Summary of Dynamical Equations

- **gravitational pot'l $\Phi(r)=-G\int\rho\,(r)/|r-r'|\,d^3r$**
- **Gravitational force $F(r)=-\nabla\Phi(r)$**
- **Poissons Eq $\nabla^2\Phi(r)=4\pi G\rho$; if there are no sources**

  Laplace Eq $\nabla^2\Phi(r)=0$
- **Gauss's theorem : $\int\nabla\Phi(r)\bullet ds^2=4\pi GM$**
- **Potential energy $U=1/2\int r\rho(r)\nabla\Phi d^3r$**

- In words Gauss's theorem says that the integral of the normal component of $\nabla\Phi$ over and closed surface equals $4\pi G$ times the mass enclosed

Potentials are Separable

- We make the fundamental assumption that the potential of a system can be decomposed into separable parts-
- This is because Poisson's equation is linear :
- differences between any two $\phi-\rho$ pairs is also a $\phi-\rho$ pair, and

differentials of $\phi-\rho$ or are also $\phi-\rho$ pairs

- e.g. $\phi_{total}=\phi_{bulge}+\phi_{disk}+\phi_{halo}$

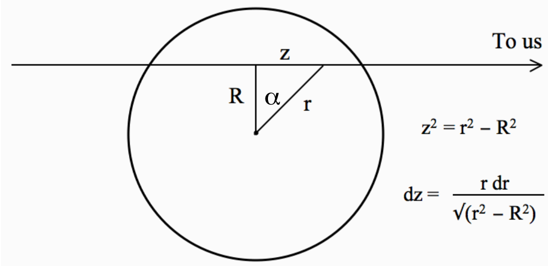
Projection Effects

from M. Whittle

http://people.virginia.edu/~dmw8f/astr5630/Topic07/Lecture_7.html

- Observed luminosity density I(R)=integral over true density distribution j(r) (in some wavelength band)
- Same sort of projection for velocity field but weighted by the density distribution of tracers
- Density distribution solution is an Abel integral (see appendix B.2 in B&T)
  - the velocity solution is also an Abel integral
- There are only a few useful I(R) & j(r) pairs that can both be expressed algebraically
  - e.g. $I(R) = I(0) / [1 + (R/r_0)^2]$ with $j(r) = I(0) / 2r_0[1 + (r/r_0)^2]^{3/2}$

$$I(R) = \int_{-\infty}^{\infty} j(r)\, dz = 2\int_R^{\infty} \frac{j(r)\, r\, dr}{\sqrt{(r^2 - R^2)}}$$



$$j(r) = \frac{-1}{\pi} \int_r^{\infty} \frac{dI}{dR} \frac{dR}{\sqrt{(R^2 - r^2)}}$$

So Far Spherical Systems

- But spiral galaxies have a significant fraction of the mass (?; at least the baryons) in a flattened system.

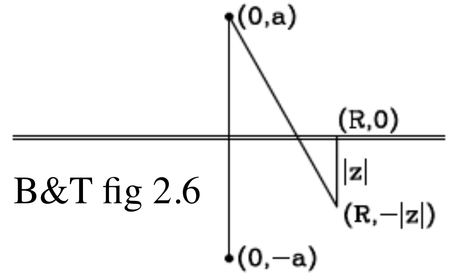
Kuzmin Disk B&T sec 2.3 S&G Prob 3.4;

- This ansatz is for a flattened system and separates out the radial and z directions
- Assume $\phi_K(z,R)$= GM/[sqrt($R^2+(a+z)^2$)] ; axisymmetric (cylindrical)

  R is in the x,y plane
- Analytically, outside the plane, ϕ_K has the form of the potential of a point mass displaced by a distance 'a' along the z axis
  - e.q. R(z)= { (0, a); z<0 ; (0, -a); z>0
- Thus $\nabla^2\Phi=0$ everywhere except along z=0- Poisson's eq
- Applying Gauss's thm $\int\nabla\Phi d^2s=4\pi GM$

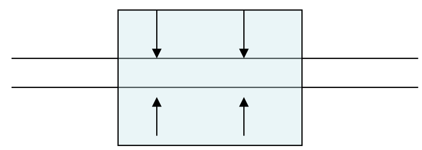
and get $\Sigma(R)=aM/[2\pi(R^2+a^2)^{3/2}]$

this is in infinitely thin disk... not too bad an approx



B&T fig 2.6

Use of Gauss's thm (divergence) the sum of all sources minus the sum of all sinks gives the net flow out of a region.



$\int\nabla\Phi d^2s=4\pi GM=2\pi G\Sigma$

as $z \longrightarrow 0$; $\Sigma=(1/2\pi)G\, d\Phi/dr$

Isothermal Sheet MBW pg 498

- simple model for the vertical structure of disk galaxies
- Allows an estimate the disk mass from a measurement of the vertical velocity dispersion, σ_z, and the radial scale length, R_d, if one knows the vertical scale height of the tracer population
- The relevant Poisson eq is $d^2\phi_z/d\,(z/z_d)^2=1/2\exp\text{-}(\phi_z)$;
- $\phi_z=\phi/\sigma^2_z$ *and* $z_d=\sigma_z/sqrt(8\pi G\rho(R,0))$

$\sigma^2_z(R)=(z/z_d)GM_dR_d\exp(-RR_d)$

- where z_d is the vertical scale height of the disk and R_d is the radial scale length
- can solve for the density distribution the disk

- *Why do we want to do this??- Estimates of the mass for face on galaxies where radial velocity data are impossible.*

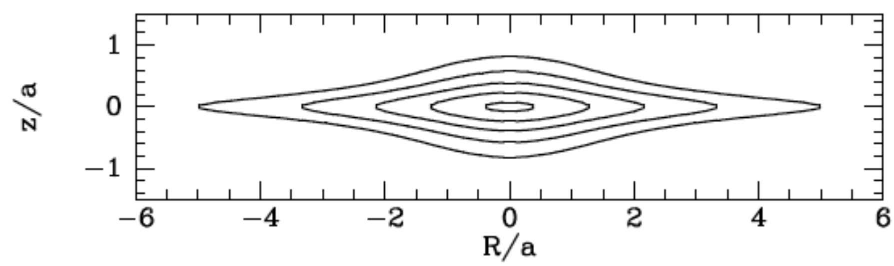
Flattened +Spherical Systems-B&T eqs

- Add the Kuzmin to the Plummer potential
- When b/a~ 0. 2, qualitatively similar to the light distributions of disk galaxies,

$$\Phi_{\rm M}(R,z) = -\frac{GM}{\sqrt{R^2 + (a+\sqrt{z^2+b^2}\,)^2}}. \qquad (2.69a)$$

When $a = 0$, $\Phi_{\rm M}$ reduces to Plummer's spherical potential (2.44a), and when $b = 0$, $\Phi_{\rm M}$ reduces to Kuzmin's potential of a razor-thin disk (2.68a). Thus, depending on the choice of the two parameters a and b, $\Phi_{\rm M}$ can represent the potential of anything from an infinitesimally thin disk to a spherical system. If we calculate $\nabla^2\Phi_{\rm M}$, we find that the mass distribution with which it is associated is (Miyamoto & Nagai 1975)

$$\rho_{\rm M}(R,z) = \left(\frac{b^2 M}{4\pi}\right)\frac{aR^2 + (a+3\sqrt{z^2+b^2}\,)(a+\sqrt{z^2+b^2}\,)^2}{\left[R^2+(a+\sqrt{z^2+b^2}\,)^2\right]^{5/2}(z^2+b^2)^{3/2}}. \qquad (2.69b)$$



Contours of equal density in the (R; z) plane for b/a=0.2

Potential of an Exponential Disk B&T sec 2.6

- As to be discussed later the light profile of the stars in most spirals has an exponential scale LENGTH

$\Sigma(R)=\Sigma_0\exp(-R/R_d)$ (this is surface brightness NOT surface mass density)- see next page for formula's

Mass of exponential disk
$M(R)= \int\Sigma(R)R\,dr = 2\pi\Sigma_0 R_d^2[1-\exp(R/R_d)(1+R/R_d)]$

when R gets large $M\sim 2\pi\Sigma_0 R_d^2$

Potential of an Exponential Disk B&T sec 2.6

.

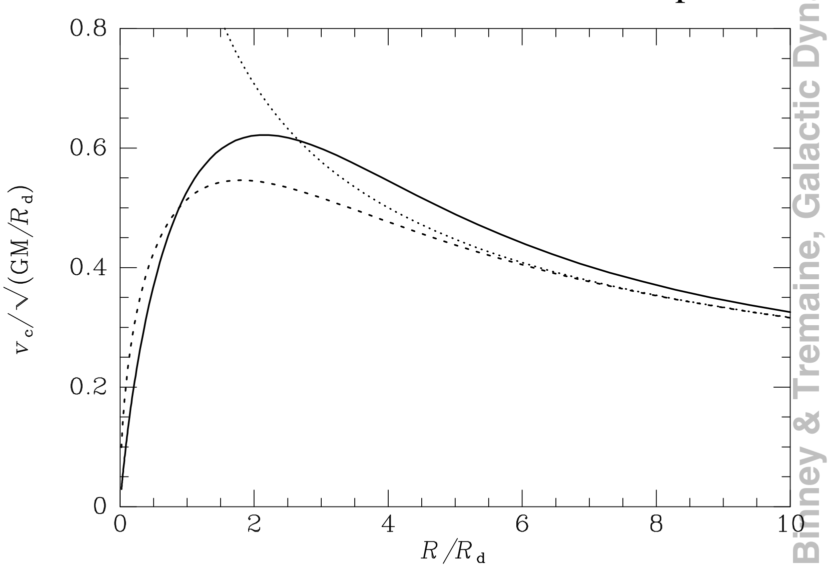
The circular velocity peaks at $R \sim 2.16\ R_d$ approaches Keplerian for a point mass at large R (eq. 11.30 in MWB) and depends only on Σ_0 and R_d

As long as the vertical scale length is much less than the radial scale the vertical distribution has a small effect - e.g. separable effects !

IF the disk is made only of stars (no DM) and and if they all have the same mass to light ratio Γ , R_d is the scale length of the stars , then the observables I_0,R_d,$v_{circ}(r)$ have all the info to calculate the mass!

Circular Velocity for 3 Potentials

Exponential disk (Solid line)
Point mass (.....)
Spherical exponential (-----)



Explaining Disks

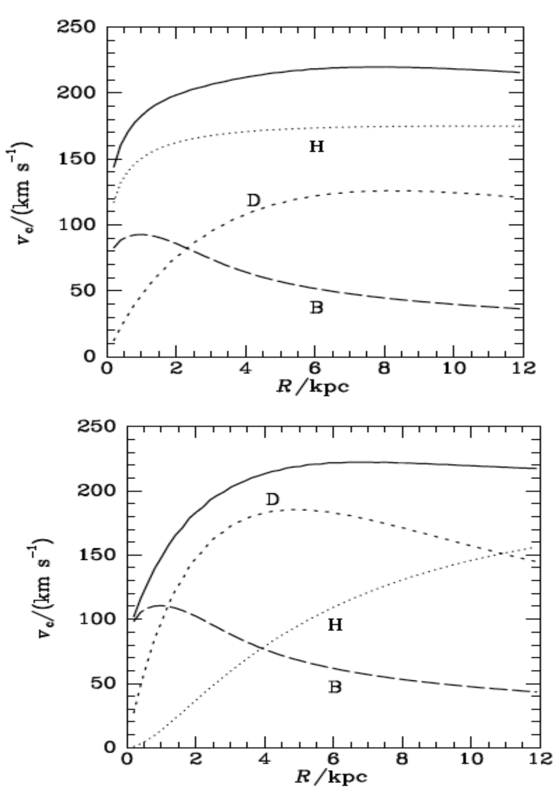
- Remember the most important properties of disk dominated galaxies (MBW pg 495)
  - Brighter disks are on average
    - larger, redder, rotate faster, smaller gas fraction
  - flat rotation curves
  - surface brightness profiles close to exponential
  - lower metallicity in outer regions

- traditional to model them as an infinitely thin exponential disk with a surface density distribution $\Sigma(R)=\Sigma_{0y}\exp(-R/R_d)$
  - This gives a potential (MBW pg 496) which is a bit messy

$$\phi(R, z)=-2\pi G\Sigma_0^2R_D\int[J_0(kR)\exp(-k|z|)]/[1+(kR_D)^2]^{3/2}dk$$

Modeling Spirals

- As indicated earlier to fit the observed density and velocity distributions in the MW one needs a 3 component mass distribution
- Traditionally this is parameterized as the sum of
  - disk $\Sigma(R) =\Sigma_0[\exp-R/a]$
  - spheroid (bulge) using $I(R)=I_0R_s^2/[R+Rs]^2$ or similar forms
  - dark matter halo $\rho(r)=\rho(0)/[1+(r/a)^2]$
- See B&T sec 2.7 for more complex forms- 2 solutions in B&T- notice extreme difference in importance of halo (H) (table 2.3)



Virial Theorem S&G pg 120, MBW pg 234

- S+G pg 120-121, MBW 5.4.4
- **$\frac{1}{2}d^2I/dt^2 = 2KE+W$** (no ext[l] forces)
  I = moment of inertia = $\Sigma m_i r_i^2$ (sum over i=1,N particles)
- A rather different derivation (H Rix)
- Consider (for simplicity) the 1-D Jeans eq in steady state (more later)
- $\partial/\partial x[\rho v^2]+\rho\partial\phi/\partial x=o$
- Integrate over velocities and then over positions...
- $-2E_{kin}=E_{pot}$ (static)
- or restating in terms of forces
- if T= total KE of system of N particles < >= time average

call the 'virial 'Q

$$Q=\frac{1}{2}\frac{dI}{dt}=m\sum r.\frac{dr}{dt}=\sum p.r$$

$$dQ/dt=\sum F.r+2T$$

$2<T>=-\Sigma(F_k \bullet r_k)$; summation over all particles k=1,N

Virial Theorem - Simple Cases

- Circular orbit:

$mV^2/r=GmM/r^2$

Multiply both sides by r, $mV^2=GmM/r$

$mV^2=2KE$; $GmM/r=-W$ so 2KE+W=0

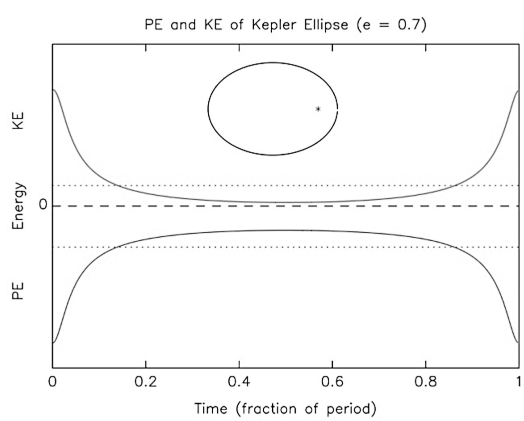
- Time averaged Keplerian orbit

define U=KE/|W|; as shown in figure i clearly changes over the orbit; but take averages:

$-W=<GMm/r>=GMm<1/r>$
$=GMm(1/a)$

$KE=<1/2mV^2>=GMm<1/r-1/2a>$
$=1/2GMm(1/a)$

So again 2KE+W=0



Red: kinetic energy (positive) starting at perigee
Blue: potential energy (negative)

Virial Theorem

- Another derivation following Bothun http://ned.ipac.caltech.edu/level5/Bothun2/Bothun4_1_1.html
- Moment of inertia, I, of a system of N particles
- $I=\Sigma m_i r_i^2$ sum over i=1,N (express r_i^2 as $(x_i^2+y_i^2+z_i^2)$)
- take the first and second time derivatives ; let d^2x/dt^2 be symbolized by x,y,z
- $\frac{1}{2}\, d^2I/dt^2 = \Sigma m_i\,(dx_i/dt)^2+(dy_i/dt)^2+(dz_i/dt)^2+\Sigma m_i(x_i\, x+y_i\, y+z_i\, z)$

mv^2 (2 KE)+Potential energy (W) [~r •(m**a**)]

after a few dynamical times, if unperturbed a system will come into Virial equilibrium–time averaged inertia will not change so 2<T>+W=0

For self gravitating systems $W=-GM^2/2R_H$; R_H is the harmonic radius- the sum of the distribution of particles appropriately weighted [$1/R_H = 1/N\ \Sigma_i\ 1/r_i$]

The virial mass estimator is $M=2\sigma^2 R_H/G$; for many mass distributions $R_H \sim 1.25\ R_{eff}$ where R_{eff} is the half light radius, σ is the 3-d velocity dispersion

Virial Thm MBW sec 5.4.4

- If I is the moment of inertia
- $\frac{1}{2} d^2I/dt^2 = 2KE+W+\Sigma$
  - where Σ is the work done by external pressure
  - KE is the kinetic energy of the system
  - W is the potential energy (only if the mass outside some surface S can be ignored)
- For a static system ($d^2I/dt^2 = 0$)
  $2KE+W+\Sigma = 0$

Using the Virial Theorm- (from J. Huchra)

- It is hard to use for distant galaxies because individual test particles (stars) are too faint
- However it is commonly used to clusters of galaxies

Assume the system is spherical. The observables are (1) the l.o.s. time average velocity:

$$\langle v^2_{R,i} \rangle_\Omega = 1/3\, v_i^2$$

projected radial v   averaged over solid angle

i.e. we only see the radial component of motion &

$$v_i \sim \sqrt{3}\, v_r$$

Ditto for position, we see projected radii R,

$R = \theta\, d$, d = distance, θ = angular separation

So taking the average projection,

$$\left\langle \frac{1}{|R_i - R_j|} \right\rangle_\Omega = \frac{1}{|r_i - r_j|} \left\langle \frac{1}{\sin \theta_{ij}} \right\rangle_\Omega$$

and

$$\left\langle \frac{1}{\sin \theta_{ij}} \right\rangle_\Omega = \frac{\int (\sin\theta)^{-1} d\Omega}{d\Omega} = \frac{\int_0^\pi d\theta}{\int_\pi^0 \sin\theta\, d\theta} = \pi/2$$

Remember we only see 2 of the 3 dimensions with R

Thus after taking into account all the projection effects, and if we assume masses are the same so that $M_{sys} = \Sigma m_i = N m_i$ we have

$$M_{VT} = \frac{3\pi}{2G} N \frac{\Sigma v_i^2}{\sum_{i<j} (1/R_{ij})}$$

this is the Virial Theorem Mass Estimator

Σv_i^2 = Velocity dispersion

$[\sum_{i<j} (1/R_{ij})]^{-1}$ = Harmonic Radius

Time Scales for Collisions (S&G 3.2)

- N particles of radius r_p; Cross section for a direction collision $\sigma_d = \pi r^2_p$
- **Definition of mean free path: $\lambda = 1/n\sigma_d$**

where n is the number density of particles (particles per unit volume), $n = N/(4\pi\ell^3/3)$

The characteristic time between collisions (Dim analysis) is $t_{collision} = \lambda/v \sim [(\ell/r_p)^2 t_{cross}/N]$ where v is the velocity of the particle. for a body of size ℓ, $t_{cross} = \ell/v$ = crossing time

Time Scales for Collisions (MBW sec 5.4.1)

So lets consider a galaxy with ℓ~10kpc, N=10^{10} stars and v~200km/sec

- if $r_p = R_{sun}$, $t_{collision}$~10^{21} yrs Therefore direct collisions among stars are completely negligible in galaxies.

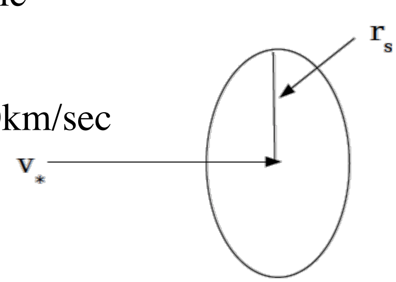
- For indirect collisions the argument is more complex (see S+G sec 3.2.2, MWB pg 231-its a long derivation-see next few pages) but the answer is the same - **it takes a very long time for star interactions to exchange energy (relaxation).**

- t_{relax}~$Nt_{cross}/10\ln N$
- It's only in the centers of the densest globular clusters and galactic nuclei that this is important

How Often Do Stars Encounter Each Other (S&G 3.2.1)

Definition of a 'strong' encounter, $GmM/r > 1/2mv^2$
potential energy exceeds KE of incoming particle
So a critical radius is $r<r_s=2GM/v^2$ eq 3.48

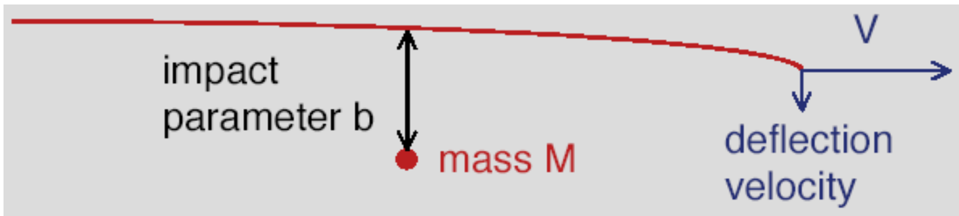
Putting in some typical numbers m~$1/2M_\odot$ v=30km/sec
r_s=1AU
So how often do stars get that close?



consider a cylinder Vol=$\pi r^2_s vt$; if have n stars per unit volume than on average the encounter occurs when
$n\pi r^2_s vt=1$, $t_s=v^3/ 4\pi nG^2M^2$
Putting in typical numbers =$\mathbf{4x10^{12}}(v/10km/sec)^3(M/M_\odot)^{-2}(n/pc^3)^{-1}$ **yr**- a very long time (universe is only 10^{10}yrs old) eq 3.55
- galaxies are essentially collisionless

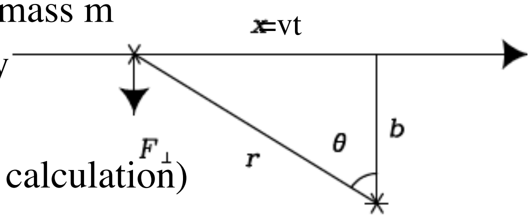
What About Collective Effects ? sec 3.2.2

For a weak encounter $b >> r_s$
Need to sum over individual interactions- effects are also small



Relaxation Times

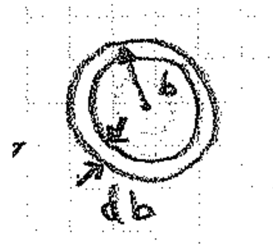
- Star passes by a system of N stars of mass m
- assume that the perturber is stationary during the encounter and that $\delta v/v<<1$ (B&T pg 33-sec 1.2.1. sec 3.1 for exact calculation)
- So δv is perpendicular to v
  - assume star passes on a straight line trajectory
- The force perpendicular to the motion is



$F_p=Gm^2\cos\theta/(b^2+x^2)=Gbm^2/(b^2+x^2)^{3/2}=(Gm^2/b^2)(1+(vt/b)^2)^{-3/2}=m(dvdt)$

so $\delta v=1/m \int F_p dt = (Gm^2/b^2)\int^{\infty}_{-\infty} dt(1+(vt/b)^2)^{-3/2}= 2GM/bv$

Relaxation Times



- In words, δv is roughly equal to the acceleration at closest approach,

  Gm/b^2, times the duration of this acceleration $2b/v$.

The surface density of stars is $\sim N/\pi R^2$
N is the number of stars and R is the galaxy radius

let δn be the number of interactions a star encounters with impact parameter
between b and δb crossing the galaxy once
$\delta n \sim (N/\pi r^2) 2\pi b \delta b = \sim (2N/r^2) b \delta b$

each encounter produces a dv but are randomly oriented to the stars intial velocity v and thus their mean is zero (vector) HOWEVER
the mean square is NOT ZERO and is
$\Sigma \delta v^2 \sim \delta v^2 \delta n = (2Gm/bv)^2 (2N/R^2)\ bdb$

Relaxation...continued (MBW pg

- now integrating this over all impact parameters from b_{min} to b_{max}
- one gets $\delta v^2 \sim 8\pi n(Gm)^2/v \ln \Lambda$; where r is the galaxy radius eq (3.54)

  $\ln \Lambda$ is the Coulomb logarithm $= \ln(b_{max}/b_{min})$ (S&G 3.55)

- For gravitationally bound systems the typical speed of a star is roughly $v^2 \sim GNm/r$

  (from KE=PE) and thus $\delta v^2/v^2 \sim 8 \ln \Lambda/N$

- For each 'crossing' of a galaxy one gets the same δv so the number of crossing for a star to change its velocity by order of its own velocity is $n_{relax} \sim N/8 \ln \Lambda$

Relaxation...continued

So how long is this??

- Using eq 3.55

$t_{relax}=V^3/[8\pi n(Gm)^2 \ln \Lambda]\sim[2x10^9 \text{ yr}/\ln\Lambda](V/10\text{km/sec})^3(m/M_\odot)^{-2}(n/10^3\text{pc}^{-3})^{-1}$

Notice that this has the same form and value as eq 3.49 (the strong interaction case) with the exception of the 2ln Λ term

- **$\Lambda \sim N/2$ (3.56)**
- **$t_{relax}\sim(0.1N/\ln N)t_{cross}$** ; if we use $N\sim10^{11}$; t_{relax} is much much longer than t_{cross}
- Over much of a typical galaxy the dynamics over timescales $t< t_{relax}$ is that of a **collisionless** system in which the constituent particles move under the influence of the gravitational field generated by a smooth mass distribution, rather than a collection of mass points
- However there are *parts* of the galaxy which 'relax' much faster

Relaxation

- Values for some representative systems

	<m>	N	r(pc)	t_{relax}(yr)	age(yrs)
Pleiades	1	120	4	$1.7x10^7$	$<10^7$
Hyades	1	100	5	$2.2x10^7$	4×10^8
Glob cluster	0.6	10^6	5	$2.9x10^9$	10^9-10^{10}
E galaxy	0.6	10^{11}	$3x10^4$	$4x10^{17}$	10^{10}
Cluster of gals	10^{11}	10^3	10^7	10^9	10^9-10^{10}

Scaling laws $t_{relax}\sim t_{cross}\sim R/v \sim R^{3/2} / (Nm)^{1/2} \sim \varrho^{-1/2}$

- Numerical experiments (Michele Trenti and Roeland van der Marel 2013 astro-ph 1302.2152) show that even globular clusters never reach energy **equipartition** (!) to quote 'Gravitational encounters within stellar systems in virial equilibrium, such as globular clusters, drive evolution over the two-body relaxation timescale. The evolution is toward a thermal velocity distribution, in which stars of different mass have the same energy). This thermalization also induces mass segregation. As the system evolves toward energy equipartition, high mass stars lose energy, decrease their velocity dispersion and tend to sink toward the central regions. The opposite happens for low mass stars, which gain kinetic energy, tend to migrate toward the outer parts of the system, and preferentially escape the system in the presence of a tidal field"

So Why Are Stars in Rough Equilibrium?

- Another process, 'violent relaxation' (MBW sec 5.5), is crucial.
- This is due to rapid change in the gravitational potential (e.g., collapsing protogalaxy)
- Stellar dynamics describes in a statistical way the collective motions of stars subject to their mutual gravity-The essential difference from celestial mechanics is that each star contributes more or less equally to the total gravitational field, whereas in celestial mechanics the pull of a massive body dominates any satellite orbits
- The long range of gravity and the slow "relaxation" of stellar systems prevents the use of the methods of statistical physics as stellar dynamical orbits tend to be much more irregular and chaotic than celestial mechanical orbits-....woops.
- to quote from MBW pg 248
- Triaxial systems with realistic density distributions are therefore difficult to treat analytically, and one in general relies on numerical techniques to study their dynamical structure