Collisionless Boltzmann Eq (= Vlasov eq)

S+G sec 3.4 (not covering 3.4.2)

- When considering the structure of galaxies, one cannot follow each individual star (10^{11} of them!),
- Consider instead stellar density and velocity distributions. However, a fluid model is not really appropriate since a fluid element has a single velocity, which is maintained by particle-particle collisions on a scale much smaller than the element.
- For stars in the galaxy, this is **not true** - stellar collisions are very rare, and even encounters where the gravitational field of an individual star is important in determining the motion of another are very infrequent
- So taking this to its limit, treat each particle as being collisionless, moving under the influence of the mean potential generated by all the other particles in the system $\phi(\mathbf{x},t)$

Collisionless Boltzmann Eq see MBW sec 5.4.2 S&G 3.4

- The distribution function is defined such that $f(\mathbf{r},\mathbf{v},t)d^3\mathbf{x}d^3\mathbf{v}$ specifies the number of stars inside the volume of phase space $d^3\mathbf{x}d^3\mathbf{v}$ centered on ($\mathbf{x},\mathbf{v}$) at time t
- Alternatively $f(\mathbf{r},\mathbf{v},t)$ is the probability that a randomly chosen star has the specified set of coordinates- f is a scalar

At time t, a full description of the state of this system is given by specifying $f(x, v, t)d^3xd^3v$

$f(\mathbf{x}, \mathbf{v}, t)$ is called the "distribution function" (or "phase space number density") in 6 dimensions (**x** and **v**) of the system.

$f \geq 0$ since no negative star densities

Since the potential is smooth, nearby particles in phase space move together-- fluid approx.

Collisionless Boltzmann Eq see MBW sec 5.4.2 S&G 3.4

For a collisionless stellar system in dynamic equilibrium, the gravitational potential, ϕ , relates to the phase-space distribution of stellar tracers $f(\mathbf{x}, \mathbf{v}, t)$, via the collisionless Boltzmann Equation

number density of particles: $n(\mathbf{x},t)=\int f(x, v, t)d^3\mathbf{v}$

average velocity:

$<v(\mathbf{x},t)>=\int f(x, v, t)\, \mathbf{v}\, d^3\mathbf{v}/\int f(x, v, t)d^3\mathbf{v}=$
$(1/n(\mathbf{x},t))\int f(x, v, t)\, \mathbf{v} d^3\mathbf{v}$

bold variables are vectors

See S&G sec 3.4

- The collisionless Boltzmann equation (CBE) is like the equation of continuity,

$dn/dt = \partial n/\partial t + \partial(nv)\partial x = 0$ (S&G 3.83) e.g particles are conserved

but it allows for changes in velocity and relates the changes in f (x, v, t) to the forces acting on individual stars

- In one dimension, the CBE is (Derivation pg 142 of S&G)

$\partial f/\partial t + v\partial f/\partial x - [\partial \phi(x, t)/\partial x]\, \partial f/\partial v = 0$ (3.86)

- holds if stars are neither created nor destroyed, and if they change their positions and velocities smoothly.
- the CBE implies that the phase-space density around a given particle remains constant

Analogy with Gas- continuity eq see MBW sec 4.1.4

- $\partial\rho/\partial t + \nabla \bullet (\rho v) = 0$ which is equiv to
- $\partial\rho/\partial t + v \bullet \nabla\rho = 0$
- In the absence of encounters f satisfies the continuity eq, flow is smooth, stars do no jump discontinuously in phase space

- Continuity equation :

define w=(x,v) pair (generalize to 3-D)

dw/dt=(v,-$\nabla\phi$) – 6-dimensional space

- $df/dt = 0$
- $\partial f/\partial t + \nabla_6(f\ \ dw/dt)=0$

Analogy of Stellar Systems to Gases

- Discussion due to Mark Whittle

- **Similarities :**

comprise many, interacting objects which act as points (separation >> size)

can be described by distributions in space and velocity eg Maxwellian velocity distributions; uniform density; spherically concentrated etc.

Stars or atoms are neither created nor destroyed -- they both obey continuity equations-not really true, galaxies are growing systems!

All interactions as well as the system as a whole obeys conservation laws (eg energy, momentum) **if isolated**

- **But** :
- The relative importance of short and long range forces is *radically different* :
  - atoms interact only with their neighbors
  - stars interact continuously with the entire ensemble via the long range attractive force of gravity
- eg uniform medium : $F \sim G(\rho\, dV)/r^{2,}$; $dV \sim r^2 dr$; $F \sim \rho\, dr$

~ equal force from all distances

Analogy of Stellar Systems to Gases

- Discussion due to Mark Whittle

- The relative frequency of strong encounters is radically different :
  -- for atoms, encounters are frequent and all are strong (ie $\delta V \sim V$)
  -- for stars, pairwise encounters are very rare, and the stars move in the smooth global potential (e.g. S+G 3.2)

- Some parallels between gas (fluid) dynamics and stellar dynamics: many of the same equations can be used as well as :
  ---> concepts such as Temperature and Pressure can be applied to stellar systems
  ---> we use analogs to the equations of fluid dynamics and hydrostatics
- there are also some interesting differences
  ---> pressures in stellar systems can be anisotropic
  ---> self-gravitating stellar systems have negative specific heat
  $2K + U = 0 \rightarrow E = K + U = -K = -3NkT/2 \rightarrow C = dE/dT = -3Nk/2<0$
  and evolve away from uniform temperature.

Full Up Equations of Motion- Stars as an Ideal Fluid

(SS+G pgs140-144, MBW pg 163)

continuity equation (particles not created or destroyed)
$d\rho/dt+\rho\nabla.v=0$; $d\rho/dt+d(\rho v)/dr=0$

Eq's of motion (Eulers eq)
$dv/dt = -\nabla P/\rho-\nabla\Phi$

Poissons eq
$\nabla^2\Phi(r) = -4\pi G\rho(r)$ (example potential)

Collisionless Boltzmann Eq

- This results in (S+G pg 143)

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f - \nabla\Phi \cdot \frac{\partial f}{\partial \mathbf{v}} = 0,$$

- the flow of stellar phase points through phase space is incompressible – the phase-space density of points around a given star is always the same
- The distribution function f is a function of seven variables (t, **x**, **v**), so solving the collisionless Boltzmann equation in general is hard. So need either simplifying assumptions (usually symmetry), or try to get insights by taking moments of the equation.
- Take moments of an eq-- multipying f by powers of v

Collisionless Boltzmann Eq- Moments (pg 143)

n(x,t) as the number density of stars at position x

The average value of a quantity Q in the neighborhood of $\mathbf{x}$ is $Q(\mathbf{x}, t) \equiv 1/n(\mathbf{x}, t)\int Q f \, d^3\mathbf{v}$,
$n(\mathbf{x}, t) \equiv \int f \, d^3\mathbf{v} = \rho(\mathbf{x}, t)/m$

Setting Q=1 we get the zeroth moment
$\partial n/\partial t+\partial(nv)/\partial x=0$; the same eq as continuity equation of a fluid

Setting Q=**v**
first moment:
$n\partial v/dt+nv\partial v/dx=-n\partial\phi/\partial x-\partial/\partial x(n\sigma^2)$

Collisionless Boltzmann Eq- Moments (pg 143)

Similar to Euler's eq for gas
σ is the velocity dispersion $\langle v^2(x, t)\rangle = \langle v(x, t)\rangle^2 + \sigma^2$;
where v is the 'coherent, streaming, velocites and σ is the 'random' velocity

But unlike fluids, we do not have thermodynamics to help out....
In general, the Jeans equations have nine unknowns (three streaming motion components v
and six independent components of σ)

Jeans Equations MBW sec 5.4.3

- Since f is a function of 7 variables, obtaining a solution is challenging
- Take moments (e.g. integrate over all **v**)
- let n be the space density of 'stars'

$\partial n/\partial t + \partial(n\langle v_i\rangle)/\partial x_i = 0$; continuity eq. zeroth moment

first moment (multiply by v and integrate over all velocities)

$\partial(n\langle v_j\rangle/\partial t) + \partial(n\langle v_i v_j\rangle)/\partial x_i + n\partial\phi/\partial x_j = 0$

equivalently

$n\partial(\langle v_j\rangle/\partial t) + n\langle v_i\rangle \partial\langle v_j\rangle/\partial x_i = -n\partial\phi/\partial x_j - \partial(n\sigma^2_{ij})/\partial x_i$

where

n is the integral over velocity of f; $n=\int f\, d^3v$

$\langle v_i\rangle$ is the mean velocity in the i^{th} direction $= (1/n) \int f\, v_i\, d^3v$

$\sigma^2_{ij} = \langle (v_i - \langle v_i\rangle)(v_j - \langle v_j\rangle)\rangle$ "stress tensor"

$= \langle v_i v_j\rangle - \langle v_i\rangle\langle v_j\rangle$

Collisionless Boltzmann Eq

astronomical structural and kinematic observations provide information only about the **projections of phase space distributions along lines of sight**,
limiting knowledge about f and hence also about ϕ.

Therefore all efforts to translate existing data sets into constraints on CBE involve simplifying assumptions.

- dynamic equilibrium,
- symmetry assumptions
- particular functional forms for the distribution function and/or the gravitational potential.

Jeans Eq

- $n\partial(\langle v_j\rangle/\partial t) + n\langle v_i\rangle \partial\langle v_j\rangle/\partial x_i = -n\partial\phi/\partial x_j - \partial(n\sigma^2_{ij})/\partial x_i$
- So what are these terms??
- Gas analogy: Euler's eq of motion

$$\rho\, \partial\mathbf{v}/\partial t + \rho\, (\mathbf{v}\, .\, \nabla)v = -\nabla P - \rho\nabla\Phi$$

- $n\partial\phi/\partial x_j$: gravitational pressure gradient
- $n\sigma^2_{ij}$ "stress tensor" is like a pressure, but may be anisotropic, allowing for different pressures in different directions - important in elliptical galaxies and bulges 'pressure supported' systems (with a bit of coordinate transform one can make this symmetric e.g. $\sigma^2_{ij}=\sigma^2_{ji}$)

Jeans Eq Cont

- $n\partial \mathbf{v}/dt + n\mathbf{v}\partial \mathbf{v}/d\mathbf{x} = -n\partial\phi/\partial\mathbf{x} - \partial/\partial\mathbf{x}(n\sigma^2)$
- Simplifications: assume isotropy, steady state, non-rotating
  → terms on the left vanish
- Jean Eq becomes: $-n \nabla\phi = \nabla(n\sigma^2)$
- using Poisson eq: $\nabla^2\phi = 4\pi G\rho$
- Generally, solve for ρ (mass density)

Spherical systems- Elliptical Galaxies and Globular Clusters

- For a steady-state non-rotating spherical system, the Jeans equations simplifies to

$$(1/n)d/dr\,(n\langle v^2_r\rangle) + 2\beta\langle v^2_r\rangle/r = -GM(R)/r^2$$

- where $d\phi/dr = GM_{tot}(r)/r^2$ and $n(r)$, $\langle v^2_r\rangle$ and $\beta(r)$ describes the 3- dimensional density, radial velocity dispersion and **orbital anisotropy** of the tracer component (stars)

  $\beta(r) = 1 - \langle v^2_\theta\rangle/\langle v^2_r\rangle$; $\beta = 0$ is isotropic, $\beta=1$ is radial

Spherical systems- Elliptical Galaxies and Globular Clusters

- We can then describe the mass profile in terms of observables as

$GM(r)=-r <v^2_r> [(d \ln n/d \ln r) + (d \ln <v^2_r> /d \ln r)+2\beta]$

where v^2_r is the radial velocity profile, n is the density and $\beta=1-<v_\theta>^2/<v_r>^2$ has to be modeled

This can be alternatively written as

$M(r)=[V^2 r/G]+(\sigma_r^2 r/G)[-(d\ln n/d\ln r)-(d\ln \sigma_r^2/d\ln r)-(1-\sigma_\theta^{2}/\sigma_r^2)-(1-\sigma_\phi^{2}/\sigma_r^2)]$

where the subsripts r,θ and ϕ refer to spherical coordinates

This will become very important for elliptical galaxies

- while apparently simple we have 3 sets of unknowns $<v^2_r>$, $\beta(r)$, n(r)
- and 2 sets of observables I(r)- surface brightness of radiation (in some wavelength band) and the lines of sight projected velocity field (first moment is velocity dispersion)
- It turns out that one has to 'forward fit'- e.g. propose a particular form for the unknowns and fit for them.

Jeans Equations: Another Formulation

- Jeans equations follow from the collisionless Boltzmann equation; Binney & Tremaine (1987), MBW 5.4.2. S+G sec 3.4 .

cylindrical coordinates and assuming an axi-symmetric and steady-state system, the accelerations in the radial (R) and vertical (Z) directions can be expressed in terms of observable quantities:

the stellar number density distribution ν_*

And 5 velocity components

- a rotational velocity v_ϕ
- 4 components of random velocities (velocity dispersion components) $\sigma_{\phi\phi}$, σ_{RR}, σ_{ZZ}, σ_{RZ}

$$a_R = \sigma^2_{RR} \times \frac{\partial(\ln\nu)}{\partial R} + \frac{\partial\sigma^2_{RR}}{\partial R} + \sigma^2_{RZ} \times \frac{\partial(\ln\nu)}{\partial Z} + \frac{\partial\sigma^2_{RZ}}{\partial Z} + \frac{\sigma^2_{RR}}{R} - \frac{\sigma^2_{\phi\phi}}{R} - \frac{\overline{v_\phi}^2}{R},$$

$$a_Z = \sigma^2_{RZ} \times \frac{\partial(\ln\nu)}{\partial R} + \frac{\partial\sigma^2_{RZ}}{\partial R} + \sigma^2_{ZZ} \times \frac{\partial(\ln\nu)}{\partial Z} + \frac{\partial\sigma^2_{ZZ}}{\partial Z} + \frac{\sigma^2_{RZ}}{R}.$$

where a_Z, a_R are accelerations in the appropriate directions-
given these values (which are
the gradient of the gravitational potential),
the dark matter contribution can be estimated after accounting for the contribution from visible matter

Use of Jeans Eqs: Surface mass density near Sun

- Select a *tracer* population of stars and measure its density $n(z)$

at height z above the disk's midplane

Measure n(z) and v_z

Looking high above the plane, $v_z n(z) \rightarrow 0$;

- Using the moment equation
- $\partial v/\partial t + v\, \partial v/\partial z = -\partial\phi/\partial x - 1/n \partial/\partial x[n\sigma^2(z, t)]$ *the terms in blue vanish* and assuming things do not change with time

$n(z)\partial\phi/\partial z - 1/\partial/\partial z[n\sigma^2(z]$

- Using Poisson's eq $4\pi G\rho(R, z) = \nabla^2\phi(R, z)$ and going to cynlidrical coordinates

$4\pi G\rho(R, z) = \partial\phi^{2/}\partial z^2 + 1/R\ (\partial\phi/\partial R\ [R\partial\phi/\partial R])$ *eq 3.92*

Use of Jeans Eqs: Surface mass density near Sun Sec 3.4.1 in S&G

Now use Jeans eq: $nF_z = -\partial(n\sigma^2_z)/\partial z + (1/R)\partial/\partial R(Rn\sigma^2_{zR})$;
if R+z are separable,
e.g $\phi(R,z) = \phi(R) + \phi(z)$ then $\sigma^2_{zR} \sim 0$ and voila! (eq 3.94 in S+G)

$\Sigma(z) = -(1/2\pi Gn)\,\partial(n\sigma^2_z)/\partial z$;

$\Sigma(< z)$ is the **surface** mass density $\Sigma(< z)$ need to observe the number density distribution of some tracer of the potential above the plane [goes as $\exp(-z/z_0)$] and its velocity dispersion distribution perpendicular to the plane to get the mass density.

Motion Perpendicular to the Plane- Alternate Analysis-

(S+G pgs140-144, MBW pg 163)

then the 1-D Poisson's eq $4\pi G\rho_{tot}(z,R) = d^2\phi(z,R)/dz^2$
where ρ_{tot} is the total mass density - put it all together to get

$4\pi G\rho_{tot}(z,R) = -dK(z)/dz$ (S+G 3.93)

$d/dz[n_*(z)\sigma_z(z)^2] = n_*(z)K(z)$

to get the data to solve this, we have to determine $n_*(z)$ and $\sigma_z(z)$ for the tracer populations(s)

Use of Jeans Eq For Galactic Dynamics

- Accelerations in the z direction from the Sloan digital sky survey for

1) all matter (top panel)
2) 'known' baryons only (middle panel)
3) ratio of the 2 (bottom panel)

Based on full-up numerical simulation from cosmological conditions of a MW like galaxy-this 'predicts' what a_Z should be near the Sun (Loebman et al 2012)

Compare with results from Jeans eq (ν is density of tracers, v_ϕ is the azimuthal velocity (rotation))

$$a_R = \sigma^2_{RR} \times \frac{\partial(\ln\nu)}{\partial R} + \frac{\partial\sigma^2_{RR}}{\partial R} + \sigma^2_{RZ} \times \frac{\partial(\ln\nu)}{\partial Z} + \frac{\partial\sigma^2_{RZ}}{\partial Z} + \frac{\sigma^2_{RR}}{R} - \frac{\sigma^2_{\phi\phi}}{R} - \frac{\overline{v_\phi}^2}{R}, \quad (1)$$

$$a_Z = \sigma^2_{RZ} \times \frac{\partial(\ln\nu)}{\partial R} + \frac{\partial\sigma^2_{RZ}}{\partial R} + \sigma^2_{ZZ} \times \frac{\partial(\ln\nu)}{\partial Z} + \frac{\partial\sigma^2_{ZZ}}{\partial Z} + \frac{\sigma^2_{RZ}}{R}. \quad (2)$$

Given accelerations $a_R(R,Z)$ and $a_Z(R,Z)$, *i.e.* the gradient of the gravitational potential, the dark matter contribution can be estimated after accounting for the contribution from visible matter.

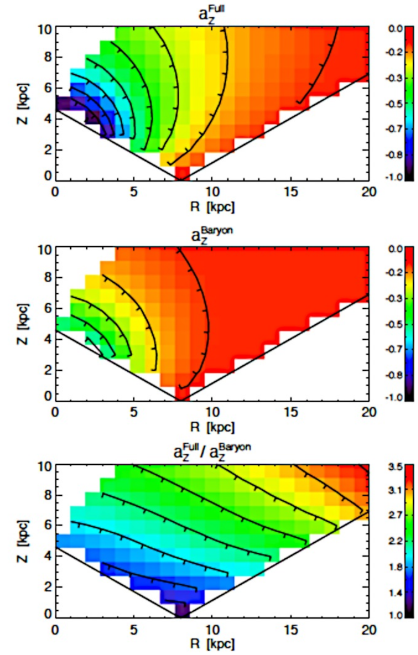
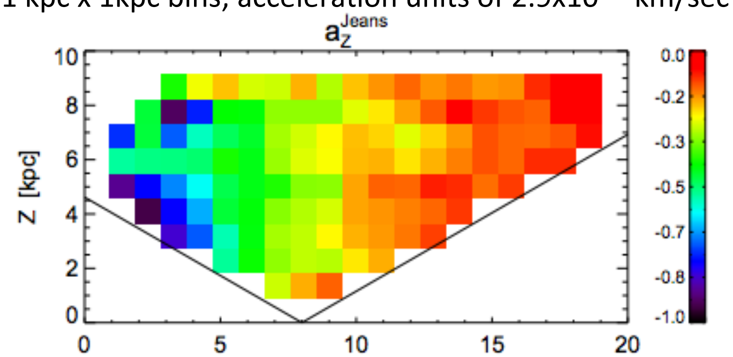


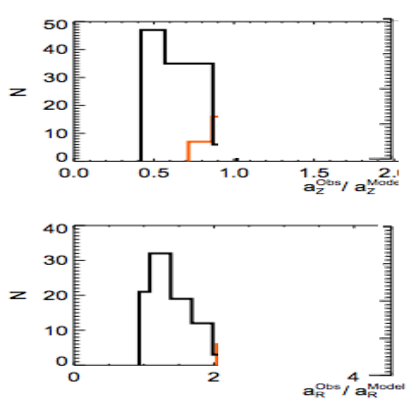
Figure 1. A comparison of the acceleration in the Z dir when all contributions are included (star, gas, and dark particles; top panel) to the result without dark matter (panel). The acceleration is expressed in units of 2.9×10^{-13} The ratio of the two maps is shown in the bottom panel. importance of the dark matter increases with the distance fr origin; at the edge of the volume probed by SDSS ($R \sim 2$

What Does One Expect The Data To Look Like

- Now using Jeans eq
- **Notice that it is not smooth or monotonic and that the simulation is neither perfectly rotationally symmetric nor steady state..**
- errors are on the order of 20-30%- figure shows comparison of true radial and z accelerations compared to Jeans model fits

1 kpc x 1kpc bins; acceleration units of 2.9×10^{-13} km/sec^2





Jeans (Continued)

- Using dynamical data and velocity data, get estimate of surface mass density in MW

Σ_{total}~70 +/- 6$M_\odot$/pc^2

Σ_{disk}~48+/-9 $M_\odot$/pc^2

Σ_{star}~35$M_\odot$/pc^2

Σ_{gas}~13$M_\odot$/pc^2

we know that there is very little light in the halo so direct evidence for dark matter

End

- Now onto the Local group !!