Astronomy 622 Cosmology **Homework Assignment #3 – Due 2:00 pm April 8th**
100 points (later arbitrarily renormalized to match Dr. McGaugh's earlier homeworks)
Please explain your answers in complete sentences where called for. Although collaboration is encouraged, remember to answer in your own words. Where math is involved, please be sure to show all your work, and answer in no more than a reasonable number of significant digits.

1. *(25) Read over section 15.2 in Peacock, and/or look at your excellent notes from class. Starting from the three equations (15.24) in Peacock, derive eq. (15.25) by linearizing the variables, dropping 2nd order terms, assuming the 0th order equations are satisfied, etc., etc. Please be sure to show all the gory detail of why you drop or alter any terms. You will notice that you have an extra term* $-\left(\frac{k c^2}{3 a^2}\delta_k\right)$ *(and you will also notice that Peacock is sloppy and doesn't use the k subscript; however, you should use it!). Why do you have this extra term?! Do not panic: read the paragraph below eq. (15.25) beginning, "In this simplified exercise..." and all will be clear (maybe).*
   I will do the hard one first, deriving 15.21; follow these steps.
   Starting with:
   $\rho=\bar{\rho}+\delta\rho,\ \ p=p_0+\delta p,\ \ \vec{v}=\vec{v}_0+\delta\vec{v},\ \ \Phi=\Phi_0+\delta\Phi$
   and noting that:
   $\nabla\bar{\rho}=\nabla p_0=0,\ \ \vec{v}_0=H\vec{x},\nabla\vec{v}_0=3H\hat{x}$
   we linearize the Euler, energy and Poisson equations from Peacock 15.7.

   (I) Euler first:
   $$\left(\frac{\partial}{\partial t}+(\vec{v}_0+\delta\vec{v})\cdot\nabla\right)(\vec{v}_0+\delta\vec{v})=\frac{-\nabla(p_0+\delta p)}{\bar{\rho}+\delta\rho}-\nabla\Phi_0-\nabla\delta\Phi$$
   Use the trick that $(\bar{\rho}+\delta\rho)^{-1}\approx(1-\delta)/\bar{\rho}$ where $\delta\equiv\frac{\delta\rho}{\bar{\rho}}$ and collect only 1st order terms (the 0th order terms are presumably already satisfied, and the 2nd order terms are presumably too small to worry about). You'll be left with Peacock's 15.12a:
   $\frac{\partial\delta\vec{v}}{\partial t}+(\vec{v}_0\cdot\nabla)\delta\vec{v}=\frac{-\nabla\delta p}{\bar{\rho}}-\nabla\delta\Phi-(\delta\vec{v}\cdot\nabla)\vec{v}_0$ Since $\frac{d}{dt}=\frac{D}{Dt}$ we can rewrite this as:
   $$\frac{d\delta\vec{v}}{dt}=\frac{-\nabla\delta p}{\bar{\rho}}-\nabla\delta\Phi-(\delta\vec{v}\cdot\nabla)\vec{v}_0 \qquad \text{(Peacock 15.12a)}$$
   We will get rid of pressure later entirely by assuming a simple equation of state = a constant sound speed: $c_S^2=\delta p/\delta\rho$ (we'll get rid of it even more easily in radiation). The last term on the RHS is a little weird, and the 0th order energy equation helps us out a little bit; Peacock's right in that using indices here is best:
   $(\delta\vec{v})_i\frac{\partial}{\partial x_i}(\vec{v}_0)_j=H(\delta\vec{v})_i\frac{\partial}{\partial x_i}(\vec{x})_j=H(\delta\vec{v})_j$ Some foreshadowing: this term will eventually give us one of the two $\frac{\dot{a}}{a}\dot{\delta}$ terms later. Anyway, we are left with:
   $$\frac{d\delta\vec{v}}{dt}=\frac{-\nabla\delta p}{\bar{\rho}}-\nabla\delta\Phi-H\delta\vec{v}$$

   (II) Similarly for the energy equation:
   $$\left(\frac{\partial}{\partial t}+(\vec{v}_0+\delta\vec{v})\cdot\nabla\right)(\bar{\rho}+\delta\rho)=-(\bar{\rho}+\delta\rho)\nabla\cdot(\vec{v}_0+\delta\vec{v})$$

collect 1st order terms only and you get Peacock's 15.12b (which seems to be missing a term?! Wait for it!):

$\left(\frac{\partial}{\partial t}+\vec{v}_0\cdot\nabla\right)\delta\rho=-\delta\rho\nabla\cdot\vec{v}_0-\bar{\rho}\nabla\cdot\delta\vec{v}$ Again, since $\frac{d}{dt}=\frac{D}{Dt}$ we can rewrite this as:

$$\frac{d\delta\rho}{dt}=-\delta\rho\nabla\cdot\vec{v}_0-\bar{\rho}\nabla\cdot\delta\vec{v} \qquad \text{(Peacock 15.12b?)}$$

We divide through by $\bar{\rho}$, naturally, but what do we do with the second term? It's missing in 15.12b! Peacock is a little sneaky, because, obvious in retrospect, the background density is changing with time in proper coordinates: $\frac{1}{\bar{\rho}}\frac{d\delta\rho}{dt}\neq\frac{d}{dt}\left(\frac{\delta\rho}{\bar{\rho}}\right)$ and this is where the "missing term" comes in. The 0th order energy equation is $\frac{d\bar{\rho}}{dt}=-\bar{\rho}\nabla\cdot\vec{v}_0$ and you are challenged to match up the missing term appropriately and absorb it into the time derivative so you really do get Peacock's 15.12b: $\frac{d\delta}{dt}=-\nabla\cdot\delta\vec{v}$

(III) The Poisson equation is trivial to linearize.

$$\nabla^2\delta\Phi=4\pi G\delta\rho=4\pi G\bar{\rho}\delta$$

(IV) Now if you take the time derivative of the energy equation and see that you have the (negative) of the divergence of the Euler:

$$\frac{d}{dt}\frac{d\delta}{dt}=-\frac{d}{dt}\nabla\cdot\delta\vec{v}=\frac{\nabla^2\delta p}{\bar{\rho}}+\nabla^2\delta\Phi+H\nabla\delta\vec{v} \qquad \text{(?!?!)}$$

and this is where things get a little strange: Peacock wanders off into a long tangent about comoving coordinates instead of proper ones. But we can handle the difference! Make the substitutions in Peacock's conventions: proper $\vec{x}=a\vec{r}$, where r is comoving, $\delta\vec{v}=a\vec{u}$ and $\nabla_r=a\nabla_x$ and note that $\nabla\bar{\rho}=0$ in comoving coordinates allowing us to simplify the pressure term a bit:

$$\frac{(1/a)\nabla_r\delta p}{\bar{\rho}}=\frac{c_S^2}{a}\nabla_r\delta$$

From here on out, divergences and gradients are in comoving and we'll drop the r subscript.

(I')The energy equation becomes:

$$\dot{\delta}=\frac{d\delta}{dt}=-\frac{1}{a}\nabla\cdot(a\vec{u})=-\nabla\cdot\vec{u} \qquad \text{(Energy Eqn.)}$$

(II') The Euler becomes:

$\frac{d(a\vec{u})}{dt}=\frac{-c_s^2}{a}\nabla\delta-\frac{1}{a}\nabla\delta\Phi-Ha\vec{u}$ by virtue of absorbing $\bar{\rho}$ into our comoving divergence (without consequences)!

So we get:

$$\frac{d\vec{u}}{dt}=-2H\vec{u}-\frac{c_s^2}{a^2}\nabla\delta-\frac{1}{a^2}\nabla\delta\Phi$$

Now we take the comoving divergence of this equation and the time derivative of the energy equation to get:

$$\ddot{\delta} = -\frac{d}{dt}\nabla\cdot\vec{u} = -\nabla\cdot\dot{\vec{u}}$$

$$= 2H\nabla\cdot\vec{u} + \frac{c_S^2}{a^2}\nabla^2\delta + \frac{1}{a^2}\nabla^2\delta\Phi$$ leading to: $$\ddot{\delta} + 2H\dot{\delta} - \frac{c_S^2}{a^2}\nabla^2\delta - 4\pi G\bar{\rho}\delta = 0$$

$$= -2H\dot{\delta} + \frac{c_S^2}{a^2}\nabla^2\delta + \frac{1}{a^2}\nabla^2\delta\Phi$$

What's left? Going into Fourier space, we can replace $\nabla \to ik$ and we get Peacock 15.21:

$$\ddot{\delta}_k + 2H\dot{\delta}_k + \left(\frac{c_S^2 k^2}{a^2} - 4\pi G\bar{\rho}\right)\delta_k = 0$$

Whew!

To get 15.25, note that you can quickly replace all pressure variables with energy density variables divided by 3 and follow the same attack as before. Some interesting departures from the analysis above include the fact that the gravity term shows up not as $4\pi G\bar{\rho}\delta_k$ but as $\frac{32\pi G\bar{\rho}}{3}\delta_k$ This should not alarm you... The missing "pressure" terms were either missing from the beginning (note how simple the Euler is in 15.24: it's missing the pressure gradient which you kept for 15.21) or absorbed in the algebra of the $\ddot{\delta}$ and $\dot{\delta}$ terms. Note the "drag" term has the same proportion to the "acceleration" term, $\ddot{\delta}/\dot{\delta} = 2H$ as it did in the MD case.

2. *(30) You know how $\rho_0 = \bar{\rho} = \rho_{background}$ evolves with time in different eras (yes, you do, it's the Friedmann equation). Below, you will drop the pressure terms in both (15.21) and your version of (15.25) and solve for $\delta_k(t)$ in the radiation (RD) and matter domination (MD) eras by presuming, like a good astrophysicist that the answer is a power law in time.* ***But first:***

   a. *(5) Explain in English at what scales, or for what reasons are you justified in dropping those terms during radiation domination and during matter domination. At which scales would you get nervous about dropping them (be specific)?*

   b. *(15) Now, drop those pressure-related terms and plug and grind (show all your work): solve for $\delta_k(t)$ during MD and then solve it during RD (or vice versa). Note how much "faster" the growth rate seems to be during RD compared to MD. BUT, see part d.*

   c. *(5) Should I care about the power laws with negative exponents? Explain.*

   At "large" scales this should be fine. pressure requires a short timescale comparable to the gravitational timescale. So if k^{-1} is near the Jeans length (pressure is comparable to gravity), then we are not justified in doing this. In RD, $\frac{32\pi G}{3}\bar{\rho} = 4H^2 = \frac{1}{t^2}$ and in MD, $4\pi G\bar{\rho} = \frac{3}{2}H^2 = \frac{2}{3t^2}$ so the solutions are pretty easy, especially if we assume a simple power law $\delta_k \propto t^n$

$$\ddot{\delta}_k + 2H\dot{\delta}_k - 4\pi G\bar{\rho}\,\delta_k = 0$$
$$n(n-1) + \frac{4}{3}n - \frac{2}{3} = 0 \text{ leads to } n = -1, \frac{2}{3}$$
$$\ddot{\delta}_k + 2H\dot{\delta}_k - \frac{32\pi G\bar{\rho}}{3}\delta_k = 0$$
$$n(n-1) + n - 1 = 0 \text{ leads to } n = \pm 1$$

and I don't care about the negative powers as they refer to decaying modes.

d. *(5) What is $\delta_k(a)$ in both RD and MD? (Ha! Now RD doesn't seem quite so fast compared to MD, does it? That explains why "t" is not the typical dependent variable of choice for cosmologists: either a or z works better. In fact, diehard theorists prefer η, the conformal time.)*
I should have asked you to show that $\delta_k(\eta) = \eta^2$ for both, but clearly wrote the problem too late at night. In terms of the scale factor, the growing modes are $\delta_k(a)(\mathrm{RD}) \propto t \propto a^2$ and $\delta_k(a)(\mathrm{MD}) \propto t^{2/3} \propto a$ as predicted in equation 15.2.

3. *(10) There's something funny going on with extremely large scales: imagine a perturbation δ_k on a **superhorizon** scale. It's "frozen" in the sense that causality won't let us do anything with it (let alone pressure). Why can I presume it will simply expand along with the background? In fact, GR folks would be very squeamish about this if we don't state that this approach is using the Newtonian gauge...*

Well, first and foremost, where did it come from if it's not causal? Okay, admittedly, a trick question: that has to wait until we discuss inflation.
Point of fact is that we can't assume it will expand along with the background until we define a coordinate system. But, we can assume that it won't enter the horizon until it is causal. This is a deep, and arguably dull philosophical point: if it's not within causal reach, is it real?

If we choose a timeslicing (the obvious one would be an inertial frame at rest with respect to the CMB (on average)), then we can discuss when each "mode" ($\lambda = 2\pi/k$) crosses the horizon $(\text{at } t \approx \lambda/c)$. Is a mode describable in a general way if it is larger than the horizon? To a "git-er-done" physicist, of course, but you actually have quite a lot of freedom in defining your coordinate system.

Traditionally, the "perturbations" to the metric (distortions in spacetime compared to the background) which are caused by the perturbations of the matter/energy density lead to one of two common choices: the Newtonian gauge vs. the Synchronous gauge. Once you've chosen one of these gauges, you can start discussing what you *mean* about perturbations beyond the horizon and whether they are sitting there, stuck waiting for the horizon to come to them, or expanding while waiting to come in or expanding.

But, all this is beyond the scope of this class since neither Stacy nor I worked with you on GR. I was curious where some of you might take it, bastard that I am. For more about this, you should read Peacock section 2.2; in any case, I'll grade this very generously.

4. *(15) As you may have heard, the CMB is remarkably "smooth." In particular, deviations from 2.73K are on the order of 10^{-5}: $\frac{\delta T}{\bar{T}} \sim 10^{-5}$. (I'll drop the "bar" over the T from here*

on.) But, this statement is angular-scale dependent. In much the way that $\delta(\mathbf{x})$ *for a point is meaningless,* $\frac{\delta T}{T}(\theta,\phi)$ *for a particular position on the sky is meaningless.*

a. *At what scale is* $\frac{\delta T}{T} \sim 10^{-5}$ *? (Hint: at what value of l is most of the angular power? What angle does that correspond to?)*
   $l \sim 200$ is where the first peak is; not surprisingly, the most power is at
   $\theta \approx \frac{180^o}{l} \sim 1^o$

b. *The deviation due to the CMB* ***dipole*** *is (one) considerably larger than this, (two) presumed to be entirely due to our relative motion and (three) is (of course) on the angular scale of half the sky! The dipole is in the mK range. Given that the overall temperature is in the O(1) K range, what* ***order of magnitude*** *(please do not overthink this!) speeds are required to get a temperature shift of mK?*

   That's a deviation of 1 part in 10^3, so, thinking in a unitless way:
   $v/c \sim 10^{-3} \rightarrow v \sim 1000$ km/s which isn't far from the real value in a handwaving way (see Peacock 9.62 and 9.63). In fact, if $v_{Earth} = 371$ km/s and $T_{CMB} = 3$ K then you should be able to tell me (to one digit accuracy) what the maximum "temperature" distortion from the dipole looks like (i.e., when $\theta = 0$ in equation 9.62).

c. *Does the temperature in that direction mean that the CMB isn't a blackbody in that direction? If not, why not? You may find section 9.4 in Peacock to be helpful with this part.*
   No, it's still a blackbody. [I'm particularly amused at the now-nostalgic sentence, "The dipole is the only term that has been detected;" just after equation 9.62. Time for a 2nd edition, John.]

5. *(10) Consider* $\frac{\delta T}{T}(\theta,\phi) = \sum_{l,m} a_l^m Y_l^m(\theta,\phi)$. *Now think about correlating the temperatures across all pairs of points* (θ,ϕ) and (θ',ϕ') *separated by the same angle on the sky. By plugging the above into "angle brackets" (i.e., averaging), show that:*
   $\left\langle \frac{\delta T}{T}(\theta,\phi)\frac{\delta T}{T}(\theta',\phi')\right\rangle = \frac{1}{4\pi}\sum_l (2l+1)C_l P_l(\cos\chi)$ *and thereby identify the connection between* C_l*'s and the* a_l^m*'s. This will also force you to look up the Spherical Harmonic Addition Theorem if you've forgotten it.*
   Start with $\left\langle \frac{\delta T}{T}(\theta,\phi)\frac{\delta T}{T}(\theta',\phi')\right\rangle = \sum_{lm}\sum_{l'm'}\langle a_{lm}a_{l'm'}\rangle Y_l^m(\theta,\phi)Y_{l'}^{m'}(\theta',\phi')$ and insist that azimuthal symmetry requires the *m*'s to be statistically uncorrelated to the *m*' s. This comes about in a sneaky way: homogeneity and isotropy says that we should get very similar $e^{im\phi}$ dependence, regardless of where we set up our "north pole." The way it's presented to unwary, unsuspecting grad students is:
   $\langle a_{lm}a_{l'm'}\rangle = C_l \delta_{l,l'}\delta_{m,m'}$ which means you should really think of C_l the way you think about δ_k^2: as a variance (and therefore like a σ^2), or as a power. At any rate, that collapses the sums down to:

$$\left\langle \frac{\delta T}{T}(\theta,\phi)\frac{\delta T}{T}(\theta',\phi')\right\rangle = \sum_l C_l \sum_m Y_l^m(\theta,\phi) Y_l^m(\theta',\phi')$$ and now you look up the spherical harmonic addition theorem (it's all over the place, e.g., Peacock 16.97 and even Wikipedia!):

$$\sum_m Y_l^m(\theta,\phi) Y_l^m(\theta',\phi') = \frac{2l+1}{4\pi} P_l(\cos\chi)$$ where

$\cos\chi = \cos\theta\cos\theta' - \sin\theta\sin\theta'\cos(\phi-\phi')$ giving you the thing you were asked for:

$$\left\langle \frac{\delta T}{T}(\theta,\phi)\frac{\delta T}{T}(\theta',\phi')\right\rangle = \frac{1}{4\pi}\sum_l (2l+1) C_l P_l(\cos\chi)$$ These are the famous C_l's plotted in any CMB paper since COBE (and some theory papers prior).

6. *(10) The Planck mission is going to probe l in the 1000's.*

   a. *What are some of the problems facing any CMB experiment trying to look at extremely low l<10? Consider statistics and real issues on the sky.*
   The main problem is that we only have one universe. The statistical error bars on the quadupole are quite large as there are only 5 truly independent "samples" of the quadrupole (when l=2, then m can range from –2 to 2, i.e., five independent coefficients of the Y_l^m's). On top of this, the quadrupole and other low l beasties are heavily polluted by the Milky Way itself.

   The theorist calculates everything thinking we have a complete sky of data (all 4π, i.e., $\sim 40{,}000$ sq.deg.) to sample. As this is not quite true, the Y_l^m's don't, in fact, supply a complete orthogonal basis. WMAP has a "preferred" skycut, for instance, which leads to using some odd combinations of the Y_l^m's which *are* (basically) orthogonal.

   b. *Pick any of your favorite mathematical software packages and see if you can plot $P_l(\cos\theta)$ for l=1000. What's are some of the issues involved?*
   Well, back in 2000 when I started messing around with Y_l^m's, I clearly had more trouble than any of you. Much code being flung around in those days involved generating functions for these kinds of orthogonal basis functions. Said iterative generating functions could easily accumulate rounding errors and Mathematica (for instance) would give me garbage if I tried to plot for l>200. Ah, those were the tough old days. You kids don't know what it was like, coding uphill both ways in the snow (ahem).

   Again, I'll grade generously here.