1. Dr. Sane has realized that the Einstein relations are woefully incomplete. They involve only radiative transitions, but of course real atoms are also subjected to collisional interactions. This changes the rates of both absorption and emission, and it means that if radiative rates are in equilibrium, the overall level populations are not. Dr. Sane has publicized his resulting “detailed imbalance” theory, in which he claims that (a) as a result of collisional processes, equilibrium is not possible, and (b) even if it were, the level populations in equilibrium would differ from the standard proportionality to \(ge^{-E/kT}\), where \(g\) is the state multiplicity and \(E\) is the standard energy of the isolated atomic state. The National Academy of Sciences is considering Dr. Sane for membership as a result, and they have brought you in as a consultant.

(a) Evaluate Dr. Sane’s claim that when collisions are included, equilibrium is not possible.

(b) Even if equilibrium is possible, evaluate Dr. Sane’s second claim, which when re-expressed says that for a two-level atom with an isolated ground state energy \(E_{\text{ground}}<0\) and an isolated excited state energy \(E_{\text{ground}}<E_{\text{excited}}<0\), with respective multiplicities \(g_{\text{ground}}\) and \(g_{\text{excited}}\), the relative number of atoms in each state in equilibrium will differ from

\[
\frac{N_{\text{excited}}}{N_{\text{ground}}} = \frac{g_{\text{excited}}}{g_{\text{ground}}} e^{-\frac{(E_{\text{excited}}-E_{\text{ground}})}{kT}}
\]

at temperature \(T\).

2. Lord Voldemort, who is showing surprising scientific curiosity, decides to perform the following experiment. A mere gesture with his yew wand produces an impermeable sphere of volume \(V\). Inside that box he puts \(N\) photons, along with matter with a temperature \(T_0\). He magically constrains the photon-matter interaction to be pure scattering; the number of photons never changes, but the energies of photons can increase or decrease as they scatter (thus the total energy in the photons can change, although the matter always is forced to remain in thermal equilibrium at temperature \(T_0\)). He finds that the photons adjust their distribution function to a Bose-Einstein form

\[
n(p) = \frac{1}{\frac{1}{\hbar^3} e^{-\mu+E(p)/kT_0} - 1}
\]

where \(\mu\) is the chemical potential (which is independent of \(p\)) and \(E(p) = pc\) is the energy of a photon of momentum \(p\). The momentum distribution is isotropic. Given this situation:

(a) Find an expression that determines \(\mu\) as a function of \(N\), \(V\), and \(T_0\), assuming that \(N/V\) is less than or equal to the number density \(n_{\text{BB}}(T_0)\) for blackbody radiation at a temperature \(T_0\). Determine \(\mu\) explicitly in the limiting cases that \(N/V \approx n_{\text{BB}}(T_0)\) (but we still have
\( N/V \leq n_{BB}(T_0) \) and \( N/V \ll n_{BB}(T_0) \). Use this to check your expressions. **Note:** we have Mathematica on the departmental computers.

(b) What happens when \( N/V \) is *greater* than the number density for blackbody radiation at a temperature \( T_0 \)?

3. **(8 points)** In our first computing assignment we will explore the Saha equation, and will test one of the assumptions we made in the notes.

(a) Assuming pure hydrogen and thermodynamic equilibrium, plot the temperature that gives an equilibrium ionization \( y = 0.5 \) for densities \( \rho \) from \( 10^{-31} \) g cm\(^{-3}\) (roughly the average baryon density of the universe) to \( 10^{-16} \) g cm\(^{-3}\) (roughly the density of the core of an interstellar cloud). Discuss the physical reason for any trends you find.

(b) We assumed in the notes that we only have to include the hydrogen ground state in our analysis. But is this true? To determine this, we will include all the relevant states of hydrogen. We will do this by noting that the binding energy \( \chi_n \) of principal quantum number \( n \) of hydrogen (where \( n = 1 \) is the ground state) is \( \chi_H/n^2 \), where \( \chi_H = m_e e^4/(2\hbar^2) \), and that the multiplicity of state \( n \) is \( 2n^2 \), so that \( e^{\chi_H/kT} \) in equation (4) in the Lecture 8 notes should be replaced by

\[
e^{\chi_H/kT} \sum_{n=1}^{n_{\text{max}}} n^2 e^{(\chi_n-\chi_H)/kT}.
\]

Here \( n_{\text{max}} \) is determined by the largest orbit that does not impinge on the next atom over. The size of a hydrogen atom of principal quantum number \( n \) is \( r_n = n^2\hbar^2/(m_e e^2) \). Let \( n_{\text{max}} \) be given by \( r_n \) such that \( r_n \) is less than or equal to half of the average distance to the next atom, at the density that is being considered. Plot the temperature for \( y = 0.5 \) at the same densities as before. Again, discuss the reason for any trends you find in comparison with part (a).

**For both parts, please send me a copy of your code before you hand in the plots on the due date.** Any language is fine as long as it compiles and runs on my departmental machine (please send me compilation/run instructions); I won’t install any libraries or download modules! Please ensure that when your code runs, it produces both a plot (which is all you need for the hardcopy) and a table of \( \log_{10}\rho \) versus \( T \). In the tables, your values of \( T \) must be correct to at least three significant figures; thus please use values of \( e, m_e, m_p, \) and \( \hbar \) that are precise.