Dynamics and how to use the orbits of stars to do interesting things

chapter 3 of S+G- parts of ch 2 of B&T and parts of Ch 11 of MWB

**Galactic Rotation- Oort Constants**

- using a bit of trig
  
  \[ R(\cos \alpha) = R_0 \sin(l) \]
  
  \[ R(\sin \alpha) = R_0 \cos(l) - d \]

  so

  \[ V_{\text{observed, radial}} = (\omega - \omega_0) R_0 \sin(l) \]

  \[ V_{\text{observed, tangential}} = (\omega - \omega_0) R_0 \cos(l) - \omega d \]

  then following the text expand \((\omega - \omega_0)\)

  around \(R_0\) and using the fact that most of

  the velocities are local e.g. \(R-R_0\) is small

  and \(d\) is smaller than \(R\) or \(R_0\) (not

  TRUE for HI) and some more trig

  get

  \[ V_{\text{observed, radial}} = A \sin(2l); V_{\text{obs, tangential}} = A \cos(2l) + Bd \]

  Where

  \(A = -1/2 R_0 \left( d \omega / dr \right) \) at \(R_0\)

  \(B = -1/2 R_0 \left( d \omega / dr - \omega \right)\)
Assume gas/star has a perfectly circular orbit

At a radius $R_0$ orbit with velocity $V_0$, another star/parcel of gas at radius R has a orbital speed $V(R)$ since the angular speed $V/R$ drops with radius, $V(R)$ is positive for nearby objects with galactic longitude $0 < l < 90$ etc etc (pg 91 bottom)

In terms of Angular Velocity

- model Galactic motion as circular motion with monotonically decreasing angular rate with distance from center.
- Simplest physics: if the mass of the Galaxy is all at center angular velocity $\omega$ at R is $\omega = M^{1/2}G^{1/2}R^{-3/2}$
- If looking through the Galaxy at an angle $l$ from the center, velocity at radius R projected along the line of site minus the velocity of the sun projected on the same line is
  
  $V = \omega R \sin d - \omega_0 R_0 \sin l$

  $\omega = \text{angular velocity at distance R}$
  $\omega_0 = \text{angular velocity at a distance } R_0$
  $R_0 = \text{distance to the Galactic center}$
  $l = \text{Galactic longitude}$
- Using trigonometric identity $\sin d = R_0 \sin (l/R)$ and substituting into equation (1)
  
  $V = (\omega - \omega_0) R_0 \sin l$

Continued

- The tangential velocity $v_T = V_o \sin \alpha - V_o \cos \lambda$
  and $R \sin \alpha = R_o \cos \lambda - d$
- a little algebra then gives
  
  $V_T = V/R(R_o \cos \lambda - d) - V_o \cos \lambda$
- re-writing this in terms of angular velocity
  
  $V_T = (\omega - \omega_o)R_o \cos \lambda - \omega d$

- For a reasonable galactic mass distribution we expect that the angular speed
  $\omega = V/R$ is monotonically decreasing at large $R$ (most galaxies have flat
  rotation curves (const $V$) at large $R$) then get a set of radial velocities as a
  function of where you are in the galaxy
- $V_T$ is positive for $0 < \lambda < 90$ and nearby objects- if $R > R_o$ it is negative
- For $90 < \lambda < 180$ $V_T$ is always negative
- For $180 < \lambda < 270$ $V_T$ is always positive (S+G sec 2.3.1)
Oort Constants S&G pg 92-93

Derivation:

- for objects near to sun, use a Taylor series expansion of $\omega - \omega_0$

\[
\omega - \omega_0 = \frac{d\omega}{dR} (R - R_0)
\]

\[
\omega = \frac{V}{R}; \quad \frac{d\omega}{dR} = \frac{d}{dR} \left( \frac{V}{R} \right) = \frac{1}{R} \frac{dV}{dr} - \frac{V}{R^2}
\]

then to first order

\[
V_r = (\omega - \omega_0)R_0 \sin l = \left[ \frac{dV}{dr} - \frac{V}{R} \right] (R - R_0) \sin l ; \quad \text{when} \quad d << R_0
\]

\[
R - R_0 = d \cos l \quad \text{which gives}
\]

\[
V_r = \left( \frac{V_0}{R_0} - \frac{dV}{dr} \right) d \sin 2l
\]

using trig identity $\sin l \cos l = \frac{1}{2} \sin 2l$

one gets the Oort formula

\[
V_r = A d \sin 2l
\]

One can do the same sort of thing for $V_T$

Oort Constants

- For nearby objects (d << R)
  \[- V(R) \sim R_0 \sin l \left( \frac{dV}{dr} \right)(R - R_0)
  \sim d\sin(2l) - R/2 \left( \frac{dV}{dr} \right) - dA \sin(2l)
\]

(l is the galactic longitude)

- A is one of 'Oorts constants'
- The other B (pg 93 S+G) is related to the tangential velocity of an object near the sun $V_t = d[A \cos(2l) + B]$
- So, stars at the same distance r will show a systematic pattern in the magnitude of their radial velocities across the sky with Galactic longitude.
- A is the Oort constant describing the shearing motion and B describes the rotation of the Galaxy

$$A = \frac{1}{2} \left[ \frac{V_0}{R_0} - \left( \frac{dV}{dR} \right) \right]$$

$$B = -\frac{1}{2} \left[ \frac{V_0}{R_0} + \left( \frac{dV}{dR} \right) \right]$$

$$A + B = -\left( \frac{dV}{dR} \right) \quad ; \quad A - B = \frac{V_0}{R_0}$$

Useful since if know A get kinematic estimate of d

Radial velocity $v_r \sim 2AR_0 (1 - \sin l)$

only valid near l ~ 90 measure

$AR_0 \sim 115 \text{km/s}$
Oort 'B'

- B measures 'vorticity' \( B = -\frac{1}{2}[R\omega/dr] = -\frac{1}{2}[(V/R)+(dV/dR)] \) angular momentum gradient
  \[ \omega = A - B = \frac{V}{R} \]; angular speed of Local standard of rest (sun's motion)

Oort constants are local description of differential rotation

Values

- A = 14.8 km/s/kpc
- B = -12.4 km/s/kpc

Velocity of sun \( V_0 = R_0(A - B) \)

I will not cover epicycles (stars not on perfect circular orbits) now (maybe next lecture): see sec pg 133ff in S&G

A Guide to the Next Few Lectures

- The geometry of gravitational potentials: methods to derive gravitational potentials from mass distributions, and visa versa.
- Potentials define how stars move
  - consider stellar orbit shapes, and divide them into orbit classes.
- The gravitational field and stellar motion are interconnected:
  - the Virial Theorem relates the global potential energy and kinetic energy of the system.
- The Distribution Function (DF):
  - the DF specifies how stars are distributed throughout the system and with what velocities.
  - For collisionless systems, the DF is constrained by a continuity equation: the Collisionless Boltzmann Equation
  - This can be recast in more observational terms as the Jeans Equation.
  - The Jeans Theorem helps us choose DFs which are solutions to the continuity equations

*Adapted from M. Whittle
A Reminder of Newtonian Physics sec 2.1 in B&T

Newton’s law of gravity tells us that two masses attract each other with a force

\[ \frac{d}{dt} (m \mathbf{v}) = -\frac{GMm}{r^3} \mathbf{r} \]

If we have a collection of masses acting on a mass \( m_\alpha \), the force is

\[ \frac{d}{dt} (m_\alpha \mathbf{v}_\alpha) = -\sum_\beta \frac{GM_\alpha M_\beta}{|\mathbf{x}_\alpha - \mathbf{x}_\beta|^3} (\mathbf{x}_\alpha - \mathbf{x}_\beta), \ \alpha \neq \beta \]

\[ \frac{d}{dt} (m \mathbf{v}) = -m \nabla \phi (\mathbf{x}), \]

with

\[ \phi (\mathbf{x}) = -\sum_\alpha \frac{GM_\alpha}{|\mathbf{x} - \mathbf{x}_\alpha|}, \text{ for } \mathbf{x} \neq \mathbf{x}_\alpha \]

the gravitational potential. If we can approximate the discrete stellar distribution with a continuous distribution \( \rho \).

\[ \phi (\mathbf{x}) = -\int \frac{G \rho (\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3 \mathbf{x}' \]

Conservation of Energy and Angular Momentum

Sec 3.1 S&G

In the absence of external forces a star will conserve energy along its orbit

\[ \mathbf{v} \cdot \frac{d}{dt} (m \mathbf{v}) = -m \mathbf{v} \cdot \nabla \phi (\mathbf{x}), \]

\[ \mathbf{v} \cdot \frac{d}{dt} (m \mathbf{v}) + m \mathbf{v} \cdot \nabla \phi (\mathbf{x}) = 0 \]

But since

\[ \frac{d}{dt} \left[ \frac{m}{2} (\mathbf{v}^2) + m \phi (\mathbf{x}) \right] = 0 \]

This is just the KE + PE

\[ \frac{dL}{dt} = \mathbf{x} \times m \frac{d\mathbf{v}}{dt} = -m \mathbf{x} \times \nabla \phi \]

Angular momentum \( L \)

Gauss’s thm \( \int \nabla \phi \cdot d\mathbf{s} = 4\pi GM \)

the Integral of the normal component over a closed surface = \( 4\pi G \times \text{mass within that surface} \)
Some Basics - M. Whittle

- The gravitational potential energy is a scalar field
- its gradient gives the net gravitational force (per unit mass) which is a vector field:

\[
\Phi(r) = -G \int_V \frac{\rho(r')}{|r' - r|} \, d^3 r'
\]

\[
\mathbf{F}(r) = -\nabla \Phi(r) = G \int_V \frac{r' - r}{|r' - r|^3} \rho(r') \, d^3 r'
\]

\[
\nabla \cdot \mathbf{F}(r) = -4\pi G \rho(r)
\]

\[
\nabla^2 \Phi(r) = 4\pi G \rho(r) \quad \text{Poisson's eq inside the mass distribution}
\]

\[
\nabla^2 \Phi(r) = 0 \quad \text{Outside the mass distribution}
\]

Poisson's Eq+ Definition of Potential Energy (W)
So the force per unit mass is \( \rho(x) \) is the density dist

\[
\mathbf{F}(x) = -\nabla \Phi(x) = \int G \rho(x') \frac{(x - x')}{|x - x'|} \, d^3 x'
\]

To get the differential form we start with the definiti of \( \Phi \) and applying \( \nabla^2 \) to both sides

\[
\nabla^2 \Phi(x) = -\nabla^2 \int \frac{G \rho(x')}{|x - x'|} \, d^3 x' = 4\pi G \rho(x) \quad \text{Poisson's equation}
\]

Potential energy \( W \)

\[
W = \frac{1}{2} \int_V \rho(r) \Phi(r) \, d^3 r = -\frac{1}{8\pi G} \int_V |\nabla \Phi|^2 \, d^3 r
\]
Derivation of Poisson's Eq

So the force per unit mass is

\[ F(x) = -\nabla \Phi(x) = \int G \rho(x') \frac{|x-x'|}{|x-x'|^3} d^3 x' \]

To get the differential form we start with the definition of \( \Phi \) and applying \( \nabla^2 \) to both sides

\[ \nabla^2 \Phi(x) = -\nabla^2 \int \frac{G \rho(x')}{|x-x'|} d^3 x' \]

\[ = 4 \pi G \rho(x) \text{ Poisson's equation.} \]

see S+G pg112 for detailed derivation

Characteristic Velocities

\[ v^2_{\text{circular}} = r \frac{d \Phi(r)}{dt} = GM/r; \ v = \sqrt{GM/r} \text{ Keplerian} \]

velocity dispersion \( \sigma^2 = \langle 1/\rho \rangle \int \rho \left( \frac{\partial \Phi(r,z)}{\partial z} \right) dz \)

or alternatively \( \sigma^2(R) = \frac{4\pi G}{3M(R)} \int r \rho(r) M(R) dr \)

escape speed \( v_{\text{esc}} = \sqrt{2 \Phi(r)} \) or \( \Phi(r) = \frac{1}{2} v_{\text{esc}}^2 \)

so choosing \( r \) is crucial
More Newton-Spherical Systems B&T 2.2

Newtons 1st theorem: a body inside a spherical shell has no net gravitational force from that shell; e.g. $\nabla\Phi(r)=0$

Newtons 2nd theorem: the gravitational force on a body outside a spherical shell is the same as if all the mass were at a point at the center of the shell.

Simple examples:
Point source of mass $M$; potential $\Phi(r)=-GM/r$;
definition of circular speed; speed of a test particle on a circular orbit at radius $r$
$v^2_{\text{circular}}=r \frac{d\Phi(r)}{dt}=GM/r$; $v_{\text{circular}}=\sqrt{GM/r} ;$ Keplerian

velocity dispersion $\sigma^2=\left(\frac{1}{\rho}\right) \int \rho \left(\frac{\partial \Phi(r,z)}{\partial z}\right) dz$
escape speed $=\sqrt{2\Phi(r)}=\sqrt{2GM/r}$; from equating kinetic energy to potential energy $1/2mv^2=|\Phi(r)|$

Escape Speed

• As $r$ goes to infinity $\phi(r)$ goes to zero
• so to escape $v^2>2\phi(r)$; e.q. $v_{\text{esc}}=\sqrt{-2\phi(r)}$
Gravity and Dynamics-Spherical Systems- Repeat

- Newtons 1\textsuperscript{st} theorem: a body inside a spherical shell has no net force from that shell $\nabla \phi = 0$
- Newtons 2\textsuperscript{nd} theorem: a body outside the shell experiences forces as if they all came from a point at the center of the shell—Gravitational force at a point outside a closed sphere is the same as if all the mass were at the center
  - This does not work for a thin disk—cannot ignore what is outside of a given radius
- One of the prime observables (especially for spirals) is the circular velocity; in general $V^2(R)/R = G(M<R)/R^2$ more accurate estimates need to know shape of potential
- so one can derive the mass of a flattened system from the rotation curve

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- point source has a potential $\phi = -GM/r$
- A body in orbit around this point mass has a circular speed $v_c^2 = r \phi'(r) = GM/r$
- $v_c = \sqrt{GM/r}$; Keplerian
- Escape speed from this potential $v_{\text{escape}} = \sqrt{2\phi} = \sqrt{2GM/r}$ (conservation of energy $KE = 1/2mv_{\text{escape}}^2$

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Homogenous Sphere  B&T sec 2.2.2

- Constant density sphere of radius a and density $\rho_0$
- $M(r) = 4\pi Gr^3 \rho_0$; $r < a$
- $M(r) = 4\pi Ga^3 \rho_0$; $r > a$
  - $\phi(R) = -d/M(M(R)) = -3/GM^2/R$; B&T 2.41
  - $R > a$ $\phi(r) = 4\pi Ga^3 \rho_0 = -GM/r$
  - $R < a$ $\phi(r) = -2\pi G\rho_0(a^2 - 1/3r^2))$;
- $v_{\text{circ}}^2 = (4\pi/3)G\rho_0 r^2$; solid body rotation $R < a$
- Orbital period $T = 2\pi r/v_{\text{circ}} = \sqrt{3\pi/16G\rho_0}$
- Dynamical time = crossing time = $T/4 = \sqrt{3\pi/16G\rho_0}$
- Potential is the same form as an harmonic oscillator with angular freq $2\pi/T$ (B&T 2.2.2(b))
- Regardless of $r$ a particle will reach $r = 0$ (in free fall) in a time $T/4$
- Eq of motion of a test particle INSIDE the sphere is
  - $dr^2/dt^2 = -GM(r)/r^2 = -(4\pi/3)G\rho_0 r$
  - General result dynamical time $\sim \sqrt{1/G\rho}$
Some Simple Cases

- **Constant density sphere** of radius $a$ and density $\rho_0$ continued

Potential energy (B&T) eq 2.41, 2.32

$$\phi(R) = -d/dr(M(R))$$

- $R>a$ \( \phi(r) = 4\pi Ga^3\rho_0 = -GM/r \)
- $R<a$ \( \phi(r) = -2\pi G\rho_0(a^2/3r^2) \)

$$v_{\text{circ}}^2 = (4\pi/3)G\rho_0 r^2$$ solid body rotation

Potential is the same form as a harmonic oscillator

e.g. the eq of motion is \( d^2r/dt^2 = -GM(r)/r = 4\pi/3G\rho \); solution to harmonic oscillator is

$$r = A\cos(\omega t + \phi)$$ with $\omega = \sqrt{4\pi/3G\rho}$

$$T = \sqrt{3\pi/G\rho_0} = 2\pi r/v_{\text{circ}}$$

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**Spherical Systems**: Homogenous sphere of radius '$a'$

**Summary**

- $M(r) = 4/3\pi r^3$ ($r<a$); $r>a$ $M(r) = 4/3\pi r^3$
- Inside body ($r<a$); $\phi(r) = -2\pi G\rho(a^2/3r^2)$ (from eq. 2.38 in B&T)

Outside ($r>a$); $\phi(r) = -4\pi G\rho(a^3/3)$

Solid body rotation $v_c^2 = -4\pi G\rho(r^3/3)$

Orbital period $T = 2\pi/v_c = \sqrt{3\pi G\rho}$;

a crossing time (dynamical time) $= T/4 = \sqrt{3\pi/16G\rho}$

potential energy $W = -3/5GM^2/a$

The motion of a test particle inside this sphere is that of a simple harmonic oscillator $d^2r/dt^2 = -G(M(r)/r^2 = 4\pi G\rho r/3$ with angular freq $2\pi/T$

no matter the intial value of $r$, a particle will reach $r=0$ in the dynamical time $T/4$

In general the dynamical time $t_{\text{dyn}} \sim 1/\sqrt{G<\rho>}$

and its 'gravitational radius' $r_g = GM^2/W$
Summary of Dynamical Equations

• gravitational pot'l $\Phi(r)=-G\int \rho(\mathbf{r})/|\mathbf{r}-\mathbf{r}'| \, d^3r$
• Gravitational force $F(\mathbf{r})=\nabla \Phi(\mathbf{r})$
• Poissons Eq $\nabla^2 \Phi(\mathbf{r})= 4\pi G \rho$; if there are no sources
  Laplace Eq $\nabla^2 \Phi(\mathbf{r})= 0$
• Gauss's theorem: $\int \nabla \Phi(\mathbf{r}) \cdot ds^2=4\pi GM$
• Potential energy $W=1/2 \int \rho(\mathbf{r}) \nabla \Phi d^3r$

• In words Gauss's theorem says that the integral of the normal component of $\nabla \Phi$ over a closed surface equals $4\pi G$ times the mass enclosed

Potentials are Separable

• We make the fundamental assumption that the potential of a system can be decomposed into separable parts-
• This is because Poisson's equation is linear:
• differences between any two $\phi-\rho$ pairs is also a $\phi-\rho$ pair, and
  differentials of $\phi-\rho$ or are also $\phi-\rho$ pairs
• e.g. $\phi_{\text{total}}=\phi_{\text{bulge}}+\phi_{\text{disk}}+\phi_{\text{halo}}$
So Far Spherical Systems

- But spiral galaxies have a significant fraction of the mass (?; at least the baryons) in a flattened system.

Kuzmin Disk B&T sec 2.3 S&G Prob 3.4;

- This ansatz is for a flattened system and separates out the radial and z directions
- Assume $\phi_K(z, R) = \frac{GM}{\sqrt{R^2 + (a+z)^2}}$; axisymmetric (cylindrical)
  - $R$ is in the x,y plane
- Analytically, outside the plane, $\phi_K$ has the form of the potential of a point mass displaced by a distance 'a' along the z axis
  - e.q. $R(z) = \begin{cases} (0, a); & z < 0 \\ (0, -a); & z > 0 \end{cases}$
- Thus $\nabla^2 \Phi = 0$ everywhere except along $z = 0$ - Poisson's eq
- Applying Gauss's thm $\int \nabla \Phi \, d^2s = 4\pi GM$
  and get $\Sigma(R) = aM/[2\pi(R^2+a^2)^{3/2}]$

this is in infinitely thin disk... not too bad an approx

\[ \int \nabla \Phi \, d^2s = 4\pi GM = 2\pi G \Sigma \]
as $z \to 0$; $\Sigma = (1/2\pi)G \frac{d\Phi}{dR}$
Flattened +Spherical Systems-Binney and Tremaine eqs

- Add the Kuzmin to the Plummer potential (S&G 113,114)
- When \( b/a \approx 0.2 \), qualitatively similar to the light distributions of disk galaxies,

\[
\Phi_M(R, z) = -\frac{GM}{\sqrt{R^2 + (a + \sqrt{z^2 + b^2})^2}}.
\] (2.69a)

When \( a = 0 \), \( \Phi_M \) reduces to Plummer’s spherical potential (2.44a), and when \( b = 0 \), \( \Phi_M \) reduces to Kuzmin’s potential of a razor-thin disk (2.68a). Thus, depending on the choice of the two parameters \( a \) and \( b \), \( \Phi_M \) can represent the potential of anything from an infinitesimally thin disk to a spherical system. If we calculate \( \nabla^2 \Phi_M \), we find that the mass distribution with which it is associated is (Miyamoto & Nagai 1975)

\[
\rho_M(R, z) = \left( \frac{b^2 M}{4\pi} \right) \frac{aR^2 + (a + 3\sqrt{z^2 + b^2})(a + \sqrt{z^2 + b^2})^2}{[R^2 + (a + \sqrt{z^2 + b^2})^2]^{5/2}(z^2 + b^2)^{3/2}}.
\] (2.69b)

![Contours of equal density in the (R; z) plane for b/a=0.2](image)

Explaining Disks

- Remember the most important properties of disk dominated galaxies (MBW pg 495)
  - More luminous disks are on average
    - larger, redder, rotate faster, smaller gas fraction
  - flat rotation curves
  - surface brightness profiles close to exponential
  - lower metallicity in outer regions
  - traditional to model them as an infinitely thin exponential disk with a surface density distribution \( \Sigma(R) = \Sigma_0 \exp(-R/R_d) \)

  - This gives a potential (MBW pg 496) which is a bit messy

\[
\phi(R, z) = -2\pi G \Sigma_0^2 R_d \int [J_0(kR)\exp(-k|z|)]/[1+(kR_d)^2]^{3/2}dk
\]

\( J_0 \) is a Bessel function order zero
Modeling Spirals

- to fit the observed density and velocity distributions in the MW one needs a 3 component mass distribution
- Traditionally this is parameterized as the sum of
  - disk $\Sigma(R) = \Sigma_0 [\exp(-R/R_d)]$
  - spheroid (bulge) using $I(R) = I_0 R_s^2/[R+R_s]^2$ or similar forms
  - dark matter halo $\rho(r) = \rho(0)/[1+(r/a)^2]$
- See B&T sec 2.7 for more complex forms- 2 solutions in B&T- notice extreme difference in importance of halo (H) (table 2.3)