# Retarded Potentials and Radiation

No, this isn't about potentials that were held back a grade :). Retarded potentials are needed because at a given location in space, a particle "feels" the fields or potentials of other charges, not where those charges are *now*, but where they *were* a light travel time ago. A few lectures ago we talked about how electromagnetism can be phrased in terms of potentials rather than fields. **Ask class:** if you have a single charge $e$ a distance $r$ away from a given point, what is the electrostatic potential there, ignoring light travel times? It's just $\phi = e/r$. Now, generalizing, suppose one had lots of (static!) charges at different locations. **Ask class:** how then would you find the potential? It's additive, so the total potential would be

$$\phi = \sum_i e_i/r_i \ , \tag{1}$$

where each source has charge $e_i$ and is a distance $r_i$ from the point in question. Now, suppose that you had a bunch of moving charges $e_i$. If you pick a particular time $t$ and you know that for each particle the distance was $r_i^{\rm ret}(t)$ a light travel time ago, **Ask class:** what will the potential be then?

$$\phi = \sum_i e_i/r_i^{\rm ret}(t) \ . \tag{2}$$

Note that more distant charges will have had a longer light travel time than nearer charges, so we can no longer evaluate all the particles simultaneously. If we take this formula and write it for a continuous charge density $\rho$, then the scalar potential at a time $t$ and location $\mathbf{r}$ is then

$$\phi(\mathbf{r}, t) = \int \frac{[\rho] d^3\mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|} \tag{3}$$

where the brackets mean to evaluate the quantity at the retarded time:

$$[Q] = Q\left(\mathbf{r}', t - \frac{1}{c}|\mathbf{r} - \mathbf{r}'|\right) \ . \tag{4}$$

Note that here we've shifted things slightly: instead of following individual charges, we're evaluating the potential from fixed points in space, but allowing the charge density to change. It amounts to the same thing.

One can go through an identical procedure for the vector potential:

$$\mathbf{A}(\mathbf{r}, t) = \frac{1}{c}\int \frac{[\mathbf{j}] d^3\mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|} \ . \tag{5}$$

Remember that the potentials have gauge freedom, so in writing $\phi$ and $\mathbf{A}$ this way we've chosen a particular gauge, in this case the Lorentz gauge, in which

$$\nabla \cdot \mathbf{A} + \frac{1}{c}\partial\phi/\partial t = 0 \ . \tag{6}$$

That's no big deal in most circumstances, but it's good to be clear.

A specific charge "distribution" that one can imagine is that of a single charge(!). If it has charge $q$ and moves along a trajectory $\mathbf{r}_0(t)$, so that its velocity is $\mathbf{u}(t) = \dot{\mathbf{r}}(t)$, then we can congratulate ourselves on being cool by writing the charge and current densities in terms of Dirac delta functions:

$$\begin{aligned} \rho(\mathbf{r},t) &= q\delta(\mathbf{r} - \mathbf{r}_0(t))\ , \\ \mathbf{j}(\mathbf{r},t) &= q\mathbf{u}(t)\delta(\mathbf{r} - \mathbf{r}_0(t))\ . \end{aligned} \tag{7}$$

Lovely. If we crank a bit (see §3.1 of Rybicki and Lightman) we get the *Liénard-Wiechart* potentials

$$\begin{aligned} \phi &= \left[\frac{q}{\kappa R}\right] \\ \mathbf{A} &= \left[\frac{q\mathbf{u}}{c\kappa R}\right]\ . \end{aligned} \tag{8}$$

Here the brackets mean an evaluation at the retarded time, and $\kappa(t') \equiv 1 - \mathbf{n}(t') \cdot \mathbf{u}(t')/c$, where $\mathbf{n} = \mathbf{R}/R$ is the unit vector in the direction to the charge. That $\kappa$ factor is mighty important. If the charge isn't moving, $\mathbf{u} = 0$, the whole thing is just 1 and you get the potentials you expect from electrostatics ($\mathbf{A} = 0$, $\phi = q/R$). On the other hand, for speeds close to the speed of light this factor produces strong intensity in the direction of motion of the particle, i.e., a beaming effect.

Let us ponder what this does for us. You may remember that the Poynting flux carried by an electromagnetic field is $\mathbf{S} = \frac{c}{4\pi}\mathbf{E} \times \mathbf{H}$. If you forget, remember that (aside from the $1/4\pi$ factor) you can get this just from units, if you remember that $B^2$ and $E^2$ have units of energy density. Anyway, think about a static charge distribution. **Ask class:** what are $\phi$ and $\mathbf{A}$? $\phi$ is just the sum of $q_i/R_i$ evaluated at the charges' *current* position (since they aren't moving), and $\mathbf{A} = 0$ because the charges are fixed (no current). **Ask class:** so, what is $\mathbf{S}$ in that case? Zero, of course. A static charge distribution has no flux.

It is therefore clear that radiation requires motion. What is less clear is that it's the retarded time that is the key. One can differentiate the potentials to get the fields (RL say that this is "straightforward but lengthy", always a warning that you may be in for an algebraic challenge). One then ends up (§3.2 in RL) with an electric field that can be written as the sum of two terms: one that does not depend on the acceleration $\dot{\beta}$ (where $\beta = \mathbf{u}/c$), and one that does. **Ask class:** which of these terms do they expect will produce radiation? From the arguments we used a few classes ago, it must be the one proportional to the acceleration. A static charge distribution emits no radiation, so a charge at constant velocity can't either (since you could transform to a frame in which it is static).

Now an interesting thing is to look at the radius dependences of the two terms for the electric field (and the corresponding magnetic field, which is just $\mathbf{B}(\mathbf{r},t) = [\mathbf{n} \times \mathbf{E}(\mathbf{r},t)]$). The "velocity field" term is proportional to $1/R^2$, whereas the "radiation field" term is proportional to $1/R$. If we consider just the velocity field then the flux $S \propto EB \propto 1/R^4$, meaning that when one integrated over a sphere at large radius the net flux would be $\propto 1/R^2$, so the system wouldn't radiate to infinity. The radiation field is necessary for radiation, hence the name. For the record, the radiation field is

$$\mathbf{E}_{\rm rad}(\mathbf{r},t) = \frac{q}{c}\left[\frac{\mathbf{n}}{\kappa^3\mathbf{R}}\times\left\{(\mathbf{n}-\beta)\times\dot{\beta}\right\}\right] \ . \tag{9}$$

Note that $\mathbf{E}\perp\mathbf{n}$, so $\mathbf{B}_{\rm rad}=\mathbf{n}\times\mathbf{E}$ has the same magnitude as $\mathbf{E}_{\rm rad}$. Also for the record, the velocity field is

$$\mathbf{E}_{\rm vel}(\mathbf{r},t) = q\left[\frac{(\mathbf{n}-\beta)(1-\beta^2)}{\kappa^3R^2}\right] \ . \tag{10}$$

The reason to write these is so that we can understand them in some limits.

Let's try the limit of small $\beta$, $\beta\ll 1$. This is nonrelativistic motion. **Ask class:** at large distances, which component of $\mathbf{E}$ do they expect will dominate? The radiation field. **Ask class:** by what power of $R$? By a single power. Let's see if this is valid by comparing the magnitudes of the velocity and radiation fields. **Ask class:** ignoring cross products and other complications, what is the magnitude of the velocity field to lowest order in $\beta$? It's $E_{\rm vel}\approx q/(\kappa^3R^2)$. Note that for $\beta\ll 1$, $\kappa\approx 1$, so this reduces to the standard electric field for an unmoving point source. **Ask class:** to lowest order in $\beta$, what is the magnitude of the radiation field? It's $E_{\rm rad}\approx(q/c)\dot{\beta}/(\kappa^3R)$. Taking the ratio and writing $\dot{\beta}=\dot{u}/c$, we have

$$E_{\rm rad}/E_{\rm vel}\approx R\dot{u}/c^2 \ . \tag{11}$$

There are several things to notice about this. First, the ratio increases like the first power of $R$, as expected. Second, when there is no acceleration there is no radiation (we've seen this several times before, but it's always good to check). Third, if the speed of light were infinite there would be no radiation. This reinforces that radiation is fundamentally a relativistic effect, even in the low-speed limit. We can get some further understanding by assuming that the particle has some characteristic frequency of oscillation $\nu$, so that $\dot{u}\approx u\nu=uc/\lambda$. In that case,

$$E_{\rm rad}/E_{\rm vel}\approx(u/c)(R/\lambda) \ . \tag{12}$$

It's often good to express a dimensionless ratio as the product of dimensionless ratios, so one can see at a glance what the dependences are. This says that in the "near zone", $R<\lambda$, the velocity field dominates by a factor $>c/u$. In the "far zone", $R\gg\lambda(c/u)$, the radiation field dominates.

We can now use this to get a handle on the simplest type of radiation: Larmor's formula for radiation from an accelerated nonrelativistic point charge. In the limit $\beta\ll 1$, an angle $\theta$ from the direction of acceleration the magnitudes of the electric and magnetic radiation fields are $E_{\rm rad}=B_{\rm rad}=(q\dot{u}/Rc^2)\sin\theta$. **Ask class:** how can we use this to compute the energy flux? The Poynting flux is $S=(c/4\pi)E_{\rm rad}^2$ in this case (because the electric and magnetic field magnitudes are equal). If we consider the energy per time emitted into a solid angle $d\Omega$ at radius $R$, then the area at that solid angle is $R^2 d\Omega$, so we have

$$\frac{dW}{dt\,d\Omega} = \frac{q^2 \dot{u}^2}{4\pi c^3} \sin^2\theta \ . \tag{13}$$

To get the total power we integrate over solid angles, which gives us the Larmor formula:

$$P = 2q^2 \dot{u}^2 / 3c^3 \ . \tag{14}$$

Again, let's look at this equation to see if it satisfies our intuition. The power must be proportional to an even power of the charge, since the sign doesn't matter. It must not depend on the velocity, but rather the acceleration, and again the sign can't matter so it must depend on an even power of the acceleration. Since (as always!) this is a relativistic effect, if $c \to \infty$ the power would vanish. All these intuitive conditions are satisfied by this formula. Two other things should be noticed. First, the radiation is in a typical dipole pattern, proportional to $\sin^2\theta$; this means that no radiation is emitted along the direction of acceleration. Second, the instantaneous direction of $\mathbf{E}_{\rm rad}$ is determined by both $\dot{\mathbf{u}}$ and $\mathbf{n}$. In particular, for linear acceleration the radiation will be 100% linearly polarized in the plane of $\dot{\mathbf{u}}$ and $\mathbf{n}$ (these last two points are taken directly from Rybicki and Lightman). By the way, note the problem this poses for atoms in classical physics. According to this formula, an electron circling around a proton would emit energy continuously and in an accelerating way, since $\dot{u}$ will increase as it spirals in. This was one of the contradictions that helped spur the development of quantum mechanics.

What if we have a bunch of particles? In general, it becomes a lot tougher because all the different particles will have different retarded times. Put another way, if we're worried about a particular frequency component of the radiation, we have to keep track of phase relations for all the individual particles. **Ask class:** can they think of a circumstance in which the phase relations are close enough to constant that this can be simplified? One way is if the frequency of interest is much lower than $c/L$, where $L$ is the characteristic dimension of the source. Then, the differences in retarded time amount to just a small fraction of a phase. Note that a particular example of this is that if the particles are moving nonrelativistically, $u \ll c$, then the characteristic frequency of their movement across the region, $u/L$, is much less than $c/L$. This means that for nonrelativistic motion we can simplify the situation dramatically. Then, the radiation field is

$$\mathbf{E}_{\rm rad} = \sum_i \frac{q_i}{c^2} \frac{\mathbf{n} \times (\mathbf{n} \times \dot{\mathbf{u}}_i)}{R_i} \ , \tag{15}$$

summed over all particles. If we think about an observation point very far from the sources, then the differences in distances are negligible and we can take some $R_0$ as characteristic of all the distances. Then $\mathbf{E}_{\rm rad} \approx \mathbf{n} \times (\mathbf{n} \times \ddot{\mathbf{d}})/c^2 R_0$, where we have defined the *dipole moment* as $\mathbf{d} \equiv \sum_i q_i \mathbf{r}_i$. In an analogous way to the Larmor formula for a single charge's power, the radiation power is $P = 2\ddot{\mathbf{d}}^2/3c^3$ in this approximation.

You may be familiar from other cases with dipole or multipole expansions. For example, in Newtonian gravity you can estimate the potential around a mass distribution by expanding it in powers of the distance, where the first term is that of a point mass at the center of mass, and higher order terms come in as well. It's similar here, with the caveat that we have to think of nonrelativistic motion to be strictly correct. The book goes into a little more detail about general multipolar expansions.

Our last topic for this lecture will be radiation reaction. Since an accelerated particle radiates, it carries away energy, linear momentum, and angular momentum. This means that the motion of the particle itself must be modified. We can get an approximate idea of what this modification does by treating it as an extra force, the force of radiation reaction.

First, though, let's figure out under what circumstances the force can be treated as a perturbation. Suppose the particle has a speed $u$, so its kinetic energy is $\sim mu^2$. Then from the Larmor formula the time in which the kinetic energy is changed substantially is

$$T \sim mu^2/P \sim (3mc^3/2e^2)(u/\dot{u})^2 \ . \tag{16}$$

Let's estimate a typical orbital time for the particle of $t_p \sim u/\dot{u}$. Then for the energy lost in an orbital period to be small, $T/t_p \gg 1$, or $t_p \gg \tau \equiv 2e^2/3mc^3 \approx 10^{-23}$ s(!). That's a mighty small time. It is about the time necessary for light to travel a distance equal to the classical electron radius $2.8 \times 10^{-13}$ cm. Now, to figure out the force our first inclination would be to set the force times the velocity equal to the power radiated, $\mathbf{F}_{\rm rad} \cdot \mathbf{u} = -2e^2\dot{u}^2/3c^3$. The problem is that (1) $\mathbf{F}_{\rm rad}$ can't depend on $\mathbf{u}$ since that would imply a preferred frame, so (2) one side of this equation depends on $\mathbf{u}$ while the other doesn't! Oops. Instead, we can see if this can be satisfied in a time-averaged sense.

$$\begin{aligned} -\int_{t_1}^{t_2} \mathbf{F}_{\rm rad} \cdot \mathbf{u}dt &= (2e^2/3c^3)\int_{t_1}^{t_2} \dot{\mathbf{u}} \cdot \dot{\mathbf{u}}dt \\ &= (2e^2/3c^3)\left[\dot{\mathbf{u}} \cdot \mathbf{u}\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \ddot{\mathbf{u}} \cdot \mathbf{u}dt\right] \ . \end{aligned} \tag{17}$$

In the second step we integrated by parts. If we assume that the motion or periodic, or at least that $\dot{\mathbf{u}} \cdot \mathbf{u}$ is the same at $t_2$ as at $t_1$, then the first term on the right hand side vanishes and we find that $\mathbf{F}_{\rm rad} = m\tau\ddot{\mathbf{u}}$ in a *time-averaged* sense. That's all very well, but as always we want to know the limits of this expression. **Ask class:** what does this say about a particle with constant linear acceleration? Then $\ddot{\mathbf{u}} = 0$, so it would imply no radiation reaction force, even though the particle does radiate (since it is accelerated). The problem is that then the expression at the limits doesn't vanish, so our approximation is not valid. For most cases, though, it is if you average over a long enough time and the motion is bounded.

If you put this into a grand equation of motion it reads

$$m(\dot{\mathbf{u}} - \tau\ddot{\mathbf{u}}) = \mathbf{F} \tag{18}$$

assuming some applied force $\mathbf{F}$. This is a little weird. One normally doesn't encounter third time derivatives like this. A problem one can encounter in such cases is spurious solutions. For example, suppose $\mathbf{F} = 0$. Then $\mathbf{u}$=constant is obviously a solution, but so is $\mathbf{u} = \mathbf{u}_0 \exp(t/\tau)$, which is a runaway solution that becomes large quickly. In such cases one must use physical or mathematical considerations to eliminate this solution. The mathematical reason is that $\dot{\mathbf{u}} \cdot \mathbf{u}(t_1) \neq \dot{\mathbf{u}} \cdot \mathbf{u}(t_2)$, so in fact the radiation reaction force would have a different form. You have to be careful in cases like this.