

Transfer Equation and Blackbodies

We are now in a position to examine what happens to a beam of radiation as it goes through matter. In general, the things that affect the intensity of the beam are:

Spontaneous emission.—Matter along the path of the beam can spontaneously emit radiation (e.g., spontaneous transitions of atoms from excited states to lower energy states). This adds to the intensity.

Stimulated emission.—If radiation passes through matter, the matter can be stimulated to emit radiation of the same frequency and direction as the initial radiation. This is the principle behind lasers, and it, too, adds to the intensity.

Absorption.—Radiation can be absorbed by the matter. This takes away from the intensity. Note that both absorption and stimulated emission are proportional to the original intensity of the beam of matter. They are therefore commonly lumped together (with stimulated emission being “negative absorption”), so we’ll group both of them under the category of “absorption”.

Scattering.—The photons can be redirected and changed in frequency without being destroyed.

Spontaneous emission

The monochromatic spontaneous emission coefficient j_ν is defined as the energy per time per volume per frequency interval emitted into a solid angle $d\Omega$:

$$dE = j_\nu dV d\Omega dt d\nu . \quad (1)$$

Note that this is independent of the intensity of the beam. In general one must consider anisotropic emission, especially when there is a global preferred direction (e.g., set by a magnetic field). However, in many situations the emission is nearly isotropic, either because the emission from individual particles/atoms/molecules is isotropic or because the preferred axes of individual particles etc. are distributed randomly. In that case, $j_\nu = P_\nu/4\pi$, where P_ν is the total emitted power per volume per frequency. On occasion you’ll see the power expressed as an *emissivity* ϵ_ν , which is the energy per mass per frequency, so $P_\nu = \rho\epsilon_\nu$ where ρ is the density.

Ask class: considering only spontaneous emission, what is the change in specific intensity I_ν after traveling a distance ds in a medium with spontaneous emission coefficient j_ν ? It is

$$dI_\nu = j_\nu ds . \quad (2)$$

This can be determined directly from the units. It also makes sense physically: it’s positive,

since intensity is being added; the higher j_ν is, the more intensity is added; and the farther one travels in the medium, the more intensity is enhanced.

Absorption (including stimulated emission)

In an analogous way, we define the absorption coefficient α_ν (dimensions cm^{-1}), by the equation

$$dI_\nu = -\alpha_\nu I_\nu ds . \quad (3)$$

Note that, unlike spontaneous emission, absorption is proportional to the intensity of the beam. This is the intensity removed from the beam after traveling a distance ds . By convention, α_ν is positive for net energy removed; note, however, that $\alpha_\nu < 0$ if stimulated emission dominates. Phenomenologically, we can understand this if we imagine a number density n of absorbers of cross section σ_ν at frequency ν . **Ask class:** based on an analysis of units, how should α_ν be related to n and σ_ν ? The only combination that gives the required cm^{-1} is $\alpha_\nu = n\sigma_\nu$. This can also be written $\alpha_\nu = \rho\kappa_\nu$, where κ_ν is the opacity. This phenomenological picture is only valid if the distance between absorbers is much larger than $\sqrt{\sigma_\nu}$ and the absorbers are distributed randomly. This is almost always the case for astrophysical situations, but for a wild violation consider metals. Metals are good thermal and electric conductors because electrons can travel great distances in them before interacting. However, if you naively consider the cross section of each atom and multiply by the number density of atoms, you'd conclude that the mean free path is tiny and so metals are terrible conductors! The loophole is that atoms in metals are distributed in highly regular lattices, and quantum analysis of the resulting periodic potential shows that electrons can travel large distances.

Scattering has some subtleties, so we'll postpone dealing with that for the moment and consider only spontaneous emission and absorption.

The Radiative Transfer Equation

We can combine the effects of spontaneous emission and absorption into one equation telling us what happens to the specific intensity along a ray:

$$dI_\nu/ds = -\alpha_\nu I_\nu + j_\nu . \quad (4)$$

Ask class: what happens when there is no absorption? Then only the second term on the right side matters, and $dI_\nu/ds = j_\nu$. Formally, this may be integrated to get

$$I_\nu(s) = I_\nu(s_0) + \int_{s_0}^s j_\nu(s) ds . \quad (5)$$

Ask class: now, what happens when there is no spontaneous emission? Then only the first term contributes and the formal solution is

$$I_\nu(s) = I_\nu(s_0) \exp \left[- \int_{s_0}^s \alpha_\nu ds \right] . \quad (6)$$

Thus, if absorption dominates over stimulated emission (i.e., $\alpha_\nu > 0$) the intensity decreases exponentially along the path. If stimulated emission dominates then the intensity *increases* exponentially, which is one reason why lasers can get to such high intensities.

The pure absorption case is particularly simple if we use the optical depth, which we discussed in an earlier class. **Ask class:** remembering that the optical depth is basically the number of mean free paths traversed by the beam, how should it be related to α_ν ? It is just $d\tau_\nu = \alpha_\nu ds$, so $I_\nu(s) = I_\nu(s_0) \exp(-\tau)$ in the pure absorption case.

Using the optical depth as a new variable, it is convenient to divide the radiative transfer equation by α_ν to get

$$dI_\nu/d\tau_\nu = -I_\nu + S_\nu, \quad (7)$$

where $S_\nu \equiv j_\nu/\alpha_\nu$ is called the *source function* (note that j_ν and α_ν are both local properties of the material, so S_ν is as well). This equation may be solved formally to get

$$I_\nu(\tau_\nu) = I_\nu(0)e^{-\tau_\nu} + \int_0^{\tau_\nu} e^{-(\tau_\nu - \tau'_\nu)} S(\tau'_\nu) d\tau'_\nu. \quad (8)$$

We can get this result qualitatively from the principle that radiation may be superposed linearly as follows. The initial radiation (intensity I_ν) is attenuated by a factor $e^{-\tau_\nu}$. There is also radiation generated along the way, and it too is attenuated by the end, by the appropriate factor $\exp(-\Delta\tau_\nu)$, where $\Delta\tau_\nu = \tau_\nu - \tau'_\nu$ for radiation generated at the “position” τ'_ν . These contributions add linearly.

For example, consider a constant source function S_ν . Then

$$\begin{aligned} I_\nu(\tau_\nu) &= I_\nu(0)e^{-\tau_\nu} + S_\nu(1 - e^{-\tau_\nu}) \\ &= S_\nu + e^{-\tau_\nu}[I_\nu(0) - S_\nu] \end{aligned} \quad (9)$$

Ask class: what does this mean for a very optically thick medium? High optical depth means $\tau_\nu \gg 1$, so the exponential factor goes to zero and $I_\nu \rightarrow S_\nu$. This is important: it means that at high optical depth, if S_ν doesn't vary much then I_ν approaches S_ν .

If you include scattering things get more difficult, because the source function depends on I_ν at all directions through a given point. For example, consider coherent, isotropic scattering (i.e., the photon energy doesn't change but its direction is completely randomized by scattering). Then $j_\nu = \alpha_\nu(\text{scatt})J_\nu$, where $\alpha_\nu(\text{scatt})$ is the scattering coefficient (similar to an absorption coefficient; R+L use σ_ν for this but I don't want to confuse it with the cross section σ) and J_ν is the mean intensity within the emitting material. Dividing, $S_\nu = j_\nu/\alpha_\nu = J_\nu$, so the transfer equation for pure scattering is

$$dI_\nu/ds = -\alpha_\nu(\text{scatt})(I_\nu - J_\nu) \quad (10)$$

and the transfer equation including thermal emission, absorption, and coherent, isotropic scattering is

$$dI_\nu/ds = -(\alpha_\nu + \alpha_\nu[\text{scatt}])(I_\nu - S_\nu), \quad (11)$$

where the source function is the weighted average of those for absorption and emission:

$$S_\nu = [\alpha_\nu B_\nu + \alpha_\nu(\text{scatt})J_\nu] / [\alpha_\nu + \alpha_\nu(\text{scatt})] . \quad (12)$$

This is an integro-differential equation and although some progress may be made using the Eddington approximation (later), in practice one solves this numerically.

Thermal Emission

A special and important case is one in which the radiation and matter are in thermal equilibrium. This gives a universal and exact solution and was the problem that led to the first glimmerings of quantum mechanics. Still, in any given problem you need to consider carefully whether the radiation and matter are in thermal equilibrium or whether nonthermal processes are important.

Let's imagine an enclosure at temperature T . We have radiation in the enclosure, and we wait a long time until equilibrium has been achieved. Note that there is no conservation law for photons, so they can be created or destroyed by many processes (equivalently, we say that photons have zero chemical potential). From thermodynamics, the specific intensity that we get must be independent of the shape of the enclosure. Otherwise, you could take two enclosures at the same temperature but with different shapes, and when you put them in contact energy would flow from one to the other, in violation of thermodynamics. Thus, the specific intensity in thermal equilibrium depends only on the temperature. This argument also shows that the specific intensity in equilibrium must be isotropic. Therefore,

$$I_\nu = B_\nu(T) \quad (13)$$

in thermal equilibrium.

If you now put thermally emitting matter at the same temperature T into the enclosure, the source function must be unchanged because we still have a blackbody enclosure at the temperature T . Therefore (Kirchoff's law), the source function in thermal equilibrium is the blackbody function

$$S_\nu = B_\nu(T) . \quad (14)$$

This also means that within the blackbody portion, $I_\nu = B_\nu$. Since blackbody radiation is homogeneous and isotropic, then from previous results we know that the pressure and energy density are related by $p = u/3$. Note that there is a useful distinction between *blackbody radiation*, where $I_\nu = B_\nu$, and *thermal radiation*, where $S_\nu = B_\nu$. Even if all the matter is radiating thermally ($S_\nu = B_\nu$), you aren't guaranteed that $I_\nu = B_\nu$ unless the medium is optically thick.

We will now skip ahead a bit. Rybicki and Lightman give some interesting derivations about blackbody radiation on pages 17-21, from classical thermodynamics as well as quantum arguments. The quantum argument, which Planck used to derive the blackbody

function (aka the Planck function), is particularly cute, and relies on photon modes being required to fit exactly (integral number of wavelengths) within a box, hence quantizing the photon energies. This was the first time that anyone had done that, and it marked a transition from classical to quantum theory that Einstein followed up five years later with the photoelectric effect. I recommend that you read the derivation, but let's focus now on properties of the function itself.

Properties of the Planck Function

A blackbody emits a flux (energy per area) of $\sigma_{\text{SB}}T^4$, where $\sigma_{\text{SB}} = 5.67 \times 10^{-5} \text{ erg cm}^{-2} \text{ K}^{-4}$ is the Stefan-Boltzmann constant. The Planck blackbody function is

$$B_\nu(T) = \frac{2h\nu^3/c^2}{\exp(h\nu/kT) - 1}, \quad (15)$$

or with wavelength $\lambda = c/\nu$ instead of ν as the primary variable,

$$B_\lambda(T) = \frac{2hc^2/\lambda^5}{\exp(hc/\lambda kT) - 1}. \quad (16)$$

Let's look at some limits of this expression.

Low frequency, $h\nu \ll kT$.—The exponent is much less than unity, so $B_\nu(T) \approx 2\nu^2 kT/c^2$. **Ask class:** there is a fundamental constant missing here; what is it? The Planck constant h , of course! This leads to an important point that will allow you to check some equations. The Planck constant will appear if and only if quantization is important. In the same way, c appears if and only if special relativity is important, and G appears if and only if gravity is important. So why isn't quantization important here? In the low-frequency, or Rayleigh-Jeans, limit, there are many photons. We therefore have a classical description. However, it was noticed immediately that if this expression continued to be valid for arbitrarily high frequency the energy would diverge (this was called the "ultraviolet catastrophe"). In the low-frequency limit the form of the Planck function (a power law, logarithmic slope 2 in frequency) is independent of the temperature.

High frequency, $h\nu \gg kT$.—Here the exponent is much greater than unity, so $B_\nu(T) \approx 2h\nu^3/c^2 \exp(-h\nu/kT)$. Now the Planck constant does appear. It's because in this limit (the Wien limit), there are very few photons and hence their discrete nature matters.

Here are some other interesting and important facts. First, there is a photon frequency at which $B_\nu(T)$ peaks, and it's at $h\nu_{\text{max}} = 2.82kT$, give or take. Incidentally, you could derive that the peak energy has to be proportional to kT simply by noticing that the only way you can get something with dimensions of energy out of h, c, k , and T is by kT ! Score one for dimensional analysis. Moving on, although the peak frequency changes, $B_\nu(T_1) > B_\nu(T_2)$ for *all* frequencies if $T_1 > T_2$.

Finally, there are some characteristic temperatures that are defined in astrophysics to relate a given arbitrary spectrum to the blackbody spectrum.

Brightness temperature.—This is the temperature a blackbody would have to have to give the observed specific intensity at a given frequency: $B_\nu(T_b) = I_\nu$. This is especially common in radio astronomy, where because of the low frequencies you're usually in the Rayleigh-Jeans limit, so $I_\nu = (2\nu^2/c^2)kT_b$.

Color temperature.—This is the temperature of a blackbody that gives the same *slope* for the spectrum as the observed slope. This is useful whenever you don't know the absolute flux, which is the case if you don't know the distance to the object (as an example).

Effective temperature.—This is the temperature of a blackbody that gives the same frequency-integrated intensity as the observed one, radiated at the source. That is, $\sigma_{\text{SB}}T_{\text{eff}}^4 = F = \int I_\nu \cos \theta d\nu d\Omega$.