is always at $x' = 0$. In addition, the perpendicular directions y and z are equal to their primed counterparts, and $t = t'$. Therefore, the coordinate transformation for Galilean relativity becomes

$$\begin{aligned} x' &= x - vt \\ y' &= y \\ z' &= z \\ t' &= t \,. \end{aligned} \tag{1}$$

We also find that Newton's laws of motion are invariant in form under these transformations. This is as expected, and is a consequence of our inability to tell whether we are moving steadily or not from purely local experiments. Among other things, this law tells us how velocities should add. Consider, for example, something that moves with speed u in the x direction as seen in the unprimed frame. Therefore, $dx/dt = u$. In the primed frame we have

$$u' = dx'/dt' = d(x - vt)/dt = dx/dt - v = u - v \,. \tag{2}$$

This is the simple, intuitive result. If a train goes by me at 100 km/hr and I throw a baseball parallel to the train at 100 km/hr, someone inside the train sees the ball not moving in that direction at all. If I throw antiparallel to the train at 100 km/hr, the person in the train sees a speed of 200 km/hr. Note, by the way, that if we want to transform from the primed frame to the unprimed frame, all we have to do is reverse the sign of v and switch the primed with unprimed variables. Very simple.

The Problem with Maxwell's Equations

In the mid-1800s, however, a problem emerged. After many people had for several decades experimented with electricity and magnetism, James Clerk Maxwell came up with a compact set of equations that beautifully described all the phenomena. To this day, Maxwell's achievement ranks among the very greatest in the history of physics. Surprisingly, though, Maxwell's equations are *not* invariant under a Galilean transformation. For example, a blatant contradiction emerges when one tries to determine the speed of light in different frames with this theory. According to this theory, the propagation speed was the same whether the source was moving or not, which violates the velocity addition law that we derived above. This is a result that could be obtained if light propagated through a medium, similarly to how the speed of sound is independent of the motion of the source (although the frequency isn't). However, the famous Michelson-Morley experiment found no evidence of any "luminiferous ether" through which light traveled. Given the overwhelming success of Newton's theories in the previous two centuries a number of people very logically tried to find formulations of Maxwell's equations that obeyed Galilean relativity. None, however, could be squared with experiment. What could be done?

The Postulates of Special Relativity

Lorentz, Poincaré, Fitz-Gerald, and others suggested essentially ad hoc ways of explaining the above results. Einstein, however, was the one who put it on a more axiomatic footing, which is why we reasonably give him the lion's share of the credit. He suggested two postulates:

- The laws of physics as derived from local experiments are the same for all inertial observers.
- All such observers measure the same speed for light in a vacuum.

The first postulate is the same one as before. The second, however, seems contradictory; how is it reconciled with normal velocity addition?

To understand this, and to adopt a perspective that has tremendous utility in general relativity, we will consider the fundamental concept of the *invariant interval.* As our first step, recall distance invariance in Euclidean geometry. Suppose we have two points in a three-dimensional space, and in a particular Cartesian coordinate system the points have coordinates (x, y, z) and $(x + dx, y + dy, z + dz)$. For the situations we consider here, dx, dy, and dz need not be infinitesimal quantities, but we write it this way for later compatibility with general relativity (where it is clearest to restrict oneself to infinitesimal distances). The distance ds between the two points is then given by

$$ds^2 = dx^2 + dy^2 + dz^2 \ . \tag{3}$$

This distance is absolutely invariant with respect to coordinate transformations. If you rotate the axes to some new x', y', z' then in general $dx' \neq dx$ and so on, but $ds'^2 = dx'^2 + dy'^2 + dz'^2 = dx^2 + dy^2 + dz^2 = ds^2$. This is also true if you go for a non-Cartesian system, e.g., spherical polar coordinates. The separation is an *invariant.*

What about when time is involved? Einstein's second postulate says that the distance light travels in a given time is measured to be the same in all frames. It's easier to deal with the squares of distances, so if at time $t = t' = 0$ the ray started out at the origin of the unprimed and primed systems (where, remember, the primed system can move relative to the unprimed system), we would find that

$$\begin{aligned} dx^2 + dy^2 + dz^2 &= c^2 dt^2 \\ dx'^2 + dy'^2 + dz'^2 &= c^2 dt'^2 \ , \end{aligned} \tag{4}$$

where c is the speed of light in a vacuum.

In fact, let's make a powerful generalization of this. Define an *event* to be something at a specific place and time, which must therefore be designated by four coordinates (t, x, y, z). Consider a nearby event $(t+dt, x+dx, y+dy, z+dz)$, and let the four-dimensional "interval" ds between the two events be given by

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2 \ . \tag{5}$$

We then postulate that, just as in Euclidean geometry the separation between points is independent of the coordinate system, the interval as defined above is an *invariant*, so all inertial observers measure the same interval between the same two events. Note that if the events are two points on the trajectory of a light ray, $ds = 0$.

To explore the consequences of this, let us again consider an unprimed frame (t, x, y, z) and a primed frame (t', x', y', z'). Suppose that, as seen in the unprimed frame, the primed frame is moving with speed v in the $+x$ direction (see Figure 1). Also suppose we have set

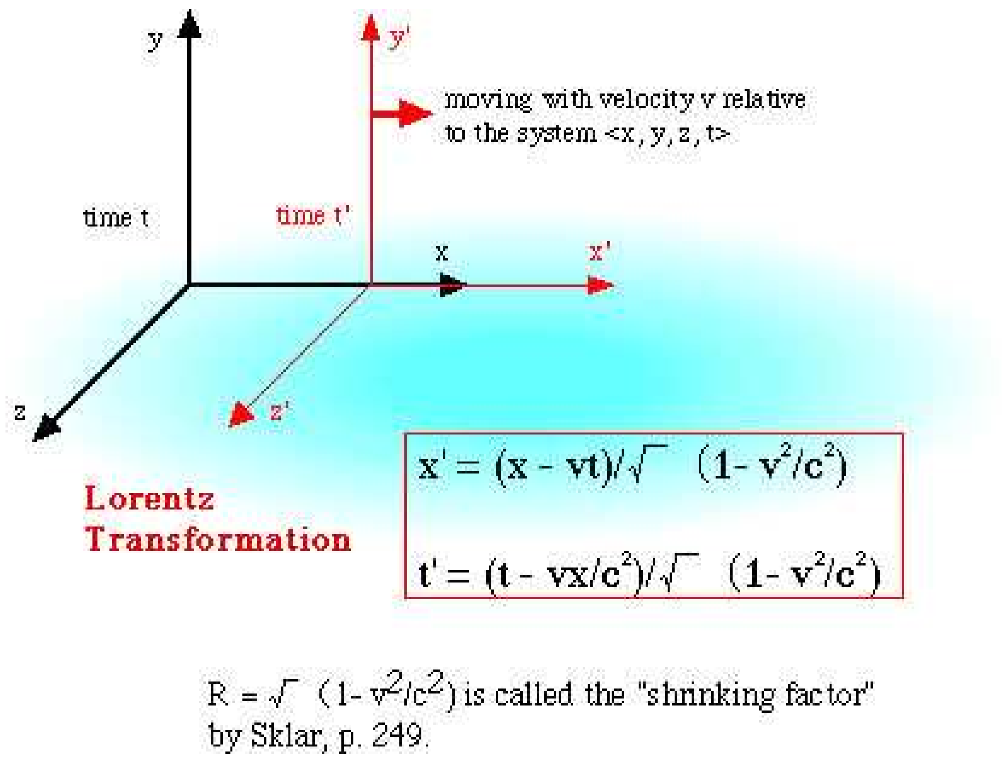

Fig. 1.— Geometry for a boost along the x-axis, and the result for a Lorentz transformation. From http://www.bun.kyoto-u.ac.jp/∼suchii/lorentz.tr.jpg

up the axes so that initially the unprimed and primed frames are coincident (i.e., x parallel to x' and so on) and $t = t' = 0$. Our postulate says that

$$-c^2dt^2 + dx^2 + dy^2 + dz^2 = -c^2dt'^2 + dx'^2 + dy'^2 + dz'^2 \ . \tag{6}$$

We can argue from symmetry that $dy = dy'$ and $dz = dz'$; this will be left as an intuition builder at the end of the class (**Hint:** consider viewing the same situation from different perspectives, and see if you can arrive at a contradiction if $dy \neq dy'$). Therefore, we are left with

$$-c^2dt^2 + dx^2 = -c^2dt'^2 + dx'^2 \ . \tag{7}$$

We now look for a transformation between the unprimed and primed frames that maintains this invariance. The simplest such transformation is linear, so that

$$\begin{aligned} x' &= a_{xx}x + a_{xt}t \\ t' &= a_{tx}x + a_{tt}t \ . \end{aligned} \tag{8}$$

We also know that when $x = vt$, $x' = 0$, because the primed frame is moving with speed v in the x direction. Therefore, it must be that $x' \propto (x - vt)$. Calling the proportionality constant γ, we find

$$\begin{aligned} x' &= \gamma(x - vt) \\ t' &= \gamma(ax + bt) \ . \end{aligned} \tag{9}$$

We choose the proportionality in the second line because we happen to know it will be simpler that way! In turn, this means that $dx' = \gamma(dx - vdt)$ and $dt' = \gamma(adx + bdt)$. Substituting this into our expression for the invariant interval, we get

$$-c^2dt^2 + dx^2 = -c^2\gamma^2(a^2dx^2 + 2abdxdt + b^2dt^2) + \gamma^2(dx^2 - 2vdxdt + v^2dt^2) \ . \tag{10}$$

Collecting terms, we have three equations: one for the dx^2 term, one for the $dxdt$ term, and one for the dt^2 term. This is encouraging, because we have three unknowns: γ, a, and b. The equations are

$$\begin{aligned} -c^2\gamma^2a^2 + \gamma^2 &= 1 \\ -2c^2\gamma^2ab - 2v\gamma^2 &= 0 \\ -c^2\gamma^2b^2 + \gamma^2v^2 &= -c^2 \ . \end{aligned} \tag{11}$$

Solving gives $\gamma = 1/\sqrt{1 - v^2/c^2}$ (the famous Lorentz factor), $a = -v/c^2$, and $b = 1$. Our transformation is therefore

$$\begin{aligned} x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \\ t' &= \gamma(-\tfrac{v}{c^2}x + t) \ . \end{aligned} \tag{12}$$

This is a *Lorentz transformation*. The generalization to arbitrary directions is straightforward. As before, to change back, we simply flip the sign of v and exchange primed for unprimed variables.

Consequences

As was discovered well before Einstein proposed special relativity, Maxwell's equations *are* invariant in form under a Lorentz transformation. That's good news. However, there are other implications that may make the cure seem worse than the disease:

Length contraction.—Suppose that in the unprimed frame we measure the length of a stick, oriented along the x axis, that is moving in the primed frame and in that frame has length l. As you may remember from other exposures to special relativity, we have to be precise in how we specify our measurement: in this case, it will be at a single time t as measured in the unprimed frame, meaning $dt = 0$. We then have

$$\begin{aligned} dx' &= \gamma(dx - vdt) \\ l &= \gamma dx \\ dx &= l/\gamma \; . \end{aligned} \tag{13}$$

Noting that $\gamma \geq 1$ for all speeds v, this means that we measure a shorter length in the frame in which the stick is moving. If instead the stick is at rest in the unprimed frame and has length l as measured there, what do we see in the primed frame? The transformation is

$$\begin{aligned} x &= \gamma(x' + vt') \\ \Rightarrow \; dx &= \gamma(dx' + vdt') \end{aligned} \tag{14}$$

hence for $dt' = 0$ we again get that in the frame in which the stick is moving, the length is contracted to l/γ.

Time dilation.—Now suppose that in the unprimed frame we look at a clock in the primed frame. In the primed frame, a time T elapses. How much time goes by in the unprimed frame? For this problem, we note that

$$\begin{aligned} t &= \gamma(\tfrac{v}{c^2}x' + t') \\ \Rightarrow \; dt &= \gamma(\tfrac{v}{c^2}dx' + dt') \; . \end{aligned} \tag{15}$$

If the clock is at rest in the primed frame then $dx' = 0$, so $dt = \gamma dt' = \gamma T$. Therefore, the elapsed time is longer as seen in a frame in which the clock is moving. Note that "clock" here is very general indeed, and refers to anything that takes time. It could be a wristwatch, a chemical process, a nuclear decay, anything at all. At this point, many people like to consider the "twin paradox": consider identical twins, one of whom stays on Earth and the other of whom blasts off in a rocket, accelerates to nearly the speed of light, travels for a year in her reference frame, then turns around and comes back. The "paradox" is posed as follows: since both twins consider themselves to be at rest, which one should be older when they meet after the journey? The resolution, which I'll let you ponder, is to determine whether there is any way that you could tell which twin you were. If something breaks the symmetry, one can be older than the other. If not, they have to be the same age.

What does it mean?.—The effects discussed above are counterintuitive, to put it mildly. The reason, of course, is that we don't travel anywhere near the speed of light relative to everyday objects, so we have evolved to be used to Newtonian mechanics. As an example, the fastest speed that most of us have ever traveled is on airplanes, perhaps up to 270 m s^{-1}. The speed of light is about $c = 3 \times 10^8$ m s^{-1}, so the Lorentz factor is $\gamma = 1/\sqrt{1 - v^2/c^2} \approx 1.00000000000041$. That's four parts in 10^{13}! This actually has been detected using atomic clocks flying on planes, but in our everyday life we'd never notice it.

Nonetheless, a lot of people are pretty uncomfortable with the implications of special relativity, which is probably one reason why it is a favorite target of crackpots (another being that Einstein personally was so famous). It is useful to remember that *in the rest frame of something* everything proceeds as normal. Aliens in some distant galaxy might see us appear to move at 90% of the speed of light, but that can't possibly affect us at all. This means, for example, that if I am moving really fast and see a star appear to be contracted by a factor of 10 in my direction of motion, it certainly doesn't imply that there really are huge pressure forces inside the star!

There are, however, real effects than can be measured, and it is this experimental confirmation that gives us confidence in the predictions of special relativity, counterintuitive though they might be. For example, consider a muon, which is a subatomic particle that decays with a characteristic lifetime of $\tau = 2.2 \times 10^{-6}$ seconds. Suppose we set one going at $v/c = 0.9$ of the speed of light. We would expect it to travel a typical distance $D = 0.9c \times 2.2 \times 10^{-6}$ s=590 meters before decaying. Instead, we find that the typical distance is 1360 meters. What is happening? We'll analyze this from two different perspectives:

- From the perspective of the particle, the length of the track on which it is traveling is smaller by a factor of $\gamma = 2.3$. That means that if, in the laboratory frame the track has a length of 1360 meters, in the particle rest frame it appears to have a length of 590 meters, so in the particle rest frame it lasts the expected 2.2 microseconds and all is well.
- From the perspective of the laboratory, the lengths are as expected but the muon decay "clock" runs slowly by a factor of γ and therefore lasts long enough to travel the longer distance.

Therefore, in both frames, observers agree on the final result. This *has* to be the case. In fact, this is a general principle that can help you navigate through tricky situations in relativity. In a given setup, think about facts or numbers on which every observer must agree. Examples might include the number of particles in a box, or whether the muon in the above example reaches the end of a track. These are, if you like, additional examples of invariants: things that are the same in all frames.

Addition of Velocities

How do velocities add relativistically? For simplicity, consider something that moves at speed u in the $+x$ direction as seen in the unprimed frame. Then we go back to

$$\begin{aligned} dx' &= \gamma(dx - vdt) \\ dt' &= \gamma(-\tfrac{v}{c^2}dx + dt) \end{aligned} \tag{16}$$

to find that

$$\frac{dx'}{dt'} = \frac{\gamma(dx - vdt)}{\gamma[dx(-v/c^2) + dt]} \ . \tag{17}$$

Since $dx/dt = u$, we can divide the numerator and denominator by dt to find

$$dx'/dt' = \frac{u - v}{1 - uv/c^2} \ . \tag{18}$$

Note that if $u = c$, we get $(c - v)/(1 - v/c) = c$, meaning that indeed the speed of light is the same in all frames (it had better be, given that this was one of our starting postulates!).

Incidentally, for this derivation and the others, we find as we should that when $v \ll c$, all the answers reduce to the familiar Newtonian ones. This is a good way to check our derivations.

Other Relativistic Effects

Aberration of light.—Suppose that in the primed frame something moves with a speed u' at an angle θ' from the direction of motion of the primed frame relative to the unprimed frame (i.e., relative to the x axis as we've set it up here). Then in the unprimed frame the angle of motion is given by

$$\tan\theta = \frac{u' \sin\theta'}{\gamma(u' \cos\theta' + v)} \ . \tag{19}$$

When v approaches c, this leads to relativistic beaming. Consider a photon (with $u' = c$) emitted at $\theta' = \pi/2$. Then we get $\tan\theta = c/(\gamma v)$. If $v \approx c$, then $\tan\theta \approx 1/\gamma$. But $v \approx c$ means $\gamma \gg 1$, so $\theta \approx 1/\gamma \ll 1$. Therefore, radiation that is spread out over half the sky in the primed frame is beamed into a solid angle $\Delta\Omega \propto 1/\gamma^2$ in the unprimed frame. This is an effect seen in relativistic jets from active galactic nuclei, and means that jets coming towards us are vastly more visible than jets going away.

Doppler effect.—Suppose a source that emits light at a frequency $\nu_{\rm source}$ comes at us with a total speed v in a direction that is an angle θ from head-on (i.e., $\theta = 0$ means directly at us). Then the observed frequency is

$$\nu_{\rm obs} = \frac{\nu_{\rm source}}{\gamma\left[1 - (v/c)\cos\theta\right]} \ . \tag{20}$$

Without the factor of γ, this is just the familiar nonrelativistic expression. With this factor, though, it means that even for a source moving perpendicular to our line of sight ($\theta = \pi/2$), the frequency is altered: $\nu_{\rm obs} = \nu_{\rm source}/\gamma$. Again using the example of AGN jets, a jet pointed towards us has higher frequency radiation than moving away.

Intuition Builder

Show by a thought experiment that distances perpendicular to the direction of motion are not altered.