http://pancake.uchicago.edu/∼carroll/notes/ has a lot of good notes on GR and links to other pages.

General Relativity

Philosophy of general relativity.—As with any major theory in physics, GR has been framed and derived in many different ways, each giving their own insight. **Ask class:** can they think of other examples in physics? $F = ma$ versus Lagrangian or Hamiltonian mechanics; wave versus matrix versus path integral quantum mechanics; quaternions(!) versus vector electromagnetism. In the case of GR, there is the geometric approach, good for insight and reasoning, and the action approach, probably better for trying to unify gravity with the other forces. In all these examples, a common theme is that the predictions had better be the same. Similarly, although, say Newtonian mechanics is based on a completely different set of philosophical principles than quantum mechanics or relativity, in the big, slow-moving, weak-gravity limit the predictions of all those theories are the same. This, the contact with observables, is the most fundamental point of theories, in my opinion. Therefore, I will present things in a way designed for calculation.

Another point about general relativity is that it is the least confirmed of our current fundamental theories. A major reason for this is that its most dramatic effects only show up in extremely strong gravity, such as near black holes and neutron stars. This gives it special status, and means that astronomical observations may have the most to contribute to fundamental physical understanding in the realm of strong gravity.

Finally, let me say that I plan to go into a little more detail about the formalism and equations of GR than I did into particle interactions. The reason is that you aren't necessarily going to see GR anywhere else, so I'd like this part of the course to be more self-contained.

Fundamental GR concepts

(1) As in special relativity, space and time are both considered as aspects of spacetime. However, whereas in special relativity spacetime is "flat" (in a sense to be defined later), in general relativity the presence of gravity warps spacetime.

(2) The natural motion of objects is to follow the warps in spacetime. "Matter tells space how to curve and space tells matter how to move." An object that is freely falling (i.e., following spacetime's warps) does not "feel" force, meaning that an accelerometer would measure zero. The path of a freely falling particle is called a geodesic.

(3) The only "force" in this sense that can be exerted by gravity is tidal force. That is, if an object has finite size, different parts of it want to follow different geodesics, and these deviate. Geodesic deviation is the GR equivalent of tidal forces.

(4) Because of this deviation, global spacetime is not flat and there is no coordinate

transformation that will make it look flat everywhere.

(5) However, THE most important principle of GR is that in a sufficiently small region of spacetime (small spatial scale, small time interval), the spacetime looks flat. This means that there is a local inertial frame that can be defined in that small patch of spacetime. In that local inertial frame, all the laws of physics are the same as they are in special relativity (electrodynamic, hydrodynamic, strong+weak nuclear, ...)!! This is called the *equivalence principle*. There is a classic elevator analogy for this principle, which says that if you are in an elevator and you feel like you are being pushed towards its floor, you can't tell whether you are at rest in a gravitational field or are being accelerated in flat spacetime (see Figure 1). The equivalence principle means that in practice one of the best ways to do calculations in GR is to do them in the local inertial frame and then use well-defined transformations between the local and the global frame.

(6) All forms of energy gravitate. In the Newtonian limit, rest mass is overwhelmingly the dominant component, but in ultradense matter other forms can be important as well.

The Mathematics of Curved Spacetime

Consider a two-dimensional space. We know that there are differences between, e.g., a flat plane and the surface of a sphere. One example of this is that on a plane, the interior angles of a triangle always add to 180°, whereas on the surface of a sphere the angles always add to something larger than 180°, but the actual value depends on the size of the triangle. Another example is that if you take a vector on a flat plane and transport it parallel to itself, you can move it around the plane to your heart's content and when you bring it back to the starting point it will have the same orientation it did before. This is not the case on a sphere!

Note, however, that (in good analogy to GR!), on a small enough region of a sphere you can treat it as flat. We need to use a formalism that can handle curvature like this, except in four dimensions (three spatial, one time). This is the formalism of geometry in curved spacetime, and we encountered the basics (scalars, events, vectors, tensors) in our second lecture on special relativity. Let us, however, emphasize one particular point about four-vectors in curved space:

Vector.—In flat space, vectors are easy. Using our previous definitions, we can simply think of a vector as an arrow connecting two events. As long as we define the arrow to be a straight line, there is no ambiguity, regardless of how far separated the two events are. Again for concreteness, let's think of two dimensions and Cartesian coordinates, so the events are labeled by their x and y coordinates and a vector between them also has an x and y component. Now, think of two points (aka events) on the surface of a sphere. How do we now define a vector? Not easy. First, we need to decide what a straight line is. A

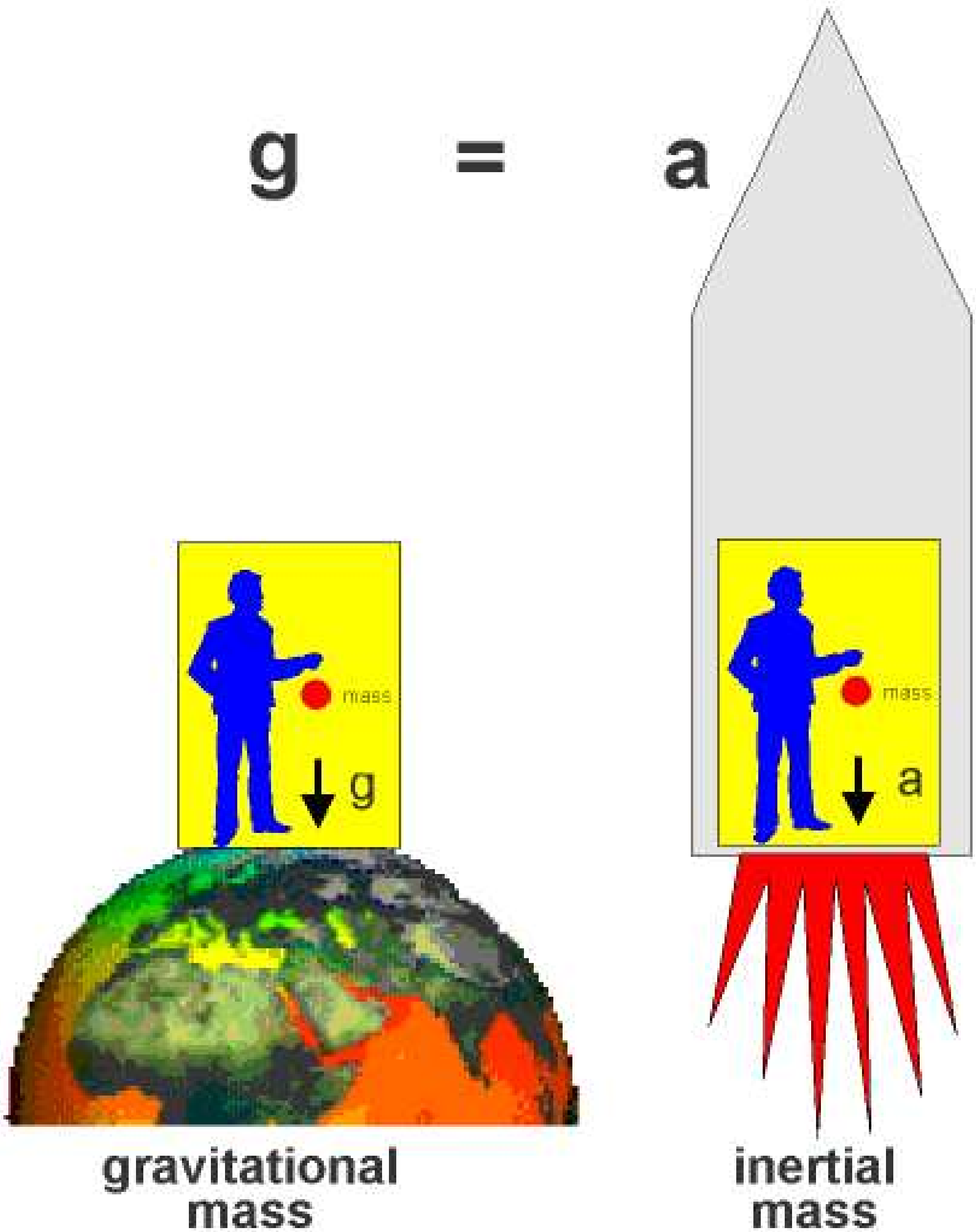

Fig. 1.— The equivalence principle: local measurements can't distinguish between acceleration and gravity. From http://www.desc.med.vu.nl/Graphics/Equiv_principle.gif

good choice is a great circle. Then, however, we have a problem. Every pair of points on a sphere is connected by at least two great circles, and antipodal points are connected by an infinite number of great circles! That shows that in curved spacetime, vectors can't be defined for two points that are distant from each other. Therefore, we define vectors only as infinitesimal quantities. Here again, we've defined a vector as a line between two points (now infinitesimally close), but a vector has its own existence independent of points or events or coordinate systems. In higher dimensions, in any case, vectors are only locally defined. They can have a magnitude, like a gradient, but they don't extend over more than an infinitesimal region.

Also, of course, the direction and magnitude of a vector field can change with position. Think of an electric or magnetic field. At each point you can define the vector (direction plus magnitude) of the field, but both direction and magnitude change with position.

Metrics

The particular metric we encountered earlier was the Minkowski metric, which has $g_{\alpha\beta} = \eta_{\alpha\beta} = (-1, 1, 1, 1)$ down the diagonal. This metric describes flat spacetime, i.e., spacetime appropriate far from any sources of gravity.

There are some metrics for which a global transformation to Minkowski is impossible. A particularly important one, which will be a prime focus for us, is the *Schwarzschild metric*

$$ds^2 = -(1 - 2GM/rc^2)c^2dt^2 + dr^2/(1 - 2GM/rc^2) + r^2(d\theta^2 + \sin^2\theta d\phi^2) \ . \tag{1}$$

Such a metric describes a fundamentally different spacetime (in this case, the spacetime around a spherically symmetric gravitating object, such as a black hole or the Earth for that matter). The nontransformability to Minkowski tells us that this spacetime is curved.

However, although you can't make such a transformation globally, you *can* make it in a small enough region of spacetime. That is, if you go into this spacetime and select an itty bitty region just Δt by Δx by Δy by Δz across, where all those are small, then in just that region you can devise a coordinate system that looks like Minkowski, except for terms of order $(\Delta t)^2$ and so on. This is an exact analogy with the fact that on a sphere, if you look at a small enough patch, you can invent a coordinate system that looks just like Euclidean plane geometry except for little terms of second order. This is one of the aspects of the equivalence principle: in a small enough region of spacetime, the geometry looks flat, which means that there is a reference frame in which the effects of gravity don't exist! The only deviations from the experiments you'd do in perfectly flat space are second-order. These are "tidal forces", known in this context as geodesic deviation. A key to doing GR calculations is to (1) do all the physics in the local Minkowski frame, where things are simple, and (2) know how to transform from that local frame to the global frame.

Spacetime and Metrics

Now let's get a little more concrete, which will eventually allow us to introduce additional concepts. We will focus on the Schwarzschild geometry (aka spacetime). It is very useful, not least because it is more general than you might think. It is the geometry outside of (i.e., in a vacuum) any spherically symmetric gravitating body. It is *not* restricted to static objects; for example, it is the right geometry outside a supernova, if that supernova is good enough to be spherically symmetric. To understand some of its aspects we'll write down the line element (i.e., the metric in a particular set of coordinates). However, the coordinates themselves are tricky, so let's start with flat space.

Ask class: what is the Minkowski spacetime in spherical coordinates?

$$ds^2 = -c^2dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\ . \tag{2}$$

Ask class: what is the meaning of each of the coordinates (this is not a trick question!)? Everything is as you expect: θ and ϕ are the usual spherical coordinates, r is radius, t is time. In particular, if you have two things at r_1 and r_2 (same t, θ, and ϕ) then the distance between them is $|r_2 - r_1|$. No sweat. You could also say that the area of a sphere at radius r is $4\pi r^2$.

Well, why all this rigamarole? It's because in Schwarzschild spacetime things get trickier. Now let's reexamine the Schwarzschild line element

$$ds^2 = -(1 - 2GM/rc^2)c^2dt^2 + dr^2/(1 - 2GM/rc^2) + r^2(d\theta^2 + \sin^2\theta d\phi^2)\ . \tag{3}$$

Ask class: are the meanings of θ and ϕ changed? No, they're the same as always. This is guaranteed by the assumption of spherical symmetry that comes into the Schwarzschild derivation. But what about r? **Ask class:** suppose $dt = d\theta = d\phi = 0$. What is the proper distance between r_1 and r_2? In this case, $ds = g_{rr}^{1/2}dr$, so the distance is

$$D = \int_{r_1}^{r_2} g_{rr}^{1/2}dr = \int_{r_1}^{r_2}(1 - 2GM/rc^2)^{-1/2}dr\ . \tag{4}$$

That means that if $r_1 = 2M$, for example, the radial distance measured is rather different than in flat space! But if you calculate the area of a sphere of radius r, you get $4\pi r^2$ as usual, and the circumference of a circle is $2\pi r$ as usual. This is one of the most extreme geometric indicators of the curved spacetime: "π"=circumference/diameter drops like a rock!

What about time? **Ask class:** what is the relation between proper time $d\tau$ at r and the coordinate time dt if $dr = d\theta = d\phi = 0$? $d\tau^2 = -ds^2 = -g_{tt}dt^2 \Rightarrow d\tau = (1-2M/r)^{1/2}dt$. Therefore, as $r \to 2M$, the elapsed proper time is tiny compared to the elapsed coordinate time. It turns out that t, the coordinate time, is the time as seen at infinity. Therefore, to a distant observer it looks like an object falling into the horizon takes an infinite time to do so. This is the origin of the term "frozen star" used by many until the 1970's for black holes. You might think, then, that if you were to look at a black hole you'd see lots of frozen surprised aliens just outside the horizon. You actually would not, but more on that later.

Conserved Quantities in Schwarzschild Spacetime

Let's take another look at the Schwarzschild spacetime. It is spherically symmetric. It is also stationary, meaning that nothing about the spacetime is time-dependent (for example, the time t does not appear explicitly in the line element). Now, in general in physics, any time you have a symmetry you have a conserved quantity. **Ask class:** for a particle or photon moving under just the influence of gravity, what are some quantities that will be conserved in the motion of that particle? As with any time-independent central force, energy and angular momentum will be conserved. These follow from, respectively, the symmetry with time and the symmetry with angle. In addition, the rest mass is conserved (more on that later). We have three conserved quantities, and four components to the motion, so if we knew one more we'd be set. Luckily, having spherical symmetry means that we can define a plane of motion for a single particle, so we only have three components to the motion (in particular, we might as well define the plane of motion to be the equatorial plane, so that $\theta = \pi/2$ and we don't have to worry about motion in the θ direction). That means we get a great boost in following (and checking) geodesic motion in the Schwarzschild spacetime from the conserved quantities.

Intuition Builder

Does the equivalence principle hold inside a black hole, assuming one is not at the singularity? That is, if you were inside the hole and falling to your doom, would local experiments over a short amount of time give you the same results as they would in flat spacetime?