

Overview of Gravitational Radiation

As direct detection of gravitational radiation draws nearer, it is useful to consider what such detections will teach us about the universe. The first such detection, of course, will be of immediate significance because it will be a direct confirmation of a dramatic prediction of general relativity: to paraphrase John Wheeler, that spacetime tells sources how to move, and moving sources tell spacetime how to ripple.

Beyond this first detection, gravitational wave detections will pass into the realm of astronomy, allowing new observational windows onto some of the most dynamic phenomena in the universe. These include merging neutron stars and black holes, supernova explosions, and possibly echoes from the very early history of the universe as a whole. They are also anticipated to provide the cleanest tests of predictions of general relativity in the realm of strong gravity.

However, there are important differences from standard astronomy. In electromagnetic observations, in every waveband there are sources so strong that they can be detected without knowing anything about the source. You don't need to understand nuclear fusion in order to see the Sun! In contrast, as we will see, most of the expected sources of gravitational radiation are so weak that sophisticated statistical techniques are required to detect them at all. These techniques involve matching templates of expected waveforms against the observed data stream. Maximum sensitivity therefore requires a certain understanding of what the sources look like, hence of the characteristics of those sources. In addition, when detections occur, it will be important to put them into an astrophysical context so that the implications of the discoveries are evident.

As an aside, it is useful to remember that historically the most interesting sources discovered with a new telescope or satellite have often been unexpected, and this is also possible with gravitational radiation. However, you can't sell a large project by appealing entirely to the unknown, so we should at least describe what we *can* imagine at this point!

Before discussing types of sources, though, we need to have a general perspective on how gravitational radiation is generated and how strong it is. We will begin by discussing radiation in a general context.

By definition, a radiation field must be able to carry energy to infinity. If the amplitude of the field a distance r from the source in the direction (θ, ϕ) is $A(r, \theta, \phi)$, the flux through a spherical surface at r is $F(r, \theta, \phi) \propto A^2(r, \theta, \phi)$. If for simplicity we assume that the radiation is spherically symmetric, $A(r, \theta, \phi) = A(r)$, this means that the luminosity at a distance r is $L(r) \propto A^2(r)4\pi r^2$. Note, though, that when one expands the static field of a source in moments, the slowest-decreasing moment (the monopole) decreases like $A(r) \propto 1/r^2$,

implying that $L(r) \propto 1/r^2$ and hence no energy is carried to infinity. This tells us two things, regardless of the nature of the radiation (e.g., electromagnetic or gravitational). First, radiation requires time variation of the source. Second, the amplitude must scale as $1/r$ far from the source.

We can now explore what types of variation will produce radiation. We'll start with electromagnetic radiation, and expand in moments. For a charge density $\rho_e(\mathbf{r})$, the monopole moment is $\int \rho_e(\mathbf{r}) d^3r$. This is simply the total charge Q , which cannot vary, hence there is no electromagnetic monopolar radiation. The next static moment is the dipole moment, $\int \rho_e(\mathbf{r}) \mathbf{r} d^3r$. There is no applicable conservation law, so electric dipole radiation is possible. One can also look at the variation of currents. The lowest order such variation (the “magnetic dipole”) is $\int \rho_e(\mathbf{r}) \mathbf{r} \times \mathbf{v}(\mathbf{r}) d^3r$. Once again this can vary, so magnetic dipole radiation is possible. The lower order moments will typically dominate the field unless their variation is reduced or eliminated by some special symmetry.

Now consider gravitational radiation. Let the mass-energy density be $\rho(\mathbf{r})$. The monopole moment is $\int \rho(\mathbf{r}) d^3r$, which is simply the total mass-energy. This is constant, so there cannot be monopolar gravitational radiation. The static dipole moment is $\int \rho(\mathbf{r}) \mathbf{r} d^3r$. This, however, is just the center of mass-energy of the system. In the center of mass frame, therefore, this moment does not change, so there cannot be electric dipolar radiation in this frame (or any other, since the existence of radiation is frame-independent). The equivalent of the magnetic dipolar moment is $\int \rho(\mathbf{r}) \mathbf{r} \times \mathbf{v}(\mathbf{r}) d^3r$. This, however, is simply the total angular momentum of the system, so its conservation means that there is no magnetic dipolar gravitational radiation either. The next static moment is quadrupolar: $I_{ij} = \int \rho(\mathbf{r}) r_i r_j d^3r$. This is not conserved, therefore there can be quadrupolar gravitational radiation.

This allows us to draw general conclusions about the type of motion that can generate gravitational radiation. A spherically symmetric variation is only monopolar, hence it does not produce radiation. No matter how violent an explosion or a collapse (even into a black hole!), no gravitational radiation is emitted if spherical symmetry is maintained. In addition, a rotation that preserves axisymmetry (without contraction or expansion) does not generate gravitational radiation because the quadrupolar and higher moments are unaltered. Therefore, for example, a neutron star can rotate arbitrarily rapidly without emitting gravitational radiation as long as it maintains axisymmetry.

This immediately allows us to focus on the most promising types of sources for gravitational wave emission. The general categories are: binaries, continuous wave sources (e.g., rotating stars with nonaxisymmetric lumps), bursts (e.g., asymmetric collapses), and stochastic sources (i.e., individually unresolved sources with random phases; the most interesting of these would be a background of gravitational waves from the early universe). We will

discuss each of these in subsequent lectures.

For now, however, it will be useful to reconsider gravitational waves from the standpoint of dynamic changes in spacetime. We will begin by reviewing some relevant aspects of general relativity.

To characterize the warping of spacetime, we define the metric tensor $g_{\alpha\beta}$ as follows. Suppose that there are two events A and B, determined by four coordinates each in a four-dimensional spacetime. Suppose in particular that these events are very close to each other, in the sense that the difference in each of their four coordinates is $x_A^\alpha - x_B^\alpha = dx^\alpha$, an infinitesimal quantity. The four-dimensional “distance” ds between these events is then given by

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta . \quad (1)$$

Here we use the convention that greek indices run over all four spacetime coordinates, and also use the Einstein summation convention by which repeated indices are summed over all coordinates, e.g., $v^\alpha u_\alpha = v^0 u_0 + v^1 u_1 + v^2 u_2 + v^3 u_3$, where 0, 1, 2, 3 represent the coordinates. The quantity ds transforms as a scalar, meaning that all observers will measure the same ds between two given events that are infinitesimally close to each other. This is the reason that ds is called the “invariant interval”. Note, incidentally, that $ds = 0$ for any two events connected by a photon. One can also define the matrix inverse $g^{\alpha\beta}$ of $g_{\alpha\beta}$, so that $g^{\alpha\beta} g_{\beta\gamma} = \delta^\alpha_\gamma$, the Kronecker delta.

It is important to remember that there is complete freedom in choosing the coordinates x^α . A point on the plane can be described by Cartesian coordinates or polar coordinates, but it’s still in the same place! That means that for a given problem we can choose coordinates based on convenience or because they bring out the physics in a clear way. This coordinate freedom is also called gauge freedom. With this perspective, a given spacetime can be represented with many different coordinate systems. The values of the metric tensor components at different events will depend on the system used, but the spacetime (also called the geometry) will be the same.

Note that no matter how warped any spacetime is, any sufficiently tiny patch looks flat unless that patch is on a singularity (which has infinite curvature). That is, at any given point in spacetime, it is always possible to come up with a local coordinate system that looks like flat spacetime to linear order, with interval (in Cartesian coordinates)

$$ds^2 = \eta_{\alpha\beta} dx^\alpha dx^\beta = -dt^2 + dx^2 + dy^2 + dz^2 . \quad (2)$$

Locally, the only deviations will be of quadratic order $\mathcal{O}(dx^\alpha dx^\beta)$. This is a statement of the equivalence principle. Another way of saying this is that for a small enough region of space and short enough period of time, if you are in free fall then you can always boost to a

frame in which all experiments give the same result as if there is no gravity. A consequence of this is that an infinitesimal particle in free fall feels no acceleration (as measured locally by an accelerometer) no matter how warped a (non-singular) spacetime is.

When we consider gravitational waves, it is convenient to separate the spacetime into a component that is time-independent (the background spacetime) and a component that varies with time (the gravitational waves). It is a useful starting point to assume that we are making measurements far enough from any mass that the background spacetime is flat. We will now assume that the time-dependent metric produced by gravitational waves is

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} , \quad (3)$$

where $h_{\alpha\beta}$ is so small that we only need to consider its linear contribution to any equation. Among other things, that means that indices are raised and lowered using η . Therefore, up and down spatial indices are identical, whereas up and down time components differ only in sign.

We can now construct the linearized version of the Einstein field equation. We will simply state the results, see a standard textbook such as *Gravitation* by Misner, Thorne, and Wheeler for the details. First, the Christoffel symbols are

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} (h^\alpha_{\beta,\gamma} + h^\alpha_{\gamma,\beta} - h_{\beta\gamma}{}^{,\alpha}) \quad (4)$$

where the commas indicate partial derivatives. The linearized Riemann tensor is

$$\begin{aligned} R^\alpha_{\beta\gamma\delta} &= \Gamma^\alpha_{\beta\delta,\gamma} - \Gamma^\alpha_{\beta\gamma,\delta} \\ &= \frac{1}{2} (h^\alpha_{\delta,\beta\gamma} + h_{\beta\gamma,\delta}{}^\alpha - h_{\beta\delta,\gamma}{}^\alpha - h^\alpha_{\gamma,\beta\delta}) . \end{aligned} \quad (5)$$

A single contraction of this forms the Ricci tensor:

$$R_{\alpha\beta} = R^\gamma_{\alpha\gamma\beta} = \frac{1}{2} (h^\gamma_{\alpha,\gamma\beta} + h_{\beta\gamma,\alpha}{}^\gamma - h_{\alpha\beta,\gamma}{}^\gamma - h^\gamma_{\gamma,\alpha\beta}) . \quad (6)$$

Note that $h^\gamma_\gamma \equiv h$ is the trace of the metric perturbation, and that the derivative ${}_{,\gamma}{}^\gamma \equiv \square = \nabla^2 - \partial_t^2$ is the wave operator. A second contraction gives the curvature scalar:

$$R = R^\alpha_\alpha = h^\gamma_{\alpha,\gamma}{}^\alpha - \square h . \quad (7)$$

Finally, we have the linearized Einstein tensor:

$$\begin{aligned} G_{\alpha\beta} &= R_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}R \\ &= \frac{1}{2} \left(h^\gamma_{\alpha,\gamma\beta} + h_{\beta\gamma,\alpha}{}^\gamma - \square h_{\alpha\beta} - h_{,\alpha\beta} - \eta_{\alpha\beta} h^\gamma_{\delta,\gamma}{}^\delta + \eta_{\alpha\beta} \square h \right) . \end{aligned} \quad (8)$$

A notation change simplifies this a bit. We used the “trace-reversed” perturbation $\bar{h}_{\alpha\beta} \equiv h_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}h$. When we change from h to \bar{h} we find that all the terms with the trace are canceled, leading to

$$G_{\alpha\beta} = \frac{1}{2} \left(\bar{h}^{\gamma}_{\alpha,\gamma\beta} + \bar{h}^{\gamma}_{\beta\gamma,\alpha} - \square \bar{h}_{\alpha\beta} - \eta_{\alpha\beta} \bar{h}^{\gamma}_{\delta,\gamma}{}^{\delta} \right). \quad (9)$$

In addition, we can make use of our coordinate freedom to impose the gauge condition $\bar{h}_{\alpha\beta}{}^{;\alpha} = 0$. With this condition we find the remarkably simple expression

$$G_{\alpha\beta} = -\frac{1}{2}\square\bar{h}_{\alpha\beta} \quad (10)$$

so that the linearized Einstein equation $G_{\alpha\beta} = 8\pi T_{\alpha\beta}$ (where T is the stress-energy tensor) becomes

$$\square\bar{h}_{\alpha\beta} = -16\pi T_{\alpha\beta} \quad (11)$$

or in a vacuum

$$\square\bar{h}_{\alpha\beta} = 0. \quad (12)$$

This is clearly a wave equation, just as in electromagnetism.

It is now standard (and appropriate) to specialize further. We specialize to globally vacuum, asymptotically flat spacetimes (where $T_{\alpha\beta} = 0$ and there is minimal curvature in the measurement region; this is an excellent approximation for any terrestrial experiments!). It then turns out that we have some remaining local gauge freedom, which we can fix completely by going into the *transverse traceless*, or TT, gauge. In this gauge the metric perturbation is purely spatial:

$$h_{tt} = h_{ti} = 0 \quad (13)$$

and traceless: $h = h_i{}^i = 0$. This is the usual gauge for gravitational wave physics. It also indicates that, given a propagation direction, there are two independent polarization components, as in electromagnetism, but they have different character than the linear polarization of EM waves.

We close this lecture with some order of magnitude estimates. What is the approximate expression for the dimensionless amplitude h of a metric perturbation, a distance r from a source? We argued that the lowest order radiation had to be quadrupolar, and hence depend on the quadrupole moment I . This moment is $I_{ij} = \int \rho r_i r_j d^3x$, so it has dimensions MR^2 , where M is some mass and R is a characteristic dimension. We also argued that the amplitude is proportional to $1/r$, so we have

$$h \sim MR^2/r. \quad (14)$$

We know that h is dimensionless, so how do we determine what else goes in here? In GR we usually set $G = c = 1$, which means that mass, distance, and time all have the same effective “units”, but we can’t, for example, turn a distance squared into a distance. Our current expression has effective units of distance squared (or mass squared, or time squared). We note that time derivatives have to be involved, since a static system can’t emit anything. Two time derivatives will cancel out the current units, so we now have

$$h \sim \frac{1}{r} \frac{\partial^2(MR^2)}{\partial t^2} . \quad (15)$$

Now what? To get back to physical units we have to restore factors of G and c . It is useful to remember certain conversions: for example, if M is a mass, GM/c^2 has units of distance, and GM/c^3 has units of time. Playing with this for a while gives finally

$$h \sim \frac{G}{c^4} \frac{1}{r} \frac{\partial^2(MR^2)}{\partial t^2} . \quad (16)$$

Since G is small and c is large, the prefactor is *tiny*! That tells us that unless M and R are large, the system is changing fast, and r is small, the metric perturbation is minuscule.

Let’s make a very rough estimate for a circular binary. Suppose the total mass is $M = m_1 + m_2$, the reduced mass is $\mu = m_1 m_2 / M$, the semimajor axis is a , and the orbital frequency Ω is therefore given by $\Omega^2 a^3 = GM$. Without worrying about precise factors, we say that $\partial^2/\partial t^2 \sim \Omega^2$ and $MR^2 \sim \mu a^2$, so

$$h \sim (G^2/c^4)(1/r)(\mu M/a) . \quad (17)$$

This can also be written in terms of orbital periods, and with the correct factors put in we get, for example, for an equal mass system

$$h \approx 10^{-22} \left(\frac{M}{2.8 M_\odot} \right)^{5/3} \left(\frac{0.01 \text{ sec}}{P} \right)^{2/3} \left(\frac{100 \text{ Mpc}}{r} \right) , \quad (18)$$

which is scaled to a double neutron star system. This is really, really, small: it corresponds to less than the radius of an atomic nucleus over a baseline the size of the Earth. That’s why it is so challenging to detect these systems!

Remarkably, though, the flux of energy is *not* tiny. To see this, let’s calculate the flux given some dimensionless amplitude h . The flux has to be proportional to the square of the amplitude and also the square of the frequency f : $F \sim h^2 f^2$. This currently has units of time squared, but the physical units of flux are energy per time per area. Replacing factors of G and c , we find that the flux is

$$F \sim (c^3/G) h^2 f^2 . \quad (19)$$

Now the prefactor is *enormous*! For the double neutron star system above, with $h \sim 10^{-22}$ and $f \sim 100$ Hz, this gives a flux of a few hundredths of an $\text{erg cm}^{-2} \text{s}^{-1}$. For comparison, the flux from Sirius, the brightest star in the night sky, is about $10^{-4} \text{ erg cm}^{-2} \text{s}^{-1}$! That means that if you could somehow absorb gravitational radiation perfectly with your eyes, you would find untold billions of sources brighter than every star except the Sun. What this really implies, of course, is that gravitational radiation interacts *very* weakly with matter, which again means that it is mighty challenging to detect.