Astronomy 622 Cosmology   Homework Assignment #4 – Due 2:00 pm April 27th
100 points (later arbitrarily renormalized to match Dr. McGaugh's earlier homeworks)
Please explain your answers in complete sentences where called for. Although collaboration is encouraged, remember to answer in your own words. Where math is involved, please be sure to show all your work, and answer in no more than a reasonable number of significant digits.

1. (15) Go to the WMAP page http://map.gsfc.nasa.gov/resources/camb_tool/index.html and take a look at the WMAP CMB Analyzer. On it, they've kindly placed the data points from WMAP's mission along with a theoretical curve you can play with. Dial the various parameters until you get the gratifying message, "Success!!" in the background. You've matched theory to observation!

   a. (5) Leaving everything else dialed "correctly," play with the redshift of reionization. Explain what you see when $z_{reionization}$ is < 11 or > 11 and why.

   b. (5) Put $z_{reionization}$ back to 11 ("but this one goes to 11!") and now leave everything else dialed "correctly" except Ω_Λ. (It's labeled more generically as dark energy.) Explain what you see when $\Omega_\Lambda < 0.74$ or > 0.74 and why.

   c. (5) Now put Ω_Λ back to 0.74 and leave everything else dialed correctly except n_S. Explain what you see in terms of large and small scale power as you change the spectral index. Is there any special reason that the pivot point is where it is?

2. (15) Wherever possible, use energy units $(c = k_B = \hbar = 1)$. Start by assuming that the energy density of the universe is the Planck energy density (Planck energy divided by Planck length cubed) at the Planck time. (Before that, who knows?) So, consider this to be the ultimate initial condition.

   a. (5) Solve for the evolution of the energy density as a function of time from the Planck time until $t = 10^{-36}$s. (Treat this entire period as a radiation dominated era.)

   b. (5) Ignoring inflation, What is the temperature as a function of time based on these values? Reduce it to a function of T, t and numbers (not constants).

   c. (5) Assume that the universe undergoes an inflationary period from $t = 10^{-36}$s until 10^{-34}s and insist that the universe has undergone 100 e-foldings in this crazily short time period. What is the value of H during this period? (Hint: can we call it a Hubble constant?)

3. (20) Problem A.3.1 in Andrew Liddle's book. See class website for a copy of Appendix A.3.

4. (25) Assume that the number densities of both neutrons and protons (n_n, n_p) follow the Maxwell-Boltzmann distribution when $T > 1$ MeV. (In this problem, mass, temperature and energy are synonymously reported in units of MeV.)

   a. (5) Write down the formula for the ratio of the two Maxwell-Boltzmann distributed number densities for two massive species at the same temperature, $\dfrac{n_1}{n_2}$ (don't plug in any numbers yet) which nicely eliminates a host of constants out in front of the exponential. (Also be sure not to bother with k_B or c^2 in the exponential since

they're equal to 1!) Now plug in for the proton and neutron masses and simplify your expression to 2 significant figures as a function of T only.

b. (5) As discussed in class, presume that the effective freezeout of neutrinos occurs at $T=0.8$ MeV (see problem 2 also). What time is it (in seconds) when $T=0.8$ MeV?

c. (5) Calculate how many neutrons per proton are left at this freezeout.

d. (5) The neutrons proceed to be bound up in nuclei, but this takes some time to last because nuclei are just as often broken up by the more energetic tail in the radiation distribution – at least until $T=0.06$ MeV. What time is it (in seconds) when $T=0.06$ MeV?

e. (5) Assume the neutrons are effectively free and decay until $T=0.06$ MeV at which point they're bound up entirely in ^{4}He. How many neutrons per proton are left now? What is the predicted mass fraction of baryons in ^{4}He?

5. (25) The simplest inflaton is a scalar field with a simple potential (and for the record, it's probably wrong).

a. (5) The energy density of such a beast is $\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi)$ and it can be shown that the pressure content of such a field goes like $p = \frac{1}{2}\dot{\phi}^2 - V(\phi)$. Assume that the universe undergoes a Λ-like period of inflation – what general constraint does this put on what $\dot{\phi}^2$ can be compared to $V(\phi)$?

b. (5) Using the fluid energy equation $\dot{\rho} + 3H(p+\rho) = 0$, plug in the expressions above and show what the fluid energy equation becomes as a function of $\phi, \dot{\phi}$ and $\ddot{\phi}$.

c. (5) From your answer to part b., treat the $\dfrac{-dV}{d\phi}$ term as a force and the $3H\dot{\phi}$ term as a friction term. In particular, presume that ϕ reaches a "terminal velocity." Given the constraint from part a. above, show that we must have:
$$\left(\frac{dV}{d\phi}\right)^2 \ll 9H^2 V$$
This is heading towards the idea of "slow roll."

d. (5) But, you know what H has to be during inflation! Rewrite it in terms of $V(\phi)$ to show that
$$\left(\frac{dV}{d\phi}\right)^2 \ll 24\pi GV^2 \quad , \text{i.e.,} \quad \left(\frac{E_{Planck}}{V}\frac{dV}{d\phi}\right)^2 \ll 1$$

e. (5) Inflation cools the universe absurdly. T still follows our favorite equation $T \sim \dfrac{1}{a}$. Assume the inflaton (whatever $V(\phi)$ might be) starts at $t=10^{-36}$s and causes 100 e-folds, what is the temperature at the end of inflation?
[We know that can't be right, so when inflation ends it has to "reheat" the universe back up to the T it had before by dumping energy into regular matter and energy at the GUT scale (10^{16} GeV).]