ASTR 601: Problem Set 5, due Thursday, November 16

1. This is your computational problem, although most of your work will be analytic. We show in the notes that because, obviously, no energy state has a lower expected energy than the ground eigenstate, any trial wavefunction gives an upper bound to the ground state energy if the expected energy of that wavefunction is minimized over the wavefunction's parameters. Here we'll give you an opportunity to explore a wavefunction family different from the true family. In particular, let the wavefunction family be a Gaussian, $\psi = Ce^{-r^2/2r_0^2}$. Using this family:

a. Determine the normalization constant C.
b. Determine the integrand that needs to be integrated to get the expected energy (explicitly, i.e., do the derivatives).
c. Perform the integral and minimize the energy as a function of r_0. How close do you get to the true answer?
d. Finally, the computational part: plot (i.e., include a hardcopy with your submission) the energy (in units of $m_e e^4/\hbar^2$) versus the characteristic radius r_0 (in units of $\hbar^2/(m_e e^2)$) for the correct waveform family (see the notes) and for the $e^{-r^2/2r_0^2}$ family that you computed here. The radius should go from 0.5 to 3 in your units of $\hbar^2/(m_e e^2)$. What trends do you notice? For example, is the energy as a function of r_0 a pure parabola? As usual, I need you to send me your code (in any language but in a form that can be compiled [please send instructions!] and run on my departmental desktop) before the class starts.

2. Shortly before the appendix in the Lecture 20 notes we discuss dipole selection rules. Let's phrase this in terms of parity, where we'll consider one dimension for simplicity. A function f has positive parity if $f(-x) = f(x)$ for any x, and negative parity if $f(-x) = -f(x)$ for any x. a. Show that whether f has positive or negative parity, $\int_{-\infty}^{\infty} f(x)xf(x)dx = 0$.
b. More generally, show that for any two functions f and g that have definite parity (meaning either negative or positive parity), $\int_{-\infty}^{\infty} f(x)xg(x)dx$ is nonzero only if f and g have opposite parity.
c. Even more generally, show that for any two functions f and g with definite parity, $\int_{-\infty}^{\infty} f(x)x^n g(x)dx$ is nonzero only if the product of the parities of f and g equals $(-1)^n$. This underlies many selection rules.

3. You have discovered an expanding molecular bubble in our galaxy, at a distance of 400 pc. You are observing an *emission* line, specifically $^{13}CO(2 \rightarrow 1)$. The bubble is optically thin to this line, i.e., you see emission from the back side as well as the front side.

(a) The observed full fractional width of the line is $\Delta\lambda/\lambda = 2.3\times10^{-4}$. What is the expansion velocity of the bubble, as measured from its center? Be careful of your factors of 2!

(b) The angular diameter of the bubble is 20", and the inferred column depth of H_2 through the bubble's center is 4×10^{23} cm^{-2}. Assuming that essentially all of the mass is in H_2 and that the molecular bubble is a perfect sphere of uniform density, what is the mass of the bubble to within 10%?

(c) From parts (a) and (b), derive limits on the *initial* kinetic energy and *current* age of the bubble, assuming that after the event that created the bubble it has expanded passively into the interstellar medium. In each case, state whether the limit is an upper limit or a lower limit, and explain your reasoning.

4. Ordinary cosmologists believe that after a redshift of $z \sim 1000$ the universe became nearly neutral (because it had cooled substantially). But Dr. Sane has realized that such people are all foolish dupes: the universe will continue to have a high ionization even after this period! His point, which is obvious after it is raised, is that if an electron and proton combine to form a hydrogen atom (we'll ignore helium and other elements), then the photon that is released of course has enough energy to ionize a neutral hydrogen atom. Thus the net ionization will remain high. A university that you dislike is considering hiring Dr. Sane onto their faculty, and they have asked for your opinions.

(a) Give two qualitatively distinct arguments for why Dr. Sane's logic is flawed. These need to be arguments at a microscopic level: that is, we are *not* allowed to use equilibrium arguments (a la the Saha equation). One of the arguments could be specific to the expanding universe, but the other would also have to be valid for an isolated box with photons and an initially highly ionized set of electrons and protons.

(b) Suggest an *observational* disproof of Dr. Sane's idea. To do this, note that (i) the number density of electrons (total, including bound and free) in the universe at redshift z is about $n_e = 2 \times 10^{-7}(1+z)^3$ cm^{-3}, (ii) the locally measured distance between redshifts z and $z + dz$, at the high redshifts that are most important, is $ds = 7.8$ Gpc $(1+z)^{-5/2}dz$ (consider $z > 100$), and (iii) we can see sub-degree angular structures in the cosmic microwave background, which has $z = 1090$.

Computational challenge problem

I think the four problems above are plenty for the homework, but for those of you who enjoy writing code I have a challenge for you. Note that this will not count in your grade at all; you don't need to do this as part of your homework, and even if you complete the challenge successfully you won't get any extra credit.

The challenge is to partially automate the problem of identifying lines in spectra. We will assume that our observations involve so many photons that we can use χ^2 statistics (which as you recall require Gaussians) with a clear conscience. Someone measures the wavelengths of n lines from a source, and also quotes their standard deviations. Thus your data are $\lambda_1, \delta\lambda_1, \lambda_2, \delta\lambda_2, \ldots, \lambda_n, \delta\lambda_n$. Assume that the probability distribution for each wavelength measurement is Gaussian. You happen to know, somehow, that these are hydrogen atomic transitions, and by the wavelength ratios you are able to figure out the initial and final principal quantum numbers n_i and n_f for each transition.

Your task is to write a code to determine the redshift z to the source, plus the 1σ uncertainty on the redshift. Recall that if you have n measurements of data d_i with standard deviation σ_i and your model predicts m_i, then

$$\chi^2 = \sum_{i=1}^{n} \frac{(m_i - d_i)^2}{\sigma_i^2} \qquad (1)$$

and that for one model parameter (z in this case), $\Delta\chi^2 = 1$ for one standard deviation, $\Delta\chi^2 = 4$ for two standard deviations, and generally $\Delta\chi^2 = m^2$ for m standard deviations (note that these numbers change if you have different numbers of parameters). To run your code you will also need to generate synthetic data; I recommend that you use the function "gasdev" from Numerical Recipes to draw from a Gaussian. We have the Numerical Recipes functions in C and FORTRAN on our system.

If this task is too easy for you, you could consider extending it to the case where you do *not* know the initial and final principal quantum numbers. If you pursue this in depth you will encounter some subtle statistical issues, and might learn a good deal about the necessity for astronomers to have priors when they try to identify lines.