Strong Gravity & Black Holes

- Outline of this lecture
  - Introduction to General Relativity
  - Spacetime metrics
  - Gravitational redshift about a black hole
  - Orbits and motions about a black hole
  - Relativistic emission lines from disks
  - The ISCO and black hole spin

I : Introduction to GR

Reminder about Special Relavitity

- Einstein postulated
  - Laws of physics are the same in all inertial frames of reference (there is no absolute reference frame)
  - The speed of light is the same in all inertial frames of reference
- Leads to some dramatic consequences
  - Time-dilation : Moving clock runs slowly by $\gamma=(1-v^2/c^2)^{-1/2}$
  - Length contraction : A moving object is compressed in the direction of motion by factor gamma$=(1-v^2/c^2)^{-1/2}$
  - Space and time dimension unified together; spacetime

$$\Delta s^2 = \Delta t^2 - (\Delta x^2 + \Delta y^2 + \Delta z^2)$$

  - Mass-energy equivalence : $E=mc^2$
  - Gravity is not included in framework of SR

I : Introduction to GR

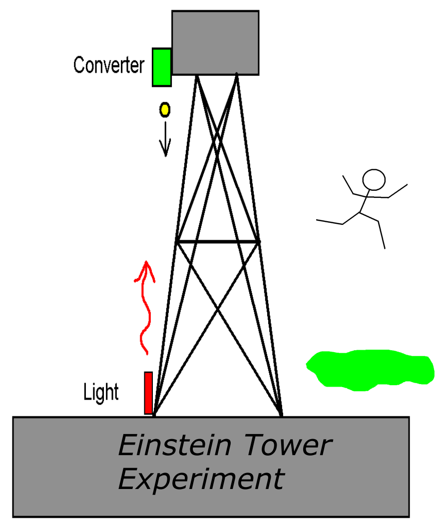
The Einstein Tower Experiment

- **How does gravity affect light?**
- Thought experiment:
  - Send photon (energy E_1) upwards in a gravitational field
  - Suppose photon at top has energy E_2. Convert energy into mass ($E_2=mc^2$) and drop
  - Convert mass back into photon. If there are no loses, final energy should be the same
- Photon must lose energy on the way up…

$$E_1 = E_2 + mgh = E_2 + \left(\frac{E_2}{c^2}\right) gh$$

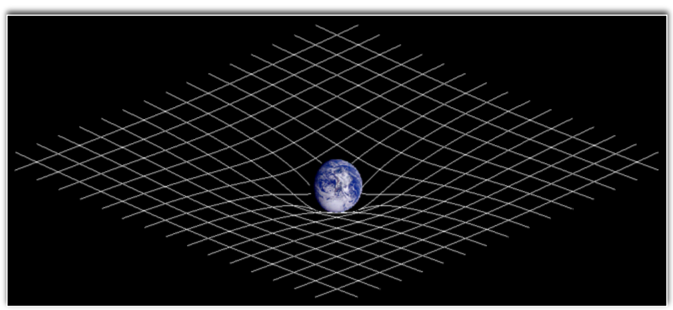
$$\Rightarrow E_2 = E_1 \left(1 + \frac{gh}{c^2}\right)^{-1}$$

- Thus, observer at the top will see frequency of photon decreased… gravitational redshift.
- **Surface of Earth is NOT inertial reference frame (in the Special Relativity sense)**
- **Free falling frames ARE inertial… Einstein Equivalent Principle**



I : Introduction to GR

Fundamentals of General Relativity



Field equations

Mass/energy distribution

Spacetime curvature

Equivalence Principle

I : Introduction to GR

Introduction to Metrics

- **Metrics tell you how to compute distances within a particular mathematical space**
- Simple 3-d space (Cartesian coordinates)

$$ds^2 = dx^2 + dy^2 + dz^2$$

- The same flat space can be described with spherical polar coordinates, giving a different looking metric

$$ds^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

- These forms describe the same metric, just in different coordinates. They both describe flat 3-d space
- An example of a fundamentally different space… spacetime of Special relativity (flat spacetime; Minkowski space)

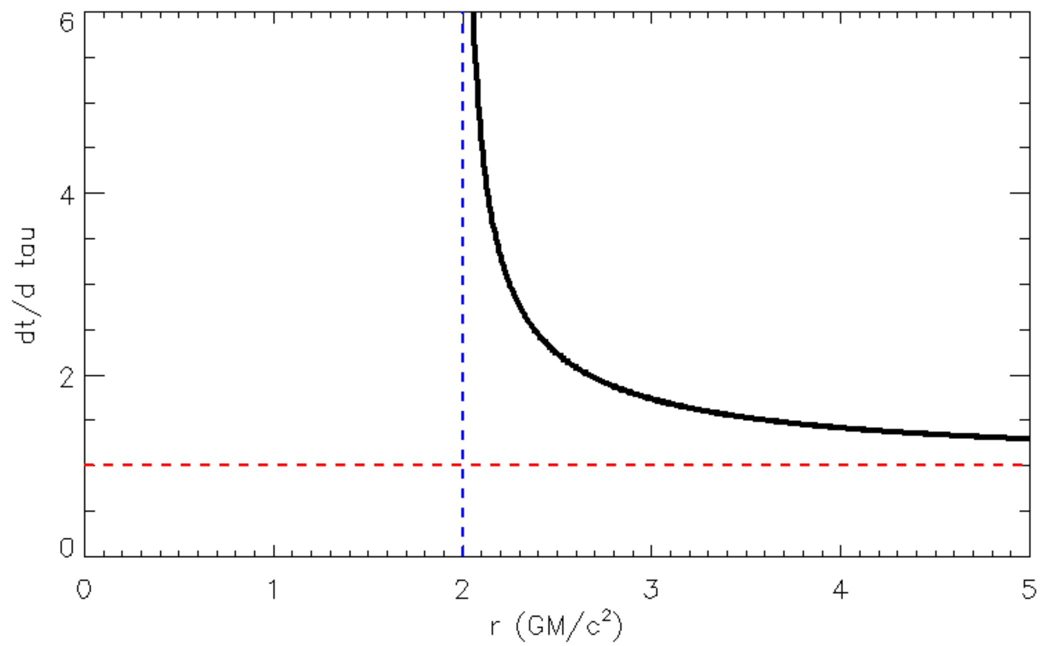
$$ds^2 = dt^2 - (dx^2 + dy^2 + dz^2)$$

II : Black holes, gravitational redshift and the event horizon

- Let's now come to black holes…
- **Start with non-spinning, uncharged black holes**
- Use "spherical polar" like coordinates, supplemented by time as measured by an observer at infinity… solving Einstein's field equations, we get the Schwarzschild metric…

$$\boxed{ds^2 = \left(1 - \frac{2GM}{c^2 r}\right) dt^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 - r^2\left[d\theta^2 + \sin^2\theta d\phi^2\right]}$$

- Gravitational Redshift
  - Imagine a clock at rest (dr=dθ=dΦ=0)
  - dt = length of clock "tick" as measured by observer at infinity
  - ds = dτ = distance along spacetime-path = proper duration of clock's "tick"
  - So… $$\frac{dt}{d\tau} = \left(1 - \frac{2GM}{c^2 r}\right)^{-1/2}$$ and… $$\frac{dt}{d\tau} \to \infty \;\; as \;\; r \searrow \frac{2GM}{c^2}$$



r=2GM/c^2 is an infinite redshift surface (Event horizon)

III : Orbits about a black hole

Basic theory

- Particles (including photons) follow **geodesics** through space-time… i.e. they follow the path that minimizes or maximizes the spacetime distance
- Start with the metric...

$$ds^2 = \left(1 - \frac{2GM}{c^2 r}\right) dt^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 - r^2 \left[d\theta^2 + \sin^2\theta d\phi^2\right]$$

- Spacetime distance traveled along path P (measured out by proper time coordinate) is

$$S = \int_P \mathcal{L}\, d\tau$$

with

$$\mathcal{L}^2 = \left(1 - \frac{2GM}{c^2 r}\right) \left(\frac{dt}{d\tau}\right)^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} \left(\frac{dr}{d\tau}\right)^2 - r^2 \left[\left(\frac{d\theta}{d\tau}\right)^2 + \sin^2\theta \left(\frac{d\phi}{d\tau}\right)^2\right]$$

III : Orbits about a black hole

Equations of motion and conserved quantities

- This kind of problem is solved using the calculus of variations
- Analysis results in set of differential equations that define the orbit of the particle/photon… Euler-Lagrange equations:

$$\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = 0 \quad \text{where} \quad \dot{x} \equiv \frac{dx}{d\tau} \qquad (x = t, r, \theta, \phi)$$

- Important special case… if metric (and hence L) has no explicit dependence on a coordinate, then

$$\frac{\partial \mathcal{L}}{\partial x} = 0 \;\Rightarrow\; \frac{\partial \mathcal{L}}{\partial \dot{x}} = \text{constant of motion}$$

- Time-independence and axisymmetry give conservation of energy and conservation of angular momentum respectively.

$$\frac{\partial \mathcal{L}}{\partial t} = 0 \;\Rightarrow\; E = \left(1 - \frac{2GM}{c^2 r}\right)\frac{dt}{d\tau} = \text{constant}$$ ***(Conservation of energy)***

$$\frac{\partial \mathcal{L}}{\partial \phi} = 0 \;\Rightarrow\; \ell = r^2 \sin^2\theta \frac{d\phi}{d\tau} = \text{constant}$$ ***(Conservation of angular momentum)***

III : Orbits about a black hole

Example: calculating path of radially infalling particle

- Assume particle is at rest at infinity. Use this fact to determine conserved energy…

$$E \equiv \left(1 - \frac{2GM}{c^2 r}\right)\frac{dt}{d\tau} = 1$$

- For a particle with mass, ds=dτ, so

$$\mathcal{L} \equiv \frac{ds}{d\tau} = 1$$

$$\Rightarrow \mathcal{L}^2 \equiv \left(1 - \frac{2GM}{c^2 r}\right)\left(\frac{dt}{d\tau}\right)^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1}\frac{dr}{d\tau}^2 = 1$$

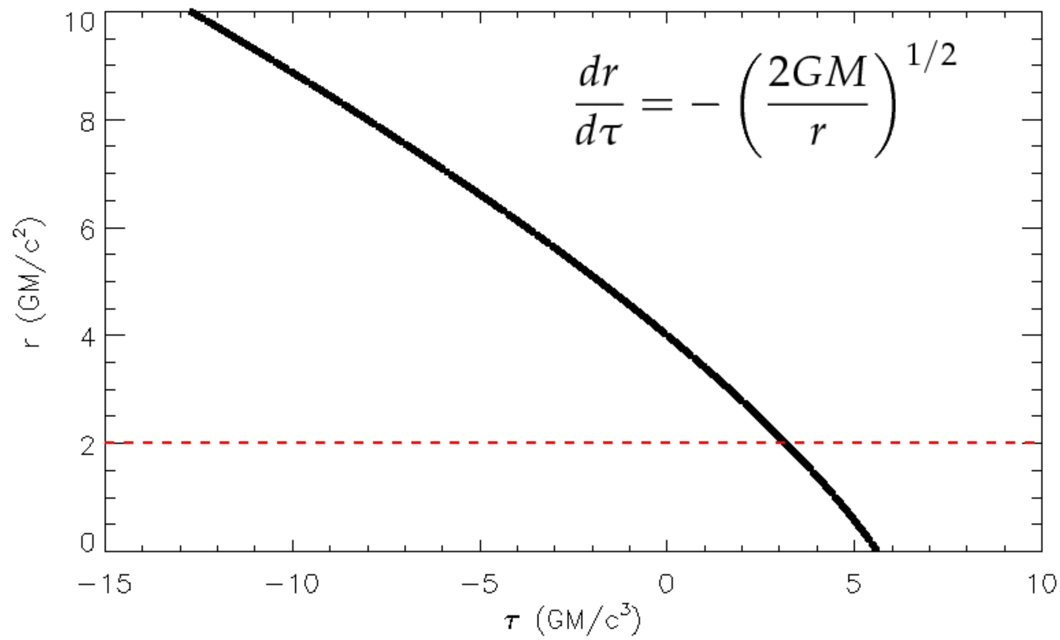
$$\Rightarrow \left(1 - \frac{2GM}{c^2 r}\right)^{-1} E^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1}\frac{dr}{d\tau}^2 = 1$$

$$\Rightarrow E^2 - \left(\frac{dr}{d\tau}\right)^2 = \left(1 - \frac{2GM}{c^2 r}\right)$$

$$\Rightarrow \left(\frac{dr}{d\tau}\right)^2 = \frac{2GM}{c^2 r}$$

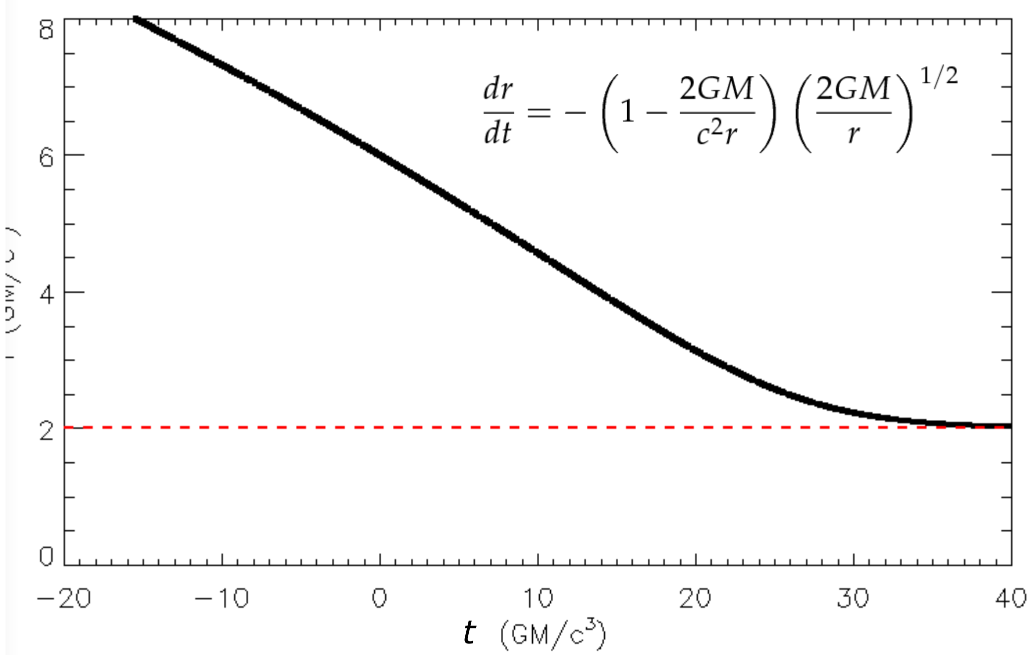
III : Orbits about a black hole

Falling radially into a black hole – victim's view



III : Orbits about a black hole

Falling radially into a black hole – external view



III : Orbits about a black hole

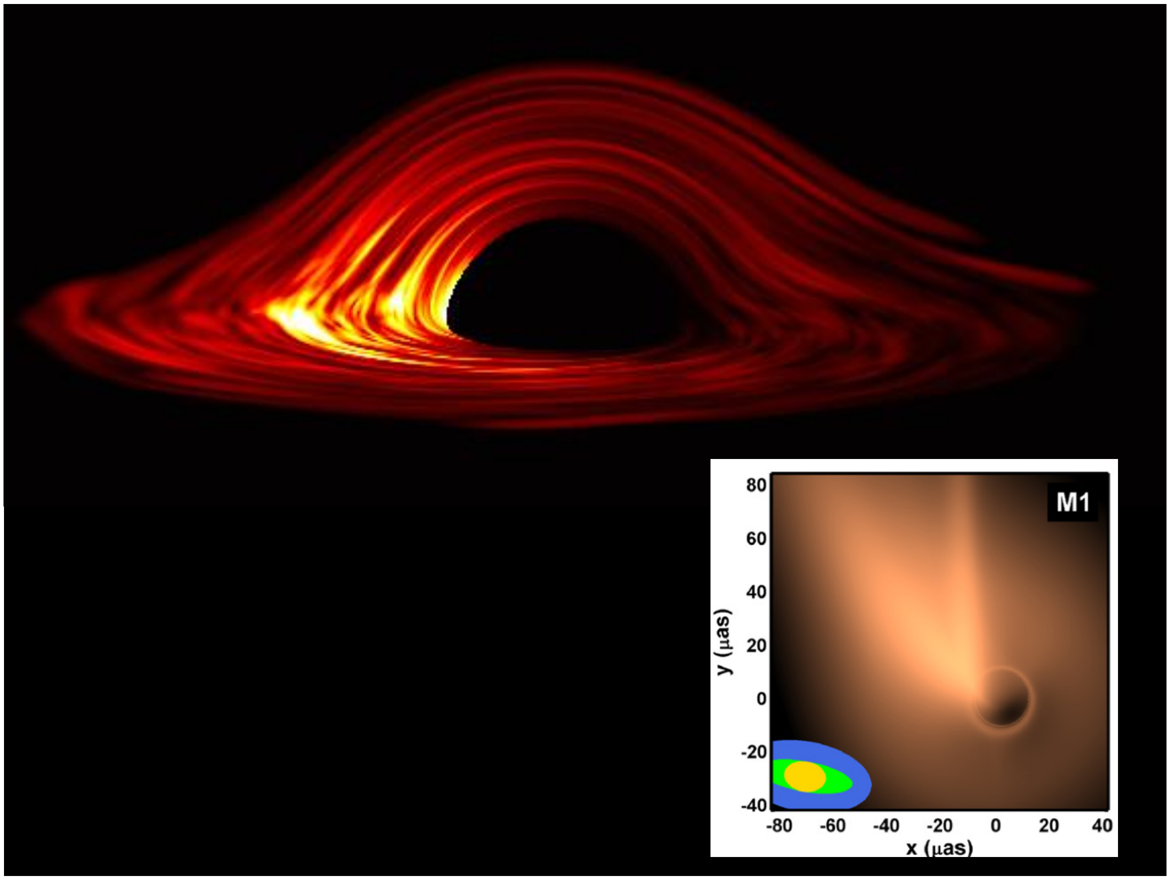
The nature of the event horizon

- So, we have learned important things about the nature of the event horizon…
- The Event Horizon is…
  - The infinite redshift surface
  - The place where infalling objects appear to "freeze" according to external observers
  - NOT a real singularity, since infalling observers pass through it unharmed
  - The boundary of the causally disconnected region (we didn't prove this!)
- The real singularity is at r=0

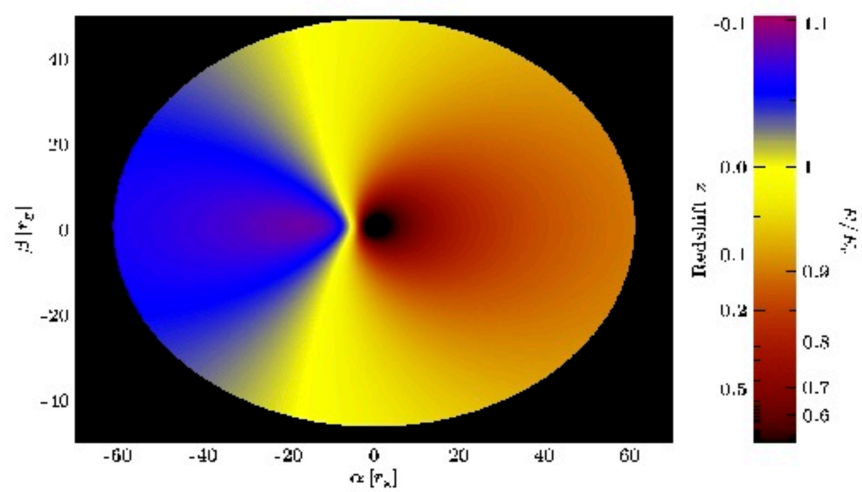
III : Orbits about a black hole

Some key results

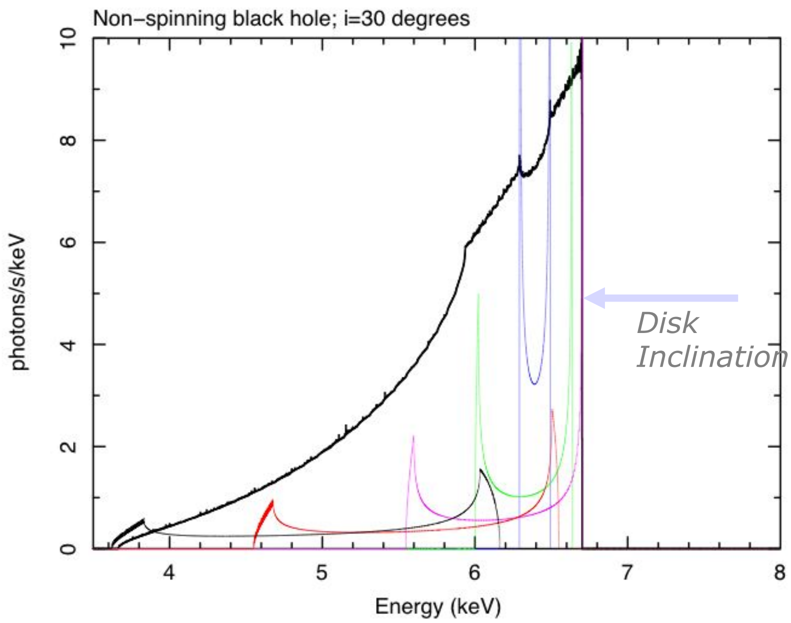
- Can identify some special radii that are relevant for orbits of particles around black holes
- For massive particles (time-like geodesics, $ds^2>0$)
  - $r=6GM/c^2$… **innermost stable circular orbit (ISCO)**
    - Beyond this radius, circular orbits are stable (as in Newtonian case)
    - Inside of this radius, circular orbit unstable and will spiral into BH
  - $r=4GM/c^2$… **marginally bound orbit**
    - Particle in circular orbit here has same energy as particle at rest at infinity.
    - Can transfer particle into this orbit from infinity with no dissipation!
- For massless particles (null geodesics, $ds^2=0$)
  - $r=3GM/c^2$… **photon circular orbit**
    - Photon grazing a black hole closer than this will fall into the black hole



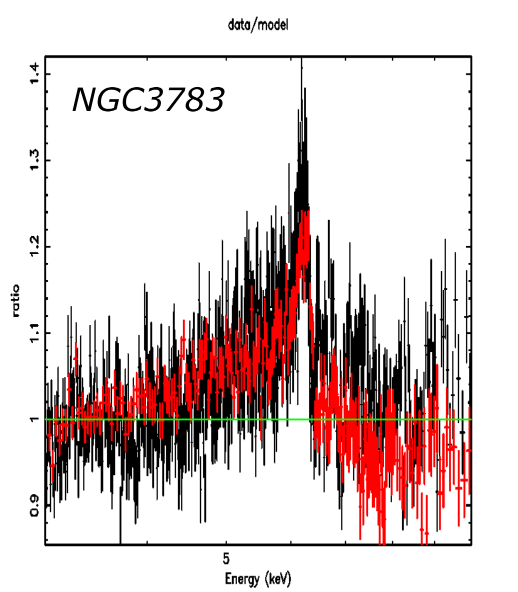
IV : Lines from relativistic disks



Courtesy T.Dauser



- Line profiles affected by
  - Doppler shift
  - Gravitational redshift
  - Other "astrophysics"… Compton broadening, blending of lines etc…
  - Of course, need good continuum modeling to study broad iron lines
- Principal parameters
  - Line energy
  - Disk inclination
  - Inner & Outer radii
  - Run of emissivity between inner and outer radii



V : Spinning black holes

- Remarkably, the equations of GR can be solved exactly even when BH is spinning… result is the Kerr Metric:

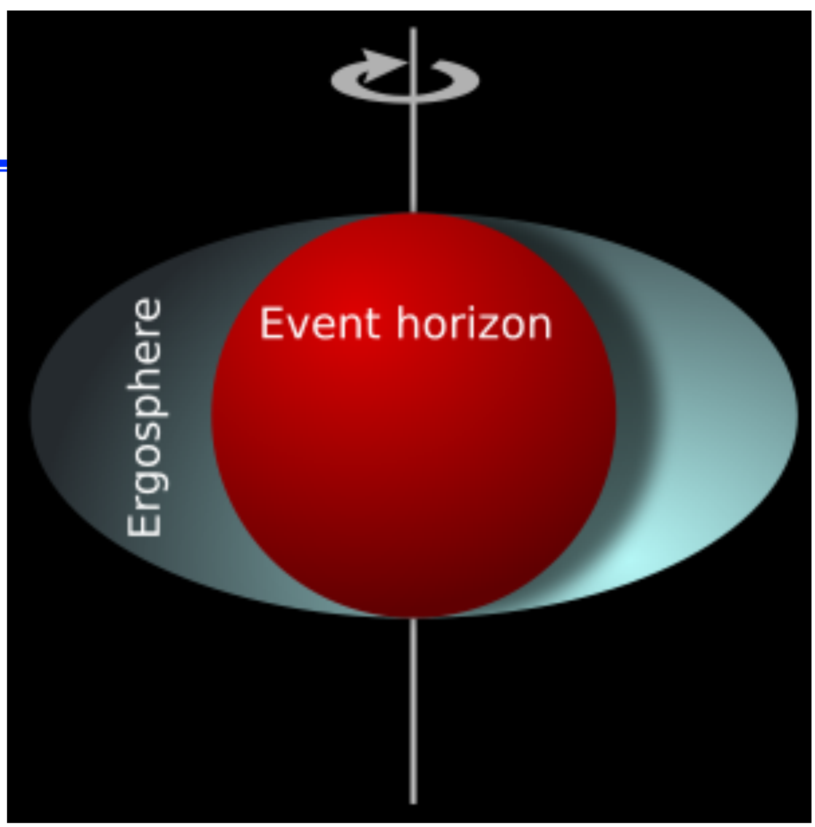
$$ds^2 = -\left(1 - \frac{2mr}{\Sigma}\right)dt^2 - \frac{4amr\sin^2\theta}{\Sigma}\,dt\,d\phi + \frac{\Sigma}{\Delta}\,dr^2 + \Sigma\,d\theta^2 + \left(r^2 + a^2 + \frac{2a^2mr\sin^2\theta}{\Sigma}\right)\sin^2\theta\,d\phi^2\,,$$

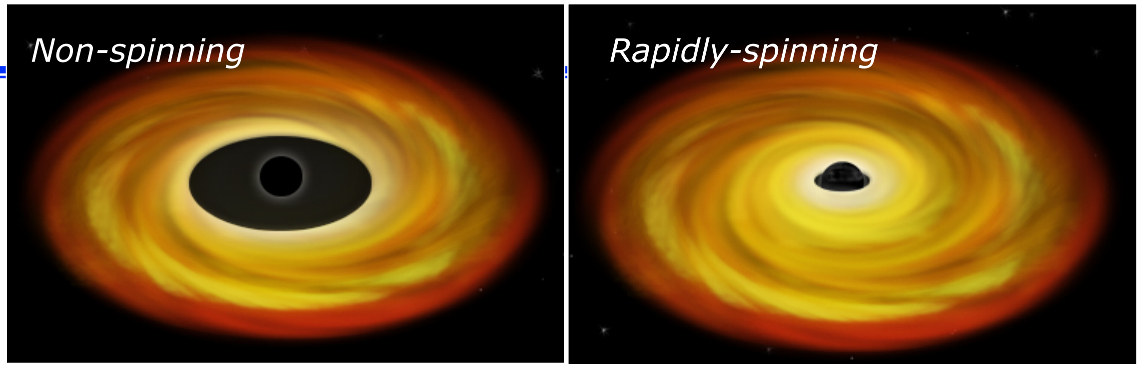
$$\Delta = r^2 - 2mr + a^2\,,$$
$$\Sigma = r^2 + a^2\cos^2\theta\,,$$

  - "a" is the angular mtm parameter (between -1 and +1)
  - Qualitatively new thing is the non-zero "cross-term" in the metric (dt dϕ)… this induces **frame-dragging**
  - Frame dragging becomes extreme where 2mr>Σ (ergosphere)
  - Event horizon is smaller than Schwarzschild

$$r_{evt} = \frac{GM}{c^2}(1 + \sqrt{1 - a^2})$$

  - ISCO is smaller for prograde orbits, larger for retrograde
  - Efficiency ($\eta = 1 - E_{isco}$) is higher for larger prograde spin





a=-1 -> $r=9GM/c^2$
a=0 -> $r=6GM/c^2$
a=1 -> $r=GM/c^2$