Collisionless Boltzmann Eq (Vlasov eq)  
S+G sec 3.4

- When considering the structure of galaxies cannot follow each individual star (10^{11} of them!),
- Consider instead stellar density and velocity distributions. However, a fluid model is not really appropriate since a fluid element has a single velocity, which is maintained by particle-particle collisions on a scale much smaller than the element. For stars in the galaxy this **not true**-stellar collisions are **very rare**, and even encounters where the gravitational field of an individual star is important in determining the motion of another are very infrequent
- So taking this to its limit, treat each particle as being collisionless, moving under the influence of the mean potential generated by **all** the other particles in the system \( \phi(x,t) \)
  - (e.g. in galaxies we use stars as 'test particles' since it seems that most of the potential is produced by dark matter)

Collisionless Boltzmann Eq S&G 3.4

- The distribution function is defined such that \( g(r,v,t)d^3xd^3v \) specifies the number of stars inside the volume of phase space \( d^3xd^3v \) centered on \((x,v)\) at time \(t\). At time \(t\) a full description of the state of this system is given by specifying the number of stars \( g(x,v,t)d^3xd^3v \)
  Then \( g(x,v,t) \) is called the "distribution function" (or "phase space number density") in 6 dimensions \((x\text{ and }v)\) of the system.

\( f \geq 0 \) since no negative star densities

Since potential is smooth nearby particles in phase space move together-- **fluid approx.**

For a collisionless stellar system in dynamic equilibrium, the gravitational potential \( \phi \), relates to the phase-space distribution of stellar tracers \( g(x,v,t) \), via the collisionless Boltzmann Equation

- number of particles
  \[ n(x,t) = \int g(x,v,t)d^3v \]

- average velocity
  \[ \langle v(x,t) \rangle = \int g(x,v,t)v d^3v g(x,v,t)d^3v = (1/n(x,t)) \int g(x,v,t)v d^3v \]
The collisionless Boltzmann equation is like the equation of continuity,
\[ \frac{\partial n}{\partial t} + \nabla (n v) = 0; \] but it allows for changes in velocity and relates the changes in \( f(x, v, t) \) to the forces acting on individual stars.

In one dimension CBE is
\[ \frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} - \frac{\partial \Phi(x, t)}{\partial x} = 0 \quad \text{eq 3.86} \]

For derivation of CBE see S&G pgs 142-143

Collisionless Boltzmann Eq

- This results in (S+G pg 143) eq 3.87

\[ \frac{\partial f}{\partial t} + v \cdot \nabla f - \nabla \Phi \cdot \frac{\partial f}{\partial v} = 0, \]

- The flow of stellar phase points through phase space is incompressible – the phase-space density of points around a given star is always the same.

define \( n(x, t) \) as the number density of stars at position \( x \)

then the first moment is
\[ \frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(n v) = 0; \] the same eq as continuity equation of a fluid

second moment
\[ n \frac{\partial v}{\partial t} + n v \frac{\partial v}{\partial x} = -n \frac{\partial \Phi}{\partial x} - \frac{\partial}{\partial x}(n \sigma^2) \]

\( \sigma \) is the velocity dispersion (eq. 3.90)

Analogous to eqs of fluid mechanics but unlike fluids do not have thermodynamics to help out.... nice math but not clear how useful
Analogy with Gas- continuity eq see MBW sec 4.1.4

- $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$ which is equiv to
  $\frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho = 0$

- In the absence of encounters $\mathcal{f}$ satisfies the continuity eq, flow is smooth, stars do no jump discontinuously in phase space

- Continuity equation:
  define $w=(x,v)$ pair (generalize to 3-D)
  $\frac{dw}{dt} = (v, \nabla \phi)$

  $\partial \mathcal{f} / \partial t + \nabla \cdot (\mathcal{f} \frac{dw}{dt}) = 0$

- the phase-space density $f(x, v, t)$ around any particular star remains constant along its orbit

Analogy of Stellar Systems to Gases
- Discussion due to Mark Whittle

- similarities:
  comprise many, interacting objects which act as points (separation $\gg$ size)
  can be described by distributions in space and velocity eg Maxwellian velocity distributions; uniform density; spherically concentrated etc.

Stars or atoms are neither created nor destroyed -- they both obey continuity equations- [not really true, galaxies are growing systems!]

All interactions, as well as the system as a whole obeys conservation laws (eg energy, momentum) **if isolated**

- But:
  - The relative importance of short and long range forces is **radically** different:
    - atoms interact only with their neighbors, however
    - stars interact continuously with the entire ensemble via the long range attractive force of gravity

- eg for a uniform medium: ~**equal force from all distances**
Analogy of Stellar Systems to Gases

- Discussion due to Mark Whittle

• The relative frequency of strong encounters is **radically different**:  
  
  • -- for atoms, encounters are frequent and all are strong (ie $\delta V \sim V$)
  
  • -- for stars, pairwise encounters are very rare, and the stars move in the smooth  
    global potential (e.g. S+G 3.2)

• Some parallels between gas (fluid) dynamics and stellar dynamics many of the  
  same equations can be used:
  
  • ---> concepts such as Temperature and Pressure can be applied to stellar systems
  
  • ---> we use analogs to the equations of fluid dynamics and hydrostatics
  
  • there are also some interesting differences
  
  • ---> pressures in stellar systems can be anisotropic
  
  • ---> stellar systems have negative specific heat and left alone can evolve away from  
    uniform temperature.

Collisionless Boltzmann Eq

astronomical structural and kinematic observations provide information only  
about the **projections of phase space distributions along lines of sight**,  
limiting knowledge about $\xi$ and hence also about $\phi$.

Therefore all efforts to translate existing data sets into constraints $\xi$ and $\phi$  
involves simplifying assumptions.

• dynamic equilibrium,

• symmetry assumptions

• particular functional forms for the distribution function and/or the  
  gravitational potential.
Jeans Equations MBW sec 5.4.3

- Since $\xi$ is a function of 7 variables obtaining a solution is challenging
- Take moments (e.g. integrate over all $v$)
- let $n$ be the space density of 'stars'

$$\frac{\partial n}{\partial t} + \frac{\partial (n<v_i>)}{\partial x_i} = 0; \text{ continuity eq. zeroth moment}$$

first moment (multiply by $v$ and integrate over all velocities)

$$\frac{\partial (n<v_j>/\partial t)}{\partial x_j} + \frac{\partial (n<v_i>v_i>)}{\partial x_i} + n\frac{\partial \phi}{\partial x_j} = 0$$

equivalently

$$n\frac{\partial (v_j>)}{\partial t} + n<v_i> \frac{\partial<v_j>}{\partial x_i} = -n\frac{\partial \phi}{\partial x_j} - \frac{\partial (n\sigma^2_{ij})}{\partial x_i}$$

$n$ is the integral over velocity of $\xi$: $n=\int f \, d^3v$

$<v_i>$ is the mean velocity in the $i^{th}$ direction $=1/n\int \xi v_i \, d^3v$

the term $-\frac{\partial (n\sigma^2_{ij})}{\partial x_i}$ is like a pressure, but allows for different pressures in different directions- important in elliptical galaxies and bulges, 'pressure supported' systems

Jeans Equations Another Formulation

- Jeans equations follow from the collisionless Boltzmann equation; Binney & Tremaine (1987), MBW 5.4.2. S+G sec 3.4.
cylindrical coordinates and assuming an axi-symmetric and steady-state system, the accelerations in the radial (R) and vertical (Z) directions can be expressed in terms of observable quantities:
the stellar number density distribution $\nu^*$
and 4 velocity components
a rotational velocity $v_\phi$
and 4 components of random velocities (velocity dispersion components)
$\sigma_{\phi\phi}, \sigma_{RR}, \sigma_{ZZ}, \sigma_{RZ}$

$$a_R = \sigma^2_{RR} \times \frac{\partial (\ln \nu)}{\partial R} + \sigma^2_{RT} \times \frac{\partial (\ln \nu)}{\partial T} + \sigma^2_{RZ} \times \frac{\partial (\ln \nu)}{\partial Z} + \frac{\partial \sigma^2_{Tz}}{\partial z} + \frac{\sigma^2_{RZ}}{R^2} - \frac{\sigma^2_{Z}}{R} - \frac{\sigma^2_{R}}{R^2},$$

$$a_Z = \sigma^2_{RZ} \times \frac{\partial (\ln \nu)}{\partial R} + \sigma^2_{RT} \times \frac{\partial (\ln \nu)}{\partial T} + \sigma^2_{ZZ} \times \frac{\partial (\ln \nu)}{\partial Z} + \frac{\partial \sigma^2_{Tz}}{\partial z} + \frac{\sigma^2_{Z}}{R^2} - \frac{\sigma^2_{R}}{R^2}.$$
Spherical systems- Elliptical Galaxies and Globular Clusters

• For a spherical system the Jeans equations simplify to

\[(1/n)d/dr (n<v^2_r>) + 2\beta<v^2_r>/r = -G(M/R)/r^2\]

• where G(M/R)/r^2 is the potential and n(r), <v^2_r> and \(\beta(r)\) describe the 3-dimensional density, radial velocity dispersion and orbital anisotropy of the tracer component (stars)

\[\beta(r) = 1 - <v^2_\theta>/v^2_r>: \beta = 0 \text{ is isotropic, } \beta = 1 \text{ is radial}\]

• We can then present the mass profile as

\[GM(r) = r <v^2_r> (d \ln n/dlnr + d\ln <v^2_r>/dlnr + 2\beta)\]

• while apparently simple we have 3 sets of unknowns \(<v^2_r>, \beta(r), n(r)\)
• and 2 sets of observables I(r)- surface brightness of radiation (in some wavelength band) and the lines of sight projected velocity field (first moment is velocity dispersion)
• It turns out that one has to 'forward fit'- e.g. propose a particular form for the unknowns and fit for them. This will become very important for elliptical galaxies

Motion Perpendicular to the Plane- Alternate Analysis-( S+G pgs140-145, MBW pg 163)

For the motion of stars in the vertical direction only-stars whose motions carry them out of the equatorial plane of the system.

\[d/dz[n\sigma_z^2] = -n_\phi \Phi(z,R)/dz; \text{ where } \phi(z,R) \text{is the vertical grav potential}\]

The study of such general orbits in axisymmetric galaxies can be reduced to a two-dimensional problem by exploiting the conservation of the z-component of angular momentum of any star

S&G EQ 3.91 \[d/dz[n\sigma_z^2] = d\phi(z)/dz n(z)\]

Use Poisson's eq \[\nabla^2 \Phi(r) = 4\pi G \rho\] and assume axisymmetry: \(\rho\) and \(\phi\) depend on (z,R)
gives

\[4\pi G \rho (R,z) = \nabla^2 \Phi(R,z) = \partial^2 \Phi/\partial z^2 + (1/R)\partial/\partial R(R\partial\Phi/\partial R)\]
Motion Perpendicular to the Plane- Alternate Analysis-( S+G pgs140-145, MBW pg 163)

the first derivative of the potential is the grav force perpendicular to the plane - call it \(K(z)\)

\(n_*(z)\) is the density of the tracer population and \(\sigma_z(z)\) is its velocity dispersion

then the 1-D Poisson's eq \(4\pi G \rho_{tot}(z,R)=d^2\phi(z,R)/dz^2\)
where \(\rho_{tot}\) is the total mass density - put it all together to get

\[4\pi G \rho_{tot}(z,R)=-dK(z)/dz\ (S+G\ 3.93)\]

\[d/dz[n_*(z)\sigma_z(z)^2]=n_*(z)K(z)\] - to get the data to solve this have to determine \(n_*(z)\) and \(\sigma_z(z)\) for the tracer populations(s)

Use of Jeans Eqs

- Surface mass density near sun
- Poissons eq \(\nabla^2 \phi=4\pi \rho G=-\nabla \mathbf{F}\)
- Use cylindrical coordinates
- \((1/R)\partial/\partial R(RF_R) + \partial F_z/\partial z = -4\pi \rho G\)

\(F_R=-v_c/R\) \(v_c\) circular velocity (roughly constant near sun) - (\(F_R\) force in R direction) so \(\partial F_R/\partial R =(-1/4\pi G)\partial F_z/\partial z; \text{ only vertical gradients count}\)

since the surface mass density \(\Sigma=2\int \rho dz=-F_z/2\pi G\)
(integrate 0 to \(+\infty\) thru plane)

Now use Jeans eq: \(nF_z\partial(n\sigma_z^2)/\partial z+(1/R)\partial/\partial R(Rn\sigma_z^2z_R)\); if \(R+z\) are separable 
\(e.g.\ \phi(R,z)=\phi(R)+\phi(z)\) then \(\sigma_z^2z_R \approx 0\) and voila! (eq 3.94 in S+G)

\(\Sigma=1/2\pi G n \partial(n\sigma_z^2)/\partial z; \text{ need to observe the number density distribution of some tracer of the potential above the plane and its velocity dispersion distribution perpendicular to the plane goes at } n \exp(-z/z_0)\)
Use of Jeans Eq For Galactic Dynamics

- Accelerations in the $z$ direction from the Sloan digital sky survey for
  1) all matter
  2) 'known' baryons only

ratio of the 2 (bottom panel)
use this data + Jeans eq (see below, to get the total acceleration
(in eqs $v$ is the density of tracers, $v_\phi$ is the azimuthal velocity (rotation)

$$a_R = \sigma^2_{RR} \frac{\partial (\ln \rho)}{\partial r} + \frac{\partial a_R}{\partial r} + \sigma^2_{RZ} \frac{\partial (\ln \nu)}{\partial R}$$

$$a_Z = \sigma^2_{ZZ} \frac{\partial (\ln \nu)}{\partial Z} + \frac{\partial a_Z}{\partial Z} + \sigma^2_{RZ} \frac{\partial (\ln \nu)}{\partial R}$$

Given accelerations $a_R(R,Z)$ and $a_Z(R,Z)$, i.e. the gradient of the gravitational potential, the dark matter contribution can be estimated after accounting for the contribution from visible matter.

Results

- Using dynamical data and velocity data get estimate of surface mass density in MW

$$\Sigma_{\text{total}} \sim 70 +/ - 6 M_\odot/pc^2$$
$$\Sigma_{\text{disk}} \sim 48 +/ - 9 M_\odot/pc^2$$
$$\Sigma_{\text{star}} \sim 35 M_\odot/pc^2$$
$$\Sigma_{\text{gas}} \sim 13 M_\odot/pc^2$$

we know that there is very little light in the halo so direct evidence for dark matter
What Does One Expect The Data To Look Like

- A full-up numerical simulation from cosmological conditions of a MW like galaxy-this 'predicts' what $a_z$ should be near the sun (Loebman et al 2012)
- Notice that it is not smooth or monotonic and the simulation is neither perfectly rotationally symmetric nor steady state..
- Errors are on the order of 20-30% - figure shows comparison of true radial and $z$ accelerations compared to Jeans model fits