Consider a population of binaries, each of which has reduced mass $\mu$ and total mass $M$. Suppose they are all circular, and that the population is in steady-state, meaning that the number in a given frequency bin is simply proportional to the amount of time they spend in that bin. Also assume that the only angular momentum loss process is gravitational radiation, rather than mass transfer or other effects. For each of the following problems, derive the answers in general and then apply the numbers to WD-WD binaries, where we assume that both masses are $0.6\,M_\odot$.

1. Using the Peters equations for circular orbits of point masses, derive the frequency $f_{\min}$ such that the characteristic inspiral time $T_{\text{insp}} \sim 1/\left[d\ln f/dt\right]$ is equal to the Hubble time $T_H \sim 10^{10}$ yr. What is the frequency specifically for a WD-WD binary?

2. Below $f_{\min}$ the distribution $dN/df$ of sources with frequency will depend on their birth population. Above it, gravitational radiation controls the distribution. Derive the dependence of $dN/df$ on $f$ for $f > f_{\min}$ (the normalization is not important).

3. Suppose there are $10^9$ WD-WD binaries at frequencies $f_{\min} < f < 0.1$ Hz. To within a factor of 2, compute the frequency $f_{\text{res}}$ above which you expect an average of less than one WD-WD binary per $df = 10^{-8}$ Hz frequency bin (this is $1/3$ yr, or about the frequency resolution expected for the LISA experiment). Very roughly speaking, above $f_{\text{res}}$ one can identify individual WD-WD binaries, whereas below it is the confusion limit.

4. Dr. I. M. N. Sane doesn't understand why everyone is so worried about white dwarf noise. He asserts that with so many WD-WD binaries in a given bin, the total flux in gravitational waves will be very stable; in particular, he believes that from frequency bin to frequency bin, the flux will vary so little that even a weak additional source will show up easily. He comes to this conclusion by taking the square root of the flux to get a measure of the amplitude.

Show Dr. Sane the error of his ways by doing the following model problem. Let there be $N$ sources in a given frequency bin. Suppose that they are all equally strong, but have random phases between 0 and $2\pi$. Add the complex amplitudes based on those random phases. Take the squared magnitude of the total amplitude as a measure of the typical flux. Determine the mean and standard deviation of the flux that results. You should find that, unlike what happens when you add sources incoherently (i.e., square the amplitudes, then add), the standard deviation of the flux is comparable to the flux, hence Dr. Sane's idea fails.

The next several questions will relate to extreme mass ratio inspirals (EMRIs), which are an important class of events expected to be observable with LISA.

5. If there is a supermassive black hole (SMBH) at the center of a galaxy, then out to a radius where the orbital speed equals the general velocity dispersion at great distance, the SMBH controls the dynamics. If the SMBH mass is $M$ and the velocity dispersion is $\sigma$, calculate the "radius of influence" $r_{\mathrm{infl}}$ inside of which the SMBH dominates.

6. When we gave the relaxation time in the notes, we assumed no dominant object in the middle. More generally, the local relaxation time is

$$t_{\mathrm{rlx}}(r) = \frac{1}{\ln \Lambda} \frac{\sigma^3(r)}{G^2 m^2 n(r)} \, , \tag{1}$$

where $\ln \Lambda \sim 10$ comes from the Coulomb integral, $\sigma(r)$ is the *local* velocity dispersion, $m$ is the typical mass of an object, and $n(r)$ is the local number density of objects. Consider a region $r < r_{\mathrm{infl}}$, where $\sigma(r)$ is given by the Keplerian orbital speed. If $n(r) \propto r^{-3/2}$ (a typical profile), how does the relaxation time depend on $r$? In contrast, for $r \gg r_{\mathrm{infl}}$, assume that $n(r) \propto r^{-2}$ and $\sigma(r)$ is constant. Then how does the relaxation time depend on $r$?

An essential concept in the study of EMRIs is the "full loss cone" and "empty loss cone" regimes. The "loss cone" is the set of directions of orbits from a certain radius that allows capture of an object by the SMBH. For example, a single stellar-mass black hole can go so close to the SMBH that it radiates lots of gravitational radiation, and its orbit undergoes an inspiral leading to a merger. Or, a binary black hole could get close enough that it is separated by the tidal field of the SMBH, leaving one member of the binary bound.

As this region is depopulated, the rapidity with which it is filled depends on how fast the angular momentum of an orbit is changed. Suppose that the loss cone as seen from radius $r$ involves orbits of angular momentum between $J = 0$ and $J = J_{\mathrm{LC}}$. Then as orbits with $J < J_{\mathrm{LC}}$ are eliminated, orbits with $J > J_{\mathrm{LC}}$ move in to fill them. If the typical change of $J$ in one orbital time $t_{\mathrm{orb}}$ is $\Delta J \gg J_{\mathrm{LC}}$ then the loss cone is refilled in a dynamical time, and this is the full loss cone regime. If $\Delta J \ll J_{\mathrm{LC}}$ then the loss cone has to be filled diffusively, which takes much longer than one orbit. This is the empty loss cone regime.

7. Here's your first question: since motion in angular momentum space is a random walk, how long does it take to diffuse by $J_{\mathrm{LC}} \ll J_{\mathrm{circ}}$, where $J_{\mathrm{circ}}$ is the angular momentum of a circular orbit with the same energy? Call this time $t_J$. Remember that $t_{\mathrm{rlx}}$ is basically the

time needed to change angular momentum by $J_{\text{circ}}$. Given that for a Keplerian orbit the angular momentum scales as $(1 - e^2)^{1/2}$, how does $t_J$ scale with $e$ for $(1 - e) \ll 1$?

8. Now for some numbers. Capture of a single $10\,M_\odot$ black hole by emission of gravitational radiation near a $10^6\,M_\odot$ SMBH requires a pericenter distance of about 0.1 AU. The standard relaxation time (time required to change by $\sim J_{\text{circ}}$) is about $10^9$ yr at 1 pc (roughly equal to $r_{\text{infl}}$). Given this, how does $t_J$ compare to $t_{\text{orb}}$ at 1 pc? If $n(r > r_{\text{infl}}) \propto r^{-2}$, how does $t_J$ compare to $t_{\text{orb}}$ at $r > r_{\text{infl}}$? If $n(r < r_{\text{infl}}) \propto r^{-3/2}$, how does $t_J$ compare to $t_{\text{orb}}$ at $r < r_{\text{infl}}$?