Polarization and Stokes Parameters

In the last class we had some discussion about the polarization of a plane wave, but now we need to go into it in more detail. Shu has more on this than Rybicki and Lightman do, so we'll follow Shu.

Let's first consider a single, monochromatic, wave. **Ask class:** if the wave is propagating in the z direction, what are the possible directions of linear polarization at any given instant? Since the wave is transverse, the linear polarization must be in the x-y plane. We could therefore break down the electric field into x and y components:

$$\mathbf{E} = \hat{\mathbf{x}}\mathcal{E}_x + \hat{\mathbf{y}}\mathcal{E}_y \tag{1}$$

where $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are unit vectors in the x and y directions. **Ask class:** is there a unique way to define the x and y directions? No, in general there isn't. That means that angles defined with respect to a specific choice of the axes are to some extent arbitrary. However, if one sticks with a particular definition of axes, the *differences* in angles between different sources *can* be meaningful. It's an important distinction to make.

Ask class: have we exhausted the possible description of the polarization of a monochromatic wave? Specifically, since we can describe the electric field vector by a particular combination of linear polarizations, will it stay with that combination forever? No, in fact there can be a time variation as well. If the frequency of the wave vector is ω, then we have

$$\mathbf{E} = \hat{\mathbf{x}}\mathcal{E}_x \cos(\omega t - \phi_x) + \hat{\mathbf{y}}\mathcal{E}_y \cos(\omega t - \phi_y) \ . \tag{2}$$

Here ϕ_x and ϕ_y are the phases at time $t = 0$. **Ask class:** are these phases independently meaningful? Again, no. Picking a different zero for the time doesn't change anything physically measureable, but it would change ϕ_x and ϕ_y. **Ask class:** Is there a combination of these phases that *is* meaningful? Yes, the difference is independent of the particular zero of time. Once again, it is important to keep these kinds of things straight; if you like, it's a form of symmetry.

Anyway, our general form of the electric field vector will trace out an ellipse over time. The major axis of the ellipse will have a tilt angle χ with respect to the x axis. We can then define new principal axes $\hat{x}'$ and $\hat{y}'$ along the axes of the ellipse, and write the electric field

$$\mathbf{E} = \hat{\mathbf{x}}' E_1 \cos\omega t + \hat{\mathbf{y}}' E_2 \sin\omega t \ . \tag{3}$$

There is also an axis ratio; we can identify $\mathcal{E}_0^2 \equiv E_1^2 + E_2^2 = \mathcal{E}_x^2 + \mathcal{E}_y^2$, and define another "angle" β so that $E_1 = \mathcal{E}_0 \cos\beta$ and $E_2 = \mathcal{E}_0 \sin\beta$. For a monochromatic wave we can then define the *Stokes parameters*, which are four quantities quadratic in the electric field components:

$$
\begin{aligned}
I &= \mathcal{E}_x^2 + \mathcal{E}_y^2 = \mathcal{E}_0^2 \\
Q &= \mathcal{E}_x^2 - \mathcal{E}_y^2 = \mathcal{E}_0^2 \cos 2\beta \cos 2\chi \\
U &= 2\mathcal{E}_x \mathcal{E}_y \cos(\phi_y - \phi_x) = \mathcal{E}_0^2 \cos 2\beta \sin 2\chi \\
V &= 2\mathcal{E}_x \mathcal{E}_y \sin(\phi_y - \phi_x) = \mathcal{E}_0^2 \sin 2\beta
\end{aligned} \tag{4}
$$

Again we emphasize that this is for a monochromatic wave; we'll get to what happens with superpositions of waves in a bit.

Note that, as expected physically, only the phase difference $\Delta\phi = \phi_y - \phi_x$ matters rather than the independent phases. Note also that since the three parameters $\mathcal{E}_0$, β, and χ determine the four parameters I, Q, U, and V, there must be a relation between the Stokes parameters. It's $I^2 = Q^2 + U^2 + V^2$ (again, this only holds for 100% polarized waves). One also has the relations

$$\tan 2\chi = U/Q, \qquad \sin 2\beta = V/I\ . \tag{5}$$

Ask class: if $V = 0$, what does that mean for β? It means that $\beta = 0$ or $\pm\pi/2$. **Ask class:** what does that tell us about the electric field? It means that it stays along one of the principal axes, meaning it's linearly polarized. If instead $Q = U = 0$ (so that $V = I$), then $\beta = \pm\pi/4$ and the axes are equal, so the electric field traces out a circle on the sky and the light is *circularly polarized*. It is called right-circularly polarized or left-circularly polarized depending on whether $\beta = \pi/4$ or $-\pi/4$, but different conventions exist so be really careful if the handedness matters to you!

In a practical sense, we can't really measure the electric field vector of monochromatic light cycle by cycle. The frequencies are extremely high (10^7 cycles per second even for decameter radio waves), so we usually have to take a time average. In addition, a detector will almost always have some finite bandwidth, so in reality we'll be averaging over a number of different frequencies. Let's denote averages over time and bandwidth by angular brackets. Then what we really measure is

$$
\begin{aligned}
\bar{I} &= \langle \mathcal{E}_x^2 + \mathcal{E}_y^2 \rangle = \langle \mathcal{E}_0^2 \rangle \\
\bar{Q} &= \langle \mathcal{E}_x^2 - \mathcal{E}_y^2 \rangle = \bar{I} \cos 2\beta \cos 2\chi \\
\bar{U} &= 2\langle \mathcal{E}_x \mathcal{E}_y \rangle \cos \Delta\phi = \bar{I} \cos 2\beta \sin 2\chi \\
\bar{V} &= 2\langle \mathcal{E}_x \mathcal{E}_y \rangle \sin \Delta\phi = \bar{I} \sin 2\beta
\end{aligned} \tag{6}
$$

We're still making the implicit assumption that the light is 100% elliptically polarized, otherwise the angles χ and β, as well as the phase difference $\Delta\phi$, would change over the bandwidth.

But now let's forego that assumption. Suppose we're looking at a general source of light. It will be a superposition of many different waves, which don't necessarily have a fixed phase relation between themselves. Therefore, we can consider the total electric field to be composed of many independent elliptically polarized waves, $\mathbf{E} = \sum_n \mathbf{E}^{(n)}$. If we assume that the different streams add incoherently (like a random walk), then the Stokes parameters (which are quadratic) are the sums of squares instead of the square of sums, meaning that

$$\bar{I} = \sum_n \bar{I}^{(n)}, \qquad \bar{Q} = \sum_n \bar{Q}^{(n)}, \qquad \bar{U} = \sum_n \bar{U}^{(n)}, \qquad \bar{V} = \sum_n \bar{V}^{(n)} . \tag{7}$$

This incoherent addition means that the relation $I^2 = Q^2 + U^2 + V^2$ no longer holds in general, but is replaced by the inequality

$$I^2 \geq Q^2 + U^2 + V^2 . \tag{8}$$

In this case (which is the only one seen in practice), all four Stokes parameters are independent and must be measured separately. We can then consider the light to be a combination of completely unpolarized light (with $\bar{Q}_u = \bar{U}_u = \bar{V}_u = 0$) and 100% elliptically polarized light in which

$$\bar{I}_p = \left(\bar{Q}_p^2 + \bar{U}_p^2 + \bar{V}_p^2\right)^{1/2} . \tag{9}$$

Then $\bar{I} = \bar{I}_p + \bar{I}_u$ and the fractional polarization is $\bar{I}_p/\bar{I}$.

The introduction of polarization produces some mild complications in radiative transfer. The key is to realize that one can think of the four Stokes parameters as different components of the electric field that propagate independently. Specifically, one can multiply $|\mathbf{E}|^2$ by a factor that converts it into the specific intensity I_ν, then do the same for the other Stokes parameters: Q_ν, U_ν, and V_ν. One can then think of the full specific intensity as a vector with these four quantities. A mild tweak used by Chandrasekhar is to define $I_\nu^+ = \frac{1}{2}(I_\nu + Q_\nu)$ and $I_\nu^- = \frac{1}{2}(I_\nu - Q_\nu)$. These represent intensities of linear polarization in two mutually orthogonal directions. The vector specific intensity is then

$$\vec{I}_\nu = (I_\nu^+, I_\nu^-, U_\nu, V_\nu) \tag{10}$$

and along a particular direction $\mathbf{k}$ the equation of radiative transfer is

$$d\vec{I}_\nu/d\tau = \vec{S}_\nu - \vec{I}_\nu . \tag{11}$$

Here the source function also has four components, hence is a vector.

Whew. Time to take stock. **Ask class:** given the above analysis, can they think of circumstances in which radiative transfer can convert initially unpolarized light into something polarized? One way is by scattering. As a specific example, consider normal Rayleigh scattering of light. As the wave hits a particle, the particle can oscillate in the plane of polarization of the light. If we are at an angle Θ from the original direction, however, we don't see any polarization from the component of the oscillation that is in our direction. If $\Theta = 0$ then we see the original polarization (none), but if $\Theta = \pi/2$ we can see only one component of the polarization, so it's 100% linearly polarized. In fact, the fractional polarization is $1 - \cos^2\Theta$. **Ask class:** suppose they are outside looking at the blue sky. Armed with only a polarization filter, how might they determine the direction of the Sun just by looking at the scattered light (rather than taking the easy way and looking at the Sun!)? One could look at each small patch of the sky with the filter, turning the filter to see how the intensity varies with filter direction. At the place where the modulation is maximal, the Sun's direction is perpendicular to the direction of maximum polarization. Another application is polarizing sunglasses, which use this principle to remove glare from water or other places where radiation scatters.

Why are clouds opaque? Molecules are then much closer together than the wavelength of light, so they act in concert, an N^2 instead of an N effect.

Ask class: can they think of a way in which radiative transfer could *decrease* the amount of polarization? This one is a lot trickier, and we'll encounter it later in plasma effects. Faraday rotation occurs when there is a magnetic field, which (among other things!) has the consequence that right circular and left circular polarization have different indices of refraction. That means they travel at different speeds, so over time their relative phase changes. Now, we can imagine an initially linearly polarized wave as being composed of some right circular and some left circular polarization. Over time, the superposition of the two circular components changes because of the relative phase, meaning that the polarization angle changes. If the length scale over which the magnetic field changes is small enough, then in a particular beam the polarizations add incoherently, meaning that the polarization fraction decreases. This decrease in polarization is a major way in which astrophysical magnetic fields are measured.