1. (a) Suppose that the surface gravitational redshift of a nonrotating neutron star is \( z \). Thus a photon that has frequency \( \nu_0 \) at the surface and travels through vacuum to infinity will have a frequency \( \nu_0/(1+z) \) at infinity. The surface gravity of a star with mass \( M \) and radius \( R \) is \( g = (GM/R^2)(1+z) \), i.e., it is enhanced by a factor \((1+z)\) compared to the Newtonian surface gravity. When the flux as measured at the surface is at a critical value called the Eddington flux, the outward acceleration due to radiation exactly balances the inward acceleration due to gravity. This flux is proportional to \( g \). If we assume that the star emits uniformly over the whole surface, you should find that without the \((1+z)\) factor, the Eddington luminosity (flux integrated over area) is independent of radius. If you do include the \((1+z)\) factor, derive how the luminosity as seen at infinity depends on \((1+z)\), assuming that the local flux as measured on the surface is the Eddington critical flux.

(b) Gravitational fields also cause light rays to bend in their path. If we define \( u \equiv GM/(rc^2) \) for some radius between \( R \) and \( \infty \), the maximum total angle traveled is

\[
\Delta \phi_{\text{max}} = \int_{0}^{GM/Re^2} \left[ (1 - 2GM/Re^2)(GM/Re^2)^2 - (1 - 2u)u^2 \right]^{-1/2} \, du .
\] (1)

For example, when \( GM/Re^2 \to 0 \) we have the Newtonian result that \( \Delta \phi = 90^\circ \) (which happens for a ray emitted tangentially to the surface). With this in mind, consider a neutron star for which only a small spot on the surface emits. Using the equation above (which you can integrate numerically or with Mathematica or a similar program), show that there is a critical radius for a given mass such that below that radius you can get a caustic. A caustic is a place where, formally, the intensity goes to infinity for a particular observer. Thus, show that for a given mass there is a radius such that a specially placed observer at infinity could in principle see a flux from a small spot that is larger by an arbitrary factor than the flux seen by any other observer at infinity. Discuss whether this violates energy conservation or the conservation law for specific intensity. If not, what is the explanation?

2. In the notes we say that, for randomly distributed scatterers that have a cross section \( \sigma \) and a number density \( n \), the mean free path is \( \ell = 1/(n\sigma) \) and the probability distribution for the distance \( r \) traveled without a scatter is \( P(r)dr = \exp(-r/\ell)dr \). Derive this by analyzing a situation in which the scatterers are identical and randomly distributed spheres of radius \( R \ll n^{-1/3} \), which therefore have cross sections \( \sigma = \pi R^2 \).

3. An independent theorist, Dr. I. M. N. Sane, has demonstrated that nuclear theorists are a bunch of establishment fools. These theorists have models of very dense matter that they
apply to neutron stars, and they have concluded that the radius of a 1.5 $M_\odot$ neutron star is between 10 km and 15 km. But Dr. Sane has proved them wrong.

He has done a careful fit of the X-ray thermal spectrum from a neutron star whose distance he knows precisely. The spectrum is very well fit by a Planck function. Based on the Planck temperature $T$ and the luminosity $L$ of the source (which he gets from the flux he observes and the known distance), he has computed the radius from the blackbody formula $L = 4\pi R^2 \sigma T^4$ to be $R = 6$ km, with negligible uncertainty. Based on this, Dr. Sane has concluded that this object is actually a “Sane star”, which is composed of a previously unknown particle that he modestly calls a saneon.

The committee for the Nobel Prize in Physics has asked you to assess Dr. Sane’s idea. In particular, they ask you to make the following assumptions that Dr. Sane does as well: (i) the whole surface of the star emits with a uniform flux, (ii) the atmosphere of the star consists of a perfect scattering layer on top of a perfect blackbody, and (iii) there is a total flux $F$ that is determined deep in the star that must escape to infinity. The scattering layer, in which photon number is conserved perfectly and scatterings do not change the photon energy at all, has a thickness such that only a fraction $1/4$ of the photons escape once they have been generated at the blackbody layer. The rest scatter around and then are reabsorbed in the blackbody layer.

Based on these assumptions, evaluate Dr. Sane’s suggestion. What would be your estimate of the radius? [Note: unlike most of my Dr. Sane problems, this one is based on something that actually was proposed, with an accompanying press release, several years ago.]

4. An evil E&M professor has assigned you the task of deriving the instantaneous power per solid angle radiated by a charge $e$ that moves in a circular orbit. At the time in question, the charge is at the origin of your coordinates, moving with speed $v = \beta c$ in the $+z$ direction, and its acceleration $\dot{v}$ is in the $+x$ direction. The Lorentz factor is $\gamma = (1 - \beta^2)^{-1/2}$ as usual. After a bleary night’s work you have come up with the cgs expression

$$\frac{dP}{d\Omega} = \frac{e^2}{4\pi c^3 (1 - \beta \cos \theta)^3} \left[ 1 - \frac{\sin^2 \theta \sec^2 \phi}{\gamma^2 (1 - \beta \cos \theta)^2} \right].$$

Here you are close to the source and your direction $\hat{n}$ from the origin is $(\theta, \phi)$, where $\theta$ is the normal polar angle measured from the $+z$ direction, and $\phi$ is the azimuthal angle; as usual, if you project $\hat{n}$ onto the $x - y$ plane, $\phi$ is 0 in the $+x$ direction, $\pi/2$ in the $+y$ direction, and so on. The homework is due in the morning, so you need to decide whether this equation satisfies constraints of units, limits, and symmetries. Please do not look up this expression.

(a) Are the units correct?
(b) Does this expression have the correct symmetries? For example, does the angular dependence satisfy your intuition? What if the signs of $\beta$ or $e$ are reversed?

(c) Does this expression have the correct limits? What if $e$, $\beta$, or other quantities go to zero or other special values? What other limits can you check? Be quantitative if possible, but at least indicate, e.g., when the power should be large and when it should be small.