ASTR 320 – Spring 2009
Prof. Ostriker

ASSIGNMENT No. 1

DUE: Tuesday, Feb. 10

1. Velocity and acceleration in cylindrical coordinates

With φ the azimuthal angle and R the cylindrical radius, the position of a particle at (x, y, z) can be expressed in component form as

$$\mathbf{r} = \hat{x} R \cos\varphi + \hat{y} R \sin\varphi + \hat{z} z.$$

Take two derivatives with respect to time to write the velocity and acceleration of the particle in terms of R, φ, $\dot{R}, \dot{\varphi}$, and $\ddot{R}, \ddot{\varphi}$ and $\hat{x}$, $\hat{y}$, $\hat{z}$. Then substitute in for

$$\hat{x} = \cos\varphi \hat{R} - \sin\varphi \hat{\varphi}, \qquad \hat{y} = \sin\varphi \hat{R} + \cos\varphi \hat{\varphi}$$

to write the velocity $\mathbf{v}$ and acceleration $\mathbf{a}$ fully in cylindrical coordinates. Compare to the answer we obtained in class directly from using $d\hat{R}/dt = \dot{\varphi}\hat{\varphi}$ and $d\hat{\varphi}/dt = -\dot{\varphi}\hat{R}$.

2. One-body solution radial coefficient

The expression for specific angular momentum conservation is

$$\dot{\varphi} R^2 = constant \equiv \Lambda,$$

and for specific energy conservation is

$$\frac{1}{2}\dot{R}^2 - \frac{GM}{R} + \frac{\Lambda^2}{2R^2} = constant \equiv \epsilon,$$

where we've use $v_\varphi = \Lambda/R$ for the azimuthal kinetic energy term $v_\varphi^2/2$ in the latter.

We showed in class that the general solution to the radial equation of motion is

$$u \equiv \frac{1}{R} = \frac{GM}{\Lambda^2}[1 + c\cos(\varphi - \varphi_0)]. \qquad (1)$$

(a) Using the above expression for $u = 1/R$, together with (*where does this come from?*) $\dot{R} = -\Lambda du/d\varphi$, to show that the energy equation can be written as

$$\epsilon = \frac{\Lambda^2}{2}\left[\frac{du}{d\varphi}^2 + u^2\right] - GMu.$$

Now substitute in for u and $du/d\varphi$ from the general form of u in terms of φ, and solve for c to show that:

$$c = \left[1 + \frac{2\epsilon\Lambda^2}{(GM)^2}\right]^{1/2} . \tag{2}$$

(b) Show that the $c = 0$ solution corresponds to an $R =$ *constant* circular orbit with angular velocity $\dot{\varphi} = \sqrt{GM/R^3}$, $v_\varphi = \sqrt{GM/R}$, $\Lambda = \sqrt{GMR}$ and $\epsilon = -GM/(2R) = -(GM)^2/(2\Lambda^2)$.

3. Bound and unbound solutions for one-body orbit

From Problem 2 above, we have a relationship between radius R and azimuthal angle φ given by equation (1), where c is written in terms of the constants of motion ϵ and Λ by equation (2). Without loss of generality, we may take $\varphi_0 = 0$. If we multiply equation (1) by $R\Lambda^2/(GM)$ and substitute in for c, we have

$$\frac{\Lambda^2}{GM} = R + \left[1 + \frac{2\epsilon\Lambda^2}{(GM)^2}\right]^{1/2} R\cos\varphi. \tag{3}$$

(a) For the case $\epsilon < 0$, show, using $R = (x^2 + y^2)^{1/2}$ and $x = R\cos\varphi$, that the above expression is equivalent to the more familiar form of an ellipse,

$$\frac{(x+f)^2}{a^2} + \frac{y^2}{b^2} = 1$$

where

$$a = \frac{GM}{2|\epsilon|}, \qquad b = \frac{\Lambda}{(2|\epsilon|)^{1/2}},$$

and

$$f = \frac{GM}{2|\epsilon|}\left[1 + \frac{2\epsilon\Lambda^2}{(GM)^2}\right]^{1/2}$$

are, respectively, the semimajor axis, the semiminor axis, and the focal length of the ellipse. Notice that $f = a \times c$ where c is defined in equation (2), showing that c is in fact *the eccentricity e of the orbit.*

(b) Draw the ellipse based on the above solution, labeling the values (x, y) of the coordinates of the center, the ends of the major and minor axes, and the position of the central mass M.

(c) When $\epsilon > 0$, show that equation (3) can instead be manipulated into the standard form of a hyperbola, and show that the slopes of the asymptotes are $dy/dx == \pm b/a = \pm\Lambda\sqrt{2\epsilon}/GM$