Compton Scattering

Initial questions: How can we detect galaxy clusters at high redshift? What information could we get out of such detections?

Last time we talked about scattering in the limit where the photon energy is much smaller than the mass-energy of an electron. However, when X-rays and gamma-rays are considered, this approximation is no longer so good. Therefore, today we'll talk about the more general process of Compton scattering. We'll start with the basic process, then discuss how it has an effect on spectra and the energetics of electrons.

Think about a photon scattering off an electron that is initially at rest. Let's treat both the photon (of initial energy $\hbar\omega_0$) and the electron as if they were particles. Suppose that after the photon scatters it moves off into a direction an angle θ from its original direction. **Ask class:** how would we figure out the new energy of the photon? We do this by energy and momentum conservation. **Ask class:** will the entire interaction take place in a single plane? Yes, it will, so that simplifies things. Before we calculate, **Ask class:** will the final photon energy be less than, equal to, or greater than the initial photon energy? Less than, since electron recoil will take some energy. This is the difference between full Compton scattering and our Thomson approximation, in which the photon energy was unchanged (thus producing coherent scattering).

We can rephrase our conservation laws a la special relativity by saying that the total four-momentum is conserved. The initial four-momentum of the photon is $\vec{P}_{\gamma i} = (\hbar\omega_0/c)(1, \mathbf{n}_i)$, where $\mathbf{n}_i$ is the initial direction of the photon. Similarly, the final four-momentum of the photon is $\vec{P}_{\gamma f} = (\hbar\omega_1/c)(1, \mathbf{n}_f)$. The initial four-momentum of the electron is $\vec{P}_{ei} = (mc, \mathbf{0})$, since it isn't moving, and the final is $\vec{P}_{ef} = (E/c, \mathbf{p})$. Setting $\vec{P}_{\gamma i} + \vec{P}_{ei} = \vec{P}_{\gamma f} + \vec{P}_{ef}$ and manipulating a bit gives

$$\hbar\omega_1 = \hbar\omega_0 / \left[1 + (\hbar\omega_0/mc^2)(1 - \cos\theta)\right] \ . \tag{1}$$

In terms of wavelength it's even easier:

$$\lambda_1 - \lambda_0 = \lambda_c(1 - \cos\theta) \tag{2}$$

where $\lambda_c \equiv h/mc = 0.02426\text{Å}(m_e/m)$ is the *Compton wavelength*.

Does this make sense to us? **Ask class:** what limits can we check? Indeed the wavelength only increases, as we guessed beforehand. If $\theta = 0$ there is no deflection and no change, as is reasonable. The change in wavelength is independent of λ, meaning that for smaller λ (larger photon energy) the fractional change is greater, which makes sense. Also, for a more massive particle (e.g., a proton instead of an electron) the wavelength change is less, meaning that the photon energy has to be larger for there to be a significant deviation from Thomson.

Something not as obvious is that the cross section is also modified by this effect. The differential cross section is

$$d\sigma/d\Omega = (r_0^2/2)(\omega_1/\omega_0)^2\left[\omega_0/\omega_1 + \omega_1/\omega_0 - \sin^2\theta\right] \; . \tag{3}$$

The main effect is to reduce the cross section from the Thomson value, but not very rapidly. Rybicki and Lightman give the expression for the total angle-integrated cross section, but it's complicated and the most important parts are the limits at low and high energy:

$$\begin{aligned} \sigma &\approx \sigma_T(1 - 2x + \ldots), \qquad x \ll 1, \\ \sigma &\approx \tfrac{3}{8}\sigma_T x^{-1}(\ln 2x + 1/2 + \ldots), \qquad x \gg 1 \; . \end{aligned} \tag{4}$$

Here we define $x \equiv \hbar\omega/mc^2$. As before, we're usually thinking of photon-electron scattering, but it's more general than that (e.g., a proton will do if we let $m = m_p$). In any case, we see that for low energies the cross section is close to (but less than) the Thomson cross section, whereas at high energies the cross section scales as $\sim (\ln x)/x$, ignoring smaller terms. Ultimately, this means that very high energy gamma rays have a substantially longer mean free path to scattering (or absorption) than lower energy photons do.

Fine, so we've done scattering off of a stationary electron. We found that in such a scattering the photon always loses energy. **Ask class:** is this still true in a frame in which the electron was moving initially? No! Think of a photon incident from the left, and a rapidly-moving electron incident from the right. If the photon doesn't have too overwhelming an energy, the result will be that the electron moves more slowly after the collision (think of an elastic collision of two balls with different masses). Energy conservation then demands that the photon *gain* energy. This is still Compton scattering, but because the energy is transferred from the electron to the photon it is usually called *inverse Compton scattering*. Note, by the way, that if the energy of the photon is large enough, the electron will gain energy as before.

As always in special relativity, the key to analyzing this is to pick a simple reference frame for your analysis, then transform to the original frame. In this case, the simple frame is the one in which the electron is originally at rest, which we already did. If the electrons are relativistic ($\gamma^2 - 1 \gg \hbar\omega/mc^2$) then, *if in the e^- rest frame we have $\hbar\omega \ll mc^2$*, the energy of the photon before scattering, the energy of the photon in the electron rest frame, and the energy of the photon after scattering are in the rough ratio $1 : \gamma : \gamma^2$. That means that a gain in energy in a single scattering can be a factor of about γ^2, which can be enormous. Of course, there are limits. **Ask class:** if the electron has energy $\gamma m_e c^2$, what is the maximum gain of energy by the photon? It can only gain $(\gamma - 1)m_e c^2$. **Ask class:** suppose that in the electron rest frame the condition $\hbar\omega \ll mc^2$ does not hold. What are the qualitative effects? One is that the electron will experience recoil, so the photon will either lose energy or not gain as much energy. Another is that the cross section will drop, so interactions will be less frequent.

Now let's ask the following question: if (as measured in a particular reference frame) one has an isotropic distribution of electrons all of which have the same speed βc, and one has an isotropic distribution of photons, what will be the net power transferred from electrons to photons by inverse Compton scattering?

Suppose that the number density of photons with energy in a range $d\epsilon$ is $(dn_\gamma/d\epsilon)d\epsilon$. Rybicki and Lightman use v instead of $dn_\gamma/d\epsilon$, but that looks too much like velocity to me :). If we ignore any change of energy of the photon in the rest frame, then Rybicki and Lightman show that the scattered energy in a direction θ with respect to the motion of the electron is

$$dE_1/dt = c\sigma_T\gamma^2\int(1-\beta\cos\theta)^2\epsilon(dn_\gamma/d\epsilon)d\epsilon\ . \tag{5}$$

This makes sense: in a direction θ, the Doppler shift is $\gamma(1-\beta\cos\theta)$, so by the $1:\gamma:\gamma^2$ rule one expects that the final energy of the photon in the lab frame will be its original energy times the square of the Doppler shift. To get the average over an isotropic distribution of photons, we need $\langle(1-\beta\cos\theta)^2\rangle$, which is $1+\beta^2/3$ (since $\langle\cos\theta\rangle=0$). Note also that this average has the right symmetry: the direction of motion of the electrons can't matter (i.e., $\beta\rightarrow-\beta$ is irrelevant), so the power must go like an even power of β. The scattered energy rate is then

$$dE_1/dt = c\sigma_T\gamma^2(1+\beta^2/3)U_{\rm ph}\ , \tag{6}$$

where $U_{\rm ph}\equiv\int\epsilon(dn_\gamma/d\epsilon)d\epsilon$ is the initial photon energy density. Subtracting from this the original energy of the photons, we have

$$\begin{aligned} dE_{\rm rad}/dt &= c\sigma_T U_{\rm ph}\left[\gamma^2(1+\beta^2/3)-1\right] \\ &= \tfrac{4}{3}\sigma_T c\gamma^2\beta^2 U_{\rm ph}\ . \end{aligned} \tag{7}$$

This is the net power transferred by inverse Compton scattering.

Let's take stock. **Ask class:** does it make sense that we only had to specify that the photons are distributed isotropically, rather than deciding on their spectrum? Yes, because in the rest frame Thomson approximation, all photons have the same fractional energy boost, so a spectrum isn't necessary. **Ask class:** how else can we test this expression? In the limit $\beta\rightarrow 0$ the power goes to zero (quadratically), as expected. In the limit $\beta\rightarrow 1$ we have $\gamma\rightarrow\infty$, so the power becomes large, as expected. The power also scales linearly with cross section and photon energy density, which makes sense.

We did, however, ignore electron recoil. **Ask class:** qualitatively, what difference would that make? If there is enough electron recoil, the photons can transfer energy to the electrons, so the Compton power could be negative. When this is accounted for, we get instead

$$P_{\rm compt} = \frac{4}{3}\sigma_T c\gamma^2\beta^2 U_{\rm ph}\left[1-\frac{63}{10}\frac{\gamma\langle\epsilon^2\rangle}{mc^2\langle\epsilon\rangle}\right]\ . \tag{8}$$

Here the angle brackets around ϵ^2 and ϵ are mean values over the photon spectrum. **Ask class:** why should the spectrum matter now? It's because electron recoil definitely does depend on the energy of the photon!

Since we have the power for electrons of a single speed, if we're given a distribution $N(\gamma)d\gamma$ of electrons per unit volume with Lorentz factors between γ and $\gamma + d\gamma$ we can just integrate over the distribution to get the total power per volume:

$$P_{\rm tot} = \int P_{\rm compt} N(\gamma) d\gamma \ . \tag{9}$$

Again, this is a *power per volume*, or erg s^{-1} cm^{-3}. For example, for a power law distribution $N(\gamma) = C\gamma^{-p}$ for γ between $\gamma_{\rm min}$ and $\gamma_{\rm max}$ and zero otherwise, the total power per volume is

$$P_{\rm tot} = \frac{4}{3}\sigma_T c U_{\rm ph} C(3-p)^{-1}(\gamma_{\rm max}^{3-p} - \gamma_{\rm min}^{3-p}) \tag{10}$$

assuming the electrons are nonrelativistic. Similarly, if the electrons are in a thermal distribution with temperature T, so that $\langle \beta^2 \rangle = 3kT/mc^2$, then

$$P_{\rm tot} = (4kT/mc^2) c\sigma_T n_e U_{\rm ph} \ . \tag{11}$$

We'd also like to figure out how a single Compton scattering affects the spectrum of an isotropic field of radiation. In principle, this poses no fundamental difficulties: from the fundamental Compton process we know the energy a photon will have if it comes in with one direction and scatters off in another, so we "merely" have to do the proper integrals. Rybicki and Lightman do this for a simplified case, in which in the electron's rest frame the scattering has the Thomson cross section but is isotropic instead of the dumbbell-shaped Thomson angular distribution. Even then, the integrals end up involving things like gamma and zeta functions. Feel free to look at Rybicki and Lightman, section 7.3, for details. We're not going to discuss the spectra to that level, but will instead mention a couple of the main points.

First, **Ask class:** if the energy of a photon is initially $\hbar\omega$, then in the Thomson limit (coherent scattering in the electron rest frame), what are limits on the energy of the photon after scattering? The maximum Doppler blueshift is when the scattering is head-on, in which case the photon energy changes by a factor $[\gamma(1+\beta)]^2$. Similarly, the minimum factor (maximum redshift) is $[\gamma(1-\beta)]^2$. Thus, for a given input photon energy, the spectrum is bounded. The second point is the form of the spectrum if the electron speeds have a power law distribution. As before, say $N(\gamma) = C\gamma^{-p}$ for γ between $\gamma_{\rm min}$ and $\gamma_{\rm max}$, and 0 otherwise. Then it turns out that, for a stretch of the spectrum, a blackbody distribution is scattered into a power law distribution in which

$$dE/(dV\, dt\, d\epsilon) \propto \epsilon^{-(p-1)/2} \tag{12}$$

where ϵ is the energy of the photon as measured in the lab frame. **Ask class:** can this power law distribution continue indefinitely to arbitrarily low or high energies? No! Integrated from 0 to ∞, *all* power laws diverge. This means that there *must* be a lower cutoff or an upper cutoff, and usually both. This is an important principle. Another important principle to mention in passing is that power laws arise a lot in problems like this, and the deep basic reason is that a power law is what you get if there is no scale in the problem. I'll just state that, but you may want to think about it. If there is no "important" energy, for example, then a power law is the most general thing that doesn't bend or otherwise have a special place in it. One example: think of a blackbody spectrum. Over the whole spectrum, the thermal energy kT is important, so it isn't a power law. But in the very low-energy portion, the exact temperature doesn't change the form of the spectrum, so in the Rayleigh-Jeans limit the spectrum is a power law. Just some food for thought. This might suggest (and in fact it's true) that a very general spectrum produced by Comptonization of, say, a Planck spectrum of temperature T_1 by thermal electrons of temperature T_2 is likely to be a power law for photon energies $kT_1 \ll \hbar\omega \ll kT_2$. This is one reason why often a high-energy source will have a near power-law spectrum.

The slope of the power law depends on the details of the scattering interaction, which brings us to our final point. Suppose you have photons in a medium involving high-energy electrons, and you'd like to know what happens after repeated scatterings. One useful quantity to know is the typical change in energy of a photon per scattering. If the electrons are in a thermal distribution of temperature T and are nonrelativistic, then the change in energy is

$$(\Delta\epsilon)_{\rm NR} = (\epsilon/mc^2)(4kT - \epsilon)\ . \tag{13}$$

This makes sense: if the photons are hotter than the electrons, the photons lose energy on average; if cooler, they gain energy on average. In the relativistic limit,

$$(\Delta\epsilon)_{\rm R} \sim (4/3)\gamma^2\epsilon\ , \tag{14}$$

where the 4/3 comes from angle averaging. If we think of a situation in which the electron temperature is much larger than the temperature of the input photons, then we can define the "Compton y parameter", which is the typical fractional gain in energy per scattering times the typical number of scatterings (in other words, it's the typical fractional change in energy by the time the photon leaves the region). From our work on random walks we know that if the electron scattering optical depth is τ_{es}, an approximation to the average number of scatterings is $\sim {\rm Max}(\tau_{es}, \tau_{es}^2)$, where the first one is appropriate for optically thin and the second for optically thick. The y parameter in the nonrelativistic and relativistic limits then becomes

$$\begin{aligned} y_{\rm NR} &= (4kT/mc^2){\rm Max}(\tau_{es}, \tau_{es}^2) \\ y_{\rm R} &= (4kT/mc^2)^2{\rm Max}(\tau_{es}, \tau_{es}^2)\ . \end{aligned} \tag{15}$$

The slope of the power-law portion of the spectrum depends on y:

$$\begin{aligned} I_\nu &\propto \nu^{3+m} \\ m &= -3/2 \pm \sqrt{9/4 + 4/y}\ , \end{aligned} \tag{16}$$

where the + root is appropriate for $y \gg 1$ and the - root for $y \ll 1$; if $y \sim 1$, a superposition is required and no power law exists. Note that even if y is determined from the spectrum, we don't know T and τ_{es} independently, just a combination. To break the degeneracy, the rollover in the high energy end of the spectrum must be measured.

Recommended Rybicki and Lightman problem: 7.1