Sources of Gravitational Radiation

We now turn our attention to sources of gravitational radiation. From the last lecture, we know that such sources need to have a time-variable quadrupole (or higher-order) mass moment. The community has divided such sources into four basic categories:

- Binaries. These obviously have a large and varying quadrupole moment, and have the additional advantage that we actually know that gravitational radiation is emitted from them at the expected level (based on observations of double neutron star binaries).
- Continuous sources. A spinning source can in principle emit gravitational waves at a single frequency for a long time, so the signal builds up in a narrow frequency bin. As a result, particularly for high frequencies observable with ground-based detectors, continuous-wave sources are interesting because they can in principle be seen even at relatively low amplitudes.
- Bursts. These refer to events of very limited duration that do not have to have any special periodicity. An example would be a core-collapse supernova.
- Stochastic sources. For these, we think in terms of broad bands of frequency with many sources, rather than the signal produced by an individual source. Examples include the huge foreground of double white dwarf binaries in our Galaxy, or possibly a background from the very early universe.

Given that binary sources are the only ones that we *know* exist at a detectable level, we'll focus mainly on them. At the end we'll have a few comments about the other types of sources.

First, let's get an idea of the frequency range available for a given type of binary. There is obviously no practical lower frequency limit (just increase the semimajor axis as much as you want), but there is a strict upper limit. The two objects in the binary clearly won't produce a signal higher than the frequency at which they touch. If we consider an object of mass M and radius R, the orbital frequency at its surface is $\sim \sqrt{GM/R^3}$. Noting that $M/R^3 \sim \rho$, the density, we can say that the maximum frequency involving an object of density ρ is $f_{\rm max} \sim (G\rho)^{1/2}$. This is actually more general than just orbital frequencies. For example, a gravitationally bound object can't rotate faster than that, because it would fly apart. In addition, you can convince yourself that the frequency of a sound wave through the object can't be greater than $\sim (G\rho)^{1/2}$. This is thus a general upper bound on dynamical frequencies.

This tells us, therefore, that binaries involving main sequence stars can't have frequencies greater than $\sim 10^{-3} - 10^{-6}$ Hz, depending on mass, that binaries involving white dwarfs can't have frequencies greater than $\sim 0.1 - 10$ Hz, also depending on mass, that for neutron stars the upper limit is $\sim 1000 - 2000$ Hz, and that for black holes the limit depends inversely on mass (and also spin and orientation of the binary). In particular, for black holes the maximum imaginable frequency is on the order of $10^4(M_\odot/M)$ Hz at the event horizon, but in reality the orbit becomes unstable at the innermost stable circular orbit (ISCO). For a nonrotating black hole the orbital frequency at the ISCO is $f_{\rm ISCO} = 2200\ {\rm Hz}(M_\odot/M)$ and thus the gravitational wave frequency for a circular orbit there is $f_{\rm GW} = 2f_{\rm ISCO}$. Prograde orbits around spinning black holes can get to higher frequencies. Black hole ringdown, i.e., the radiation of non-Kerr horizon structure, can get to a factor of a few higher frequencies.

Now suppose that the binary is well-separated, so that the components can be treated as points and we only need take the lowest order contributions to gravitational radiation. Temporarily restricting our attention to circular binaries, how will their frequency and amplitude evolve with time?

Let the masses be m_1 and m_2, and the orbital separation be R. We argued in the previous lecture that the amplitude a distance $r \gg R$ from this source is $h \sim (\mu/r)(M/R)$, where $M \equiv m_1 + m_2$ is the total mass and $\mu \equiv m_1 m_2/M$ is the reduced mass. We can rewrite the amplitude using $f \sim (M/R^3)^{1/2}$, to read

$$\begin{aligned} h &\sim \mu M^{2/3} f^{2/3}/r \\ &\sim M_{ch}^{5/3} f^{2/3}/r \end{aligned} \tag{1}$$

where M_{ch} is the "chirp mass", defined by $M_{ch}^{5/3} = \mu M^{2/3}$. The chirp mass is named that because it is this combination of μ and M that determines how fast the binary sweeps, or chirps, through a frequency band. When the constants are put in, the dimensionless gravitational wave strain amplitude (i.e., the fractional amount by which a separation changes as a wave goes by) measured a distance r from a circular binary of masses M and m with a binary orbital frequency $f_{\rm bin}$ is (Schutz 1997)

$$h = 2(4\pi)^{1/3} \frac{G^{5/3}}{c^4} f_{\rm GW}^{2/3} M_{ch}^{5/3} \frac{1}{r} \ , \tag{2}$$

where $f_{\rm GW}$ is the gravitational wave frequency. Redshifts have not been included in this formula.

The luminosity in gravitational radiation is then

$$\begin{aligned} L &\sim 4\pi r^2 f^2 h^2 \\ &\sim M_{ch}^{10/3} f^{10/3} \\ &\sim \mu^2 M^3/R^5 \ . \end{aligned} \tag{3}$$

The total energy of a circular binary of radius R is $E_{\rm tot} = -G\mu M/(2R)$, so we have

$$\begin{aligned} dE/dt &\sim \mu^2 M^3/R^5 \\ \mu M/(2R^2)(dR/dt) &\sim \mu^2 M^3/R^5 \\ dR/dt &\sim \mu M^2/R^3 \;. \end{aligned} \tag{4}$$

What if the binary orbit is eccentric? The formulae are then more complicated, because one must then average properly over the orbit. This was done first to lowest order by Peters and Matthews (1963) and Peters (1964), by calculating the energy and angular momentum radiated at lowest (quadrupolar) order, and determining the change in orbital elements that would occur if the binary completed a full Keplerian ellipse in its orbit. That is, they assumed that to lowest order, they could have the binary move as if it experienced only Newtonian gravity, and integrate the losses along that path.

Before quoting the results, we can understand one qualitative aspect of the radiation when the orbits are elliptical. From our derivation for circular orbits, we see that the radiation is emitted much more strongly when the separation is small, because $L \sim R^{-5}$. Consider what this would mean for a very eccentric orbit $(1-e) \ll 1$. Most of the radiation would be emitted at pericenter, hence this would have the character of an impulsive force. With such a force, the orbit will return to where the impulse was imparted. That means that the pericenter distance would remain roughly constant, while the energy losses decreased the apocenter distance. As a consequence, the eccentricity decreases. Typically, gravitational radiation will decrease the eccentricity of an orbit, although near the ISCO there are some other effects that enter.

The Peters formulae bear this out. If the orbit has semimajor axis a and eccentricity e, their lowest-order rates of change are

$$\langle \frac{da}{dt} \rangle = -\frac{64}{5}\frac{G^3\mu M^2}{c^5 a^3 (1-e^2)^{7/2}}\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right) \tag{5}$$

and

$$\langle \frac{de}{dt} \rangle = -\frac{304}{15}e\frac{G^3\mu M^2}{c^5 a^4 (1-e^2)^{5/2}}\left(1+\frac{121}{304}e^2\right) \tag{6}$$

where the angle brackets indicate an average over an orbit. One can show that these rates imply that the quantity

$$ae^{-12/19}(1-e^2)\left(1+\frac{121}{304}e^2\right)^{-870/2299} \tag{7}$$

is constant throughout the inspiral.

Do we have evidence that these formulae actually work? Yes! Nature has been kind enough to provide us with the perfect test sources: binary neutron stars. Several such systems are known, all of which have binary separations orders of magnitude greater than the size of a neutron star, so the lowest order formulae should work. Indeed, the da/dt predictions have been verified to better than 0.1% in a few cases. The de/dt predictions will be much tougher to verify, though. The reason for the difference is that de/dt has to be measured by determining the eccentricity orbit by orbit, whereas da/dt has a manifestation in the total phase of the binary, so it accumulates quadratically with time. These systems provide really spectacular verification of general relativity in weak gravity. In particular, in late 2003 a double pulsar system was detected, that in addition has the shortest expected time to merger of any known system (only about 80 million years). Having two pulsars means that extra quantities can be measured (such as the relative motion, which gives us the mass ratio), and in fact the system is now dramatically overconstrained (more things measured than there are parameters in the theory). The tests of GR by observations of binary neutron star systems deservedly resulted in the 1993 Nobel Prize in physics going to Hulse and Taylor, who discovered the first such binary.

We are therefore quite confident that, at least in weak gravity, gravitational radiation exists as advertised. What happens in strong gravity? That is a question that we hope will be answered clearly via direct detections of gravitational waves. In the current, necessarily theoretical, treatments, the process of coalescence of two black holes is usually divided into three phases: (1) inspiral, which is typically considered to be down to an orbit or so before the horizons overlap, (2) merger, which follows from the inspiral phase and which involves the overlap of the horizons, and (3) ringdown, in which the now single horizon settles into the Kerr configuration. Phases (1) and (3) can be approached analytically (although great sophistication is needed for (1)). Phase (2) requires high-precision numerical simulations, which first became possible in 2005. There is now excellent agreement between different codes. If the objects are two neutron stars or a neutron star and a black hole, the numerics become more complex; for example, if the neutron stars have low enough mass, it could be that their merger produces a "hypermassive" neutron star, which holds off on collapse to a black hole only because it rotates rapidly and differentially. Comparisons of the numerically predicted waveforms in the merger and ringdown phases of BH-BH coalescence with what is observed will provide the most direct possible tests of models of strong gravity.

Continuous sources

Let us now think briefly about continuous sources. For these, our model will be a spinning neutron star that has a nonaxisymmetric lump or wave.

What amplitude can we expect? From the first lecture we know that if the moment of inertia is I, then the amplitude is

$$h \sim (G/c^4)(1/r)(\partial^2 I/\partial t^2)\ . \tag{8}$$

For binaries we argued that $I \sim MR^2$, and also had a relation between $\Omega^2 \sim \partial^2/\partial t^2$ and M and R. However, for a spinning source these relations do not have to hold. For a gravitationally bound source (e.g., a neutron star and not a strange star, which is self-bound and can therefore in principle rotate faster), Ω cannot be greater than the Keplerian angular velocity, but it can certainly be less. In addition, unlike for binaries, not the entire moment of inertia is involved in gravitational wave generation (indeed, if the spinning source is axisymmetric, no gravitational radiation is emitted). Let us say that some fraction ϵ of the moment of inertia is nonaxisymmetric. Generically this could be, e.g., a lump or a wave. Therefore, $h \sim (G/c^4)(1/r)\Omega^2\epsilon I$.

The luminosity is then

$$\begin{aligned} L &\sim r^2h^2f^2 \\ &= (32/5)(G/c^5)\epsilon^2 I_3^2\Omega^6\ , \end{aligned} \tag{9}$$

where we have put in the correct factors for rotation around the minor axis of an ellipsoid (here I_3 is the moment of inertia around that axis), and we are now defining ϵ to be the ellipticity in the equatorial plane: $\epsilon = (a-b)/(ab)^{1/2}$, where the principal axes of the ellipsoid are $a \geq b > c$.

Note the extremely strong dependence on Ω. The rotational energy is $E_{\rm rot} = \frac{1}{2}I\Omega^2$, so if the part of the star generating the gravitational waves (e.g., a lump) is coupled to the rest of the star then we have

$$\begin{aligned} I\Omega\dot{\Omega} &= -(32/5)(G/c^5)\epsilon^2 I_3^2\Omega^6 \\ \dot{\Omega} &= -(32/5)(G/c^5)\epsilon^2 I_3\Omega^5\ . \end{aligned} \tag{10}$$

For pulsars, we can relate this to the dimensionless period derivative $\dot{P} = -2\pi\dot{\Omega}/\Omega^2$, which is between $\sim 10^{-13}$ for young pulsars and $\sim 10^{-21} - 10^{-22}$ for the most stable of the millisecond pulsars. Therefore, we have

$$\dot{P} = (64\pi/5)(G/c^5)\epsilon^2 I\Omega^3\ . \tag{11}$$

For a typical neutron star moment of inertia $I \approx 10^{45}$ g cm^2 and a young pulsar like the Crab with $\Omega \approx 200$ rad s^{-1} and $\dot{P} \approx 10^{-13}$, this implies $\epsilon < 3 \times 10^{-4}$. The reason for the inequality is that the observed spindown can also be caused by other effects, notably magnetic braking. By the same argument, a millisecond pulsar with $\Omega \approx 2000$ rad s^{-1} and $\dot{P} \approx 10^{-21}$ has $\epsilon < 10^{-9}$.

It is not expected that gravitational waves from any isolated pulsars will be seen with Advanced LIGO (although limits better than spindown have already been set with initial

LIGO observations of the Crab pulsar, realistic strains are below detectable levels). There has been some discussion about whether actively accreting neutron stars might produce detectable gravitational waves. The argument is that these stars have a maximum frequency (620 Hz for currently accreting stars, 716 Hz maximum for their descendents the millisecond pulsars) that is much lower than it could have been, so maybe gravitational wave emission limits the spin. It's possible, but the known magnetic fields $B \sim 10^{8-9}$ G in these sources are adequate to explain the limits (consider a very strong field; it would grab onto matter a long way from the star, where the Keplerian frequency is small, so the star would not be able to spin up to high rates). In addition, in at least two sources the contribution of gravitational waves is limited to at most tens of percent of the spindown by observations between accretion outbursts. Nonetheless, it would be exciting to find continuous gravitational waves and thus the search will go on!

Burst sources

Data analysis for these will be very challenging indeed, but since they are by definition associated with violent events, we could potentially learn a great deal from detection of gravitational radiation. Let's consider a few of the more commonly discussed possibilities.

Core-collapse supernovae. When the core of a massive star collapses, it will not do so in a perfectly symmetric fashion. For example, convection will introduce asymmetries. What fraction of the mass-energy will therefore be released as gravitational radiation? This is a question that has to be answered numerically, but it is an extraordinarily challenging problem. Convection is important, so simulations have to be done in three dimensions. Radiation transfer is also essential, as is a good treatment of neutrino transport. To make things even worse, it seems likely that magnetic fields will play a major role, and a wide range of scales could influence each other! Nonetheless, the current best guess is that only a very small fraction of the total mass-energy will come out in gravitational radiation, perhaps $\sim 10^{-10}$. If so, supernovae outside our galaxy will be undetectable. However, the rate of core-collapse supernovae in our Milky Way is estimated to be one per few decades, which means that there is a probability of tens of percent per decade that a supernova will occur within ~ 10 kpc. Current calculations suggest that the strain amplitude at 10 kpc could be $h \sim 10^{-20}$ for a few milliseconds, which would be detectable with advanced ground-based instruments. There have also been proposals that a much higher fraction of energy is emitted during the collapse, which brings us to the next topic.

Gamma-ray bursts (which for our current purposes are a subset of supernovae). These are short (milliseconds to minutes), high intensity bursts of gamma rays. After a long and interesting history (starting with their detection with US spy satellites!), it has been established that there are two categories of GRBs, the long (tens of seconds) and the short (less than a second, typically). The long bursts are convincingly associated with a type of supernova, but the detailed mechanism for their production is uncertain. Some people believe that GRBs are the birth events for rapidly rotating black holes. If so, the rapid rotation could be a path to much more substantial gravitational wave production. For example, in a massive disk there are bar instabilities that could produce rotating nonaxisymmetric structures. If these emit a lot of gravitational radiation and can be identified with particular bursts, then we have a wonderful situation: extremely bright events at cosmological distances whose redshift can be determined based on the electromagnetic signal, and whose luminosity distance can be determined based on the gravitational wave signal. The difficulty is that to be detectable at cosmological distances (at least 3 Gpc is needed to be interesting), a truly enormous fraction of the mass-energy needs to emerge in gravitational waves (at least tens of percent). This currently seems unlikely, but it is obviously worth pursuing from the observational standpoint.

Stochastic background sources

These can be pictured as the overlap of many sources, which are independently unresolvable. For example, there could easily be 10^8 double white dwarf binaries in our Galaxy that have frequencies $> 10^{-4}$ Hz and would thus fall into the detection bands of proposed space-based detectors such as eLISA.

The most interesting possibilities, however, are sources from the early universe. It is predicted (albeit with abundant wiggle room) that the inflationary epoch of the universe will have produced tensor modes that generate nonzero curl in the polarization from low-ℓ modes in the cosmic microwave background. The expected levels will be such that ground-based detectors should see them in the next few years (if Planck doesn't see them first, which is possible but not at all guaranteed).

If phase transitions in the early universe (e.g., from a quark-gluon plasma to baryonic matter) are first-order, then by definition some variables are discontinuous at the transition. If the transition occurs in localized regions ("bubbles") in space, collisions between the bubbles could produce gravitational radiation. In addition, turbulent magnetic fields produced by the fluid motion could generate secondary gravitational radiation, but these are weaker. The most optimistic estimates put the contribution at $h_0^2\Omega_{\rm GW} \sim 10^{-10}$, peaking in the millihertz range. This would be detectable with LISA, but don't bet on it.

A more recent suggestion has been that gravitational radiation could be produced by cosmic strings. Cosmic strings, if they exist, are one-dimensional topological defects. Assuming a network of cosmic strings exists, it would have strings of all sizes and therefore contribute gravitational radiation at a wide range of frequencies. Recently, some work has been done on the possibility that cusps or kinks in cosmic strings could produce beams of gravitational radiation.

If any of these scenarios comes true and in fact there is a cosmological background of gravitational waves detected with planned instruments, this will obviously be fantastic news. However, what if it isn't seen? That won't be a surprise, but there has been discussion about missions to go after weaker backgrounds. It is often thought that the 0.1-1 Hz range is likely to be least "polluted" by foreground vermin (i.e., the rest of the universe!). This may be, but it is worth remembering that there are an enormous number of sources out there in even that frequency range, and that to see orders of magnitude below them will required *extremely* precise modeling of all those sources. Either way, whether we see a background or "merely" detect a large number of other sources, gravitational wave astronomy has wonderful prospects to enlarge our view of the cosmos.