Exponential Disks

- Motivated by the exponential surface brightness profiles of disks examine a potential that is generated by such a distribution (B+T 2.162)

$\Sigma(R)=\Sigma_0\exp(-R/R_D)$ which gives a mass distribution

$M(R)=2\pi\int\Sigma(R)RdR=2\pi\Sigma_0R^2{}_D[1-\exp(-R/R_D)(1+R/R_D)]$;

as shown in detail in eqs B&T 2.153-2.157 one gets a potential in the form of Bessel functions

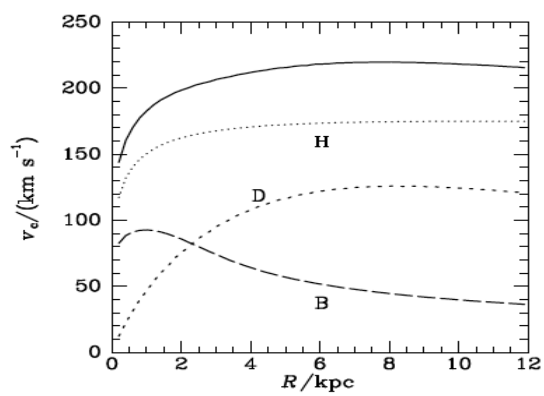
This comes from the use of Hankel functions (analogs of Fourier transforms but for cylindrically symmetric systems)

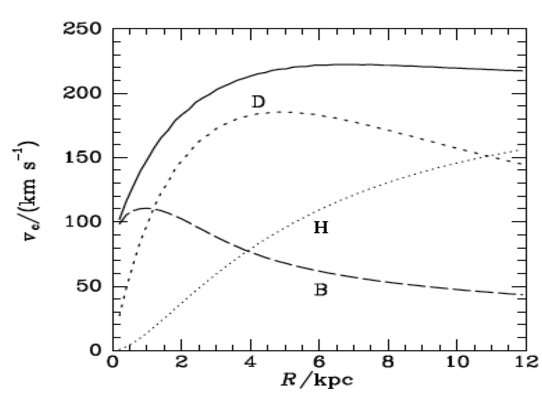
$S(k)=-2\pi G\int J_0(kR)\Sigma(R)RdR$; J_0 is a Bessel function order zero

$\phi(R, z)=-2\pi G\Sigma_0\,{}^2R_D\int[J_0(kR)\exp(-k|z|)]/[1+(kR_D)^2]^{3/2}dk$

Modeling Spirals

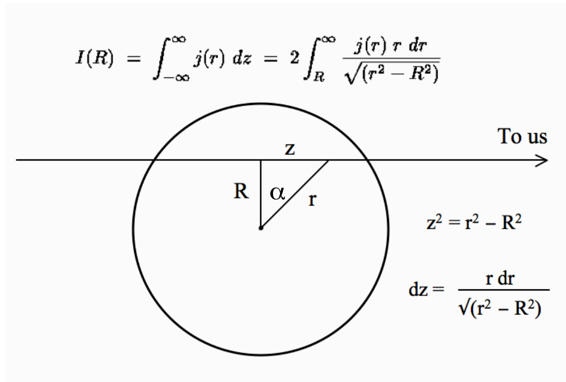
- As indicated earlier to fit the observed density and velocity distributions in the MW one needs a 3 component mass distribution
- Traditionally this is parameterized as the sum of
  - disk $\Sigma(R)=\Sigma_0[\exp-R/a]$
  - spheroid (bulge) using $I(R)=I_0R_s{}^2/[R+Rs]^2$ or similar forms
  - dark matter halo $\rho(r)=\rho(0)/[1+(r/a)^2]$
- See B&T sec 2.7 for more complex forms- 2 solutions in B&T- notice extreme difference in importance of halo (H) (table 2.3)





Projection Effects
from M. Whittle
http://www.astro.virginia.edu/class/whittle/astr553/Topic07/t7_projection.ht

- Observed luminosity density I(R)=integral over true density distribution j(r) (in some wavelength band)
- Same sort of projection for velocity field but weighted by the density distribution of tracers
- $I(R)\sigma(r)^2 = 2\int[(v_r \cos\alpha - v_\theta \sin\alpha)^2 nr]/\mathrm{sqrt}(r^2-R^2)$
- Density distribution solution is an Abel integral (see appendix B.2 in B&T) with solution of the form
- while the velocity solution is also an Abel integral
- There are a few useful I(R) & j(r) pairs that can both be expressed algebraically



$$\rightarrow j(r) = \frac{-1}{\pi}\int_r^{\infty} \frac{dI}{dR}\frac{dR}{\sqrt{(R^2-r^2)}}$$

Orbits in a **static** spherical potential B& T sec 3.1

The radial acceleration in a centrally directed gravitational field is

- $d^2\mathbf{r}/dt^2 = -\nabla\Phi(r) = g(r)\mathbf{e}_r$ ($\mathbf{e}_r$ is the unit vector in radial direction)
- $d/dt(\mathbf{r} \times d\mathbf{r}/dt) = (d\mathbf{r}/dt \times d\mathbf{r}/dt) + \mathbf{r} \times d^2\mathbf{r}/dt^2 = g(r) r \times \mathbf{e}_r = 0$;

  → conservation of angular momentum $\mathbf{L} = \mathbf{r} \times d\mathbf{r}/dt$ (eqs. 3.1-3.5)

- Since this vector is constant, we conclude that the star moves in a plane, the orbital plane.
- This simplifies the determination of the star's orbit, since the star moves in a plane, we may use plane polar coordinates
  - stars move in a plane (orbital plane) $\theta=0$
  - for which the center is at $r = 0$ and φ is the azimuthal angle in the orbital plane
  - Use plane polar coordinates (R,φ,z) (Appendix B)
  - eqs of orbits

R coordinate: $d^2R/dt^2 - R(d\varphi/dt)^2 = g(R)$

φ coordinate :$2(dR/dt)(d\varphi/dt) + R(d^2\varphi/dt^2) = 0$ [since $L = R^2 (d\varphi/dt) = $ constant]

R equation of motion is $(d^2R/dt^2) - (L^2/R^3) = g(R)$

MBW 5.4 Collisionless Dynamics

- The discussion in this section is rather different than what I chose to follow and is rather mathematical.

Stellar Dynamics B&T ch 3, S&G 3

- Orbits in a static spherical potential:

  angular momentum is conserved

  – $d^2r/dt^2=g(r)e_r$ e_r is the unit vector in radial direction; the radial acceleration $g=d^2r/dt^2$

  – $d/dt(r \times dr/dt)=(dr/dt \times dr/dt)+r \times d^2r/dt^2=g(r) r \times e_r =0$; conservation of angular momentum $L=r \times dr/dt$ (eqs. 3.1-3.5)around the z axis

Conservation of energy:

total energy =PE+KE or in above formalism

  – stars move in a plane (orbital plane) $\theta=0$

  – Use plane polar coordinates (R,φ,z) (Appendix B)

  – eqs of orbits

  R coordinate: $d^2R/dt^2-R(d\varphi/dt)^2 = g(R)$

  ψ coordinate :$2(dR/dt)(d\varphi/dt)+R(d^2\varphi/dt^2)=0$ (since $L = R^2 (d\varphi/dt) =$ constant)

  equation of motion is $(d^2R/dt^2)-(L^2/R^3) = g(R)$

Stellar Dynamics B&T ch 3

total energy =PE+KE or in above formalism

- if substitute $u = 1/R$, the energy equation takes the form :
- $2E/L^2 = 2\phi/L^2 + u^2 + (du/d\varphi)^2$ (eqn 3.12)
  - Bound orbits are those in which the radius R is always finite. Thus, for bound orbits $u = 1/R$ is finite while for unbound orbits u tends to zero.
- In a bound orbit the condition $du/d\varphi = 0$ occurs when
  - $u^2 + 2\,[\phi(1/u) - E\,] / L^2 = 0$.
- This equation normally has 2 roots, u_1 and u_2 . And thus a " bound " star orbiting in a conservative potential will thus move in an orbit twixt 2 radii $R_1 = 1/u_1$ and $R_2 = 1/u_2$; the pericenter and apocenter

B&T pgs 143-147

- Since $L=R^2 d\varphi/dt \rightarrow d/dt = (L/R^2)\, d/d\varphi$
- $(L^2/R^2) d/d\varphi(1/R^2 dR/d\varphi) - L^2/R^3 = g(R)$;
- using u=1/R, the equation of motion is

$$\boxed{d^2u/d\varphi^2 + u = -g(1/u)/L^2u^2}$$ eq. 3.11 in B&T

now putting in a spherical potential

$g = -GM(<R)/R^2$ and substituting, we get

$$d^2u/d\varphi^2 + u = GM(<R)/L^2$$

two general solutions, bound (u>0) and unbound (u=0)

bound orbits have $du/d\varphi = 0$ and are confined between pericenter and apocenter

- Aside: For a halo with outer radius r_h, a flat rotation curve, and circular velocity V_c the escape velocity at R is

$V_{esc}(R)^2 = 2V_c^2 \ln(1+r_h/R)$ (Binney & Tremaine)

Some Simple Cases

- Point source potential (pg 147 B&T; eq 3.23 and following)

$g(R)=-GM/R^2 \rightarrow g(u)=-GMu^2$

using

$\mathbf{d^2u/d\varphi^2+u=}$-g $\mathbf{(1/u)/L^2u^2}$

$\mathbf{d^2u/d\varphi^2+u= -GM/L^2}$

general solution: $\mathbf{u(\varphi)=C\cos(\varphi-\varphi_0)+GM/L^2}$

C and φ_0 are constants

Nature of solutions: [reminder v_{circ}=sqrt(GM/r)]

C=0 circular orbits (B&T define the eccentricity as CL^2/GM so if C=0, eccentricity=0)

if C> GM/L^2 unbound orbit; e.g. u = 0 is possible → r goes to infinity

C<GM/L^2 bound orbit; we know this solution(!); ellipse with pt source at one focus and complete a radial period in $\Delta\varphi=2\pi$

Constant Density Sphere

From before…

Potential energy (B&T) eq 2.41, 2.32, 2.43

R>a $\phi(R)=-4\pi Ga^3\rho_0 / 3R = -GM/R$

R<a $\phi(R)=-2\pi G\rho_0(a^2- 1/3\ R^2))$;

$F_R = -d\phi(R)/dR = -4\pi GR\rho_0/3$

Use Cartesian coordinates $x=R\cos\varphi$, $y=R\sin\varphi$

$F_x=-4\pi GR\rho_0\cos(\varphi)e_x/3=-4\pi Gx\rho_0 e_x/3$

$F_y=-4\pi Gy\rho_0 e_y/3$

need to transform $d^2\mathbf{R}/dt^2=e_x d^2\mathbf{x}/dt^2+e_y d^2\mathbf{y}/dt^2$

define $\Omega^2=(4\pi/3)G\rho_0$;$d^2\mathbf{x}/dt^2=-\Omega^2 x$; $d^2\mathbf{y}/dt^2=-\Omega^2 y$

this is the harmonic oscillator general solution

$x=A\cos(\Omega t+k_x)$; $y=B\cos(\Omega t+k_y)$;

A,B are amplitudes and k's the initial phases

going backwards to polar coordinates

$R=\mathrm{sqrt}[A^2\cos^2(\Omega t+k_x)+ B^2\cos^2(\Omega t+k_y)]$

$\varphi=\tan^{-1}[B\cos(\Omega t+k_y)/A\cos(\Omega t+k_x);]$

R and φ define a closed ellipse on the center of the sphere; A and B are the major and semi-major axes.

Complete radial period in $\Delta\varphi=\pi$

Most mass distributions will lie between a pt mass and a uniform sphere so radial and azimuthal periods are not the same ; **rosette pattern for orbits**

'Real' Orbits

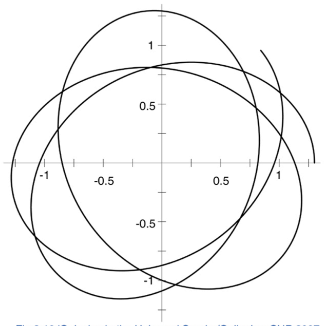
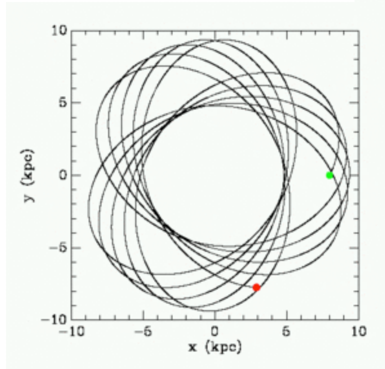
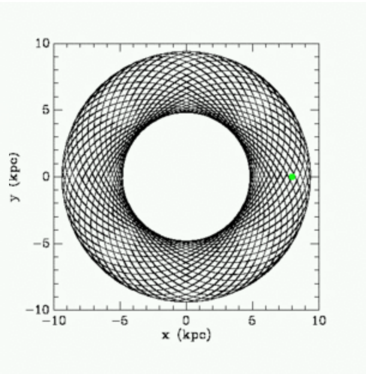


Fig 3.10 'Galaxies in the Universe' Sparke/Gallagher CUP 2007





- A few orbits, ~2 Gyr of orbits- 20 Gyrs from C. Flynn

Stellar Dynamics Summary B&T ch 3; S&G 3.3

Orbits of disk stars

only the component of angular momentum parallel to symmetry axis is constant.

- Since **L** is conserved, stars move in a plane - can use polar coordinates (R,φ)

(do not need z, appendix B B&T B.24)

- R eq of motion $dR^2/dt^2 - R(d\varphi/dt)^2 = g(R)$
- φ eq of motion $(2dR/dt * d\varphi/dt) + R d\varphi^2/dt^2 = 0$; $\mathbf{L} = R^2 d\varphi/dt$ is a constant
- total equation of motion $d^2R/dt^2 - \mathbf{L}/R^3 = g(R)$

- Stars whose motions are confined to the equatorial plane of an axisymmetric galaxy 'feel' only an effectively spherically symmetric potential
- Therefore their orbits will be identical with those discussed previously
- the radial coordinate R of a star on such an orbit oscillates between the peri and apo-galacticon as the star revolves around the center, and the orbit forms a rosette figure.

Orbits in Axisymmetric Potentials- B&T 3.2, S&G 3.3

cylindrical coordinate system (R; φ; z) with origin at the galactic center, the z axis is the galaxy's symmetry axis.

- Stars in a axisymmetric galaxy 'see' a potential which is spherically symmetric. orbits will be identical to those in such a potential
- The situation is much more complex for stars whose motions carry them out of the equatorial plane of the system.
- orbits in axisymmetric galaxies can be reduced to a two-dimensional problem by exploiting the conservation of the z-component of angular momentum
- **S&G give nice physical description** (see eq 3.29 and S&G pgs 133-136)

- $d^2\mathbf{r}/dt^2 = -\nabla\Phi\,(R,z)$; which can be written in cylindrical coordinates as
- $d^2R/dt^2 - R(d\varphi/dt)^2 = -\partial\Phi/\partial R$
- Motion in the φ direction : $d/dt\,(R^2\,d\varphi/dt)=0$; $L_z = R^2(d\varphi/dt)$ = constant
- z direction : $d^2z/dt^2 = -\partial\Phi/\partial z$

Orbits in Axisymmetric Potentials- B&T 3.2

- Eliminating $d\varphi/dt$ and putting in angular momentum
- $d^2R/dt^2 - L^2_z/R^3 = -\partial\Phi/\partial R$ - if we define an effective potential $\Phi_{eff} = \Phi(R,z) + L^2_z/2R^2$
- $d^2R/dt^2 = -\partial\Phi_{eff}/\partial R$ (see S&G 3.36-3.68)

- Unless it has enough energy to escape from the Galaxy, each star must remain within some apogalactic outer limit.

Orbits in Axisymmetric Potentials- B&T 3.2

The three-dimensional motion of a star in an axisymmetric potential (R; z) can be reduced to the two dimensional motion of the star in the (R; z) plane (the meridional plane)

- Since the change in ang mom in the z direction is zero (planar orbits)

$\partial/\partial z(L^2_z/2R^2) = 0$; $d^2z/dt^2 = -\partial\Phi_{eff}/\partial z$; and

- $\Phi_{eff} \equiv \Phi(R, z) + L_z^2/2R^2$ (S&G 3.65)

The effective potential is the sum of gravitational potential and KE in the φ direction

rises very steeply near the z axis

Orbits in Axisymmetric Potentials- B&T 3.2

The minimum in Φ_{eff} has a "simple" physical meaning (see next page)

$d^2R/dt^2 = 0 = \partial\Phi_{eff}/\partial R = \partial\Phi/\partial R - L^2_z/R^3$; which is satisfied at a particular radius - <u>the guiding center radius R_G</u> where

$(\partial\Phi/\partial R)|_{R_G} = L^2_z/R^3 = R_G\,(d\varphi/dt)^2$

and $0 = \partial\Phi_{eff}/\partial z$ which is satisfied in the equatorial plane

these are the conditions for a circular orbit with angular speed $d\varphi/dt$

Orbits in Axisymmetric Potentials- B&T 3.2

the minimum of Φ_{eff} occurs at the radius at which a circular orbit has angular momentum L_z, and the value of Φ_{eff} at the minimum is the energy of this circular orbit

- Unless Φ has a special form these eq's cannot be solved analytically

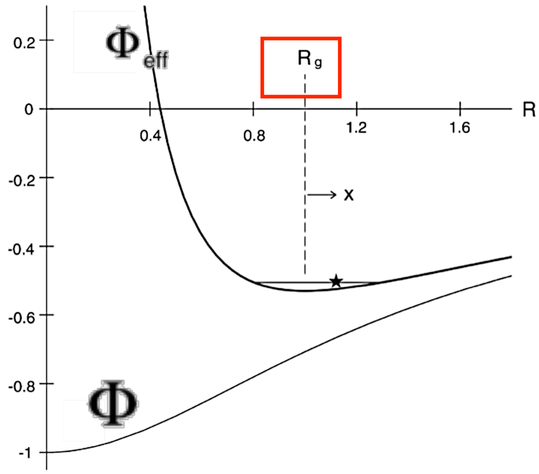


Fig 3.8 'Galaxies in the Universe' Sparke/Gallagher CUP 2007

The vertical dashed line marks the guiding center R_g; **the star oscillates about R_g between inner and outer limiting radii.**

Φ_{eff} for the Plummer potential S&G fig 3.8

Orbits in Axisymmetric Potentials- B&T 3.2.3- S&G pgs 136-139

If assume in disk galaxies that the orbits are *nearly* circular
What approx can we make to the orbits??

let $x = R - R_g$; where $R_g(L_z)$ is the guiding-center radius for an orbit of angular momentum L_z (eq. 3.72).

Expand Φ_{eff} around x (see B&T eq 3.76) ; the epicyclic approx ignores all terms of xz^2 or higher

Then define 2 new quantities:
$\kappa^2(R_G) = (\partial^2\Phi_{eff}/\partial R^2)$; $\nu^2(R_G) = (\partial^2\Phi_{eff}/\partial z^2)$; then keeping the lower orders $d^2x/dt^2 = -\kappa^2 x$; $d^2z/dt^2 = -\nu^2 z$; these are the harmonic oscillator eq's around x and z with frequencies κ and ν.
κ is the epicycle freq and ν the vertical frequency
this gives a vertical period $T = 2\pi/\nu \sim 6\times10^7$ yrs for the MW

EpiCycles B&T,S&G 3.3

- Remember the Oort constants??
- Well in the same limit (remember $v_{circ} = R\Omega(R)$)

  Ω=A-B; κ^2 =-4B(A-B)=-4BΩ (eq 3.84); using the measured values of these constants one finds that near the sun $\kappa_0{}^2$= 37km/sec/kpc and the ratio of the freq of the sun's orbit around the GC and the radial freq κ_0/Ω_0=2 sqrt(−B/(A−B))= 1.35
- Stellar orbits do not close on themselves in an inertial frame, but form a rosette figure like those discussed above for stars in spherically symmetric potentials

- The ratio ν^2/κ^2 ~3/2 $\rho/\langle\rho\rangle$ a measure of how concentrated the mass is near the plane ($\langle\rho\rangle$ = mean density within the sphere of radius R)
- The value of this approximation is in its ability to describe the motions of stars in the disk plane (does not work well for motion perpendicular to the plane): $\nu/\kappa \sim 2$ for the Sun.

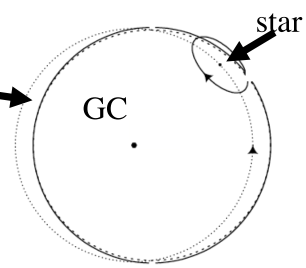
- The angular momentum on a circular orbit is $R^2\Omega(R)$;

  if it increases outward at radius R, the circular orbit is stable. This condition always holds for circular orbits in galaxy-like potentials.

Motion in Both Coordinates B&T 3.91-3.94

- $d^2x/dt^2=-\kappa^2x$; $d^2z/dz^2=-\nu^2z$; these are the harmonic oscillator eq's around x and z with frequencies κ and ν.
- and the general solution is
  - $x(t)=C\cos(\kappa t+A)$; C>0 and A are arbitrary constants
- the solution for the φ direction is a bit messier and is
  - $\varphi=(L_z/R^2_g)t-(2\Omega/\kappa)(C/R_g)\sin(\kappa t+A)+\varphi_0$
- B&T go back to Cartesian coordinates (argh!) and define
- $y= R_g(\varphi - \varphi_0) = -(2\Omega/\kappa)C\sin(\kappa t+A)=Y\sin(\kappa t+A)$
  - In the (x; y) plane the star moves on an ellipse called the epicycle around the guiding center
  - see fig 3.10 in S&G

circular motion of guiding center



$(2\Omega/\kappa) = 1 - 2$ in real galaxies

Epicycles

- Why did we go thru all that??
- **Want to understand how to use stellar motions determine where the mass is.**
- the orbits of stars take them through different regions of the galaxies -their motions at the time we observe them have been affected by the gravitational fields through which they have travelled earlier.
- use the equations for motion under gravity to infer from observed motions how mass is distributed in those parts of galaxies that we cannot see directly.
- The motions we have considered so far are the simplest !
- Using epicycles, we can explain the observed motions of disk stars near the Sun.

$(2\Omega/\kappa) = 1 - 2$ in real galaxies

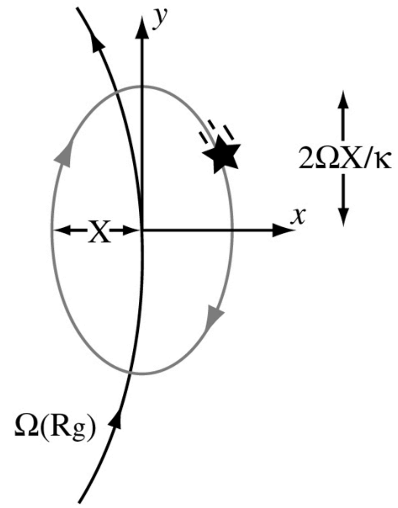


Fig 3.9 'Galaxies in the Universe' Sparke/Gallagher CUP 2007

Virial Theorem S&G pg 120, MBW pg 234

- S+G pg 120-121, MBW 5.4.4, B&T pg 360
- **$\frac{1}{2}d^2I/dt^2 = 2KE+W$** (no ext' forces) I = moment of inertia = $\Sigma m_i r_i^2$ (sum over i=1,N particles)
- A rather different derivation (H Rix)
- Consider (for simplicity) the 1-D Jeans eq in steady state (more later)
- $\partial/\partial x[\rho v^2]+\rho\partial\phi/\partial x=0$
- Integrate over velocities and then over positions...
- $-2E_{kin}=E_{pot}$ (static)
- or restating in terms of forces
- if T= total KE of system of N particles < >= time average

call the 'virial 'Q

$$Q= \frac{1}{2}\frac{dI}{dt} = m\sum r.\frac{dr}{dt} = \sum p.r$$

$$dQ/dt= \sum F.r + 2T$$

$2<T>=-\Sigma(F_k \bullet r_k)$; summation over all particles k=1,N

Virial Theorem - Simple Cases

- Circular orbit:

$mV^2/r=GmM/r^2$

Multiply both sides by r, $mV^2=GmM/r$

$mV^2=2KE$; $GmM/r=-W$ so $2KE+W=0$

- Time averaged Keplerian orbit

define U=KE/|W|; as shown in figure i clearly changes over the orbit; but take averages:

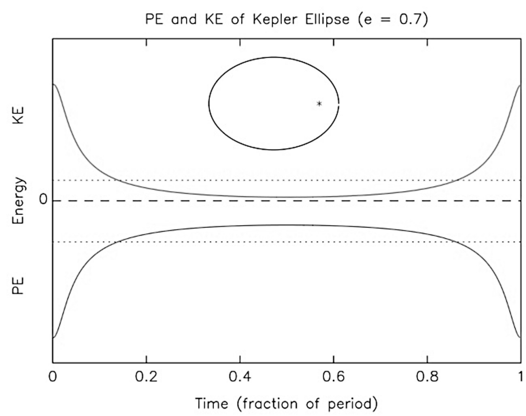
$-W=<GMm/r>=GMm<1/r>$

$=GMm(1/a)$

$KE=<1/2mV^2>=GMm<1/r-1/2a>$

$=1/2GMm(1/a)$

So again $2KE+W=0$



Red: kinetic energy (positive) starting at perigee

Blue: potential energy (negative)

Virial Theorem

- Another derivation following Bothun http://ned.ipac.caltech.edu/level5/Bothun2/Bothun4_1_1.html
- Moment of inertia, I, of a system of N particles
- $I=\Sigma m_i r_i^2$ sum over i=1,N (express r_i^2 as $(x_i^2+y_i^2+z_i^2)$)
- take the first and second time derivatives ; let d^2x/dt^2 be symbolized by x,y,z
- $½\, d^2I/dt^2 =\Sigma m_i (dx_i/dt)^2+(dy_i/dt)^2+(dz_i/dt)^2+\Sigma m_i(x_i x+y_i y+z_i z)$

mv^2 (2 KE)+Potential energy (W) [~r •(m**a**)]

after a few dynamical times, if unperturbed a system will come into Virial equilibrium–time averaged inertia will not change so 2<T>+W=0

For self gravitating systems $W=-GM^2/2R_H$; R_H is the harmonic radius- the sum of the distribution of particles appropriately weighted [$1/R_H =1/N\ \Sigma_i\ 1/r_i$]

The virial mass estimator is $M=2\sigma^2R_H/G$; for many mass distributions R_H~1.25 R_{eff} where R_{eff} is the half light radius, σ is the 3-d velocity dispersion

Virial Thm MBW sec 5.4.4

- If I is the moment of inertia
- $\frac{1}{2} d^2 I/dt^2 = 2KE + W + \Sigma$
  - where Σ is the work done by external pressure
  - KE is the kinetic energy of the system
  - W is the potential energy (only if the mass outside some surface S can be ignored)
- For a static system ($d^2I/dt^2 = 0$)

  $2KE + W + \Sigma = 0$

Time Scales for Collisions (MBW sec 5.4.1)

- N particles of radius r_p; Cross section for a direction collision $\sigma_d = \pi r^2_p$
- Definition of mean free path:

  $\lambda = 1/n\sigma_d$

  where n is the number density of particles (particles per unit volume), $n = N/(4\pi\ell^3/3)$

  The characteristic time between collisions (Dim analysis) is $t_{collision} = \lambda/v \sim [\,(\ell/r_p)^2\, t_{cross}/N]$ where v is the velocity of the particle.
  for a body of size ℓ, $t_{cross} = \ell/v$ = crossing time

Time Scales for Collisions (MBW sec 5.4.1)

So lets consider a galaxy with ℓ~10kpc, N=10^{10} stars and v~200km/sec

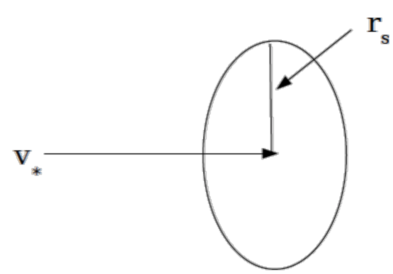
- if $r_p = R_{sun}$, $t_{collision}$~10^{21} yrs Therefore direct collisions among stars are completely negligible in galaxies.

- For indirect collisions the argument is more complex (see S+G sec 3.2.2, MWB pg 231-its a long derivation-see next few pages) but the answer is the same - it takes a very long time for star interactions to exchange energy (relaxation).

- t_{relax}~$Nt_{cross}/10\ln N$
- It's only in the centers of the densest globular clusters and galactic nuclei that this is important

How Often Do Stars Encounter Each Other

For a 'strong' encounter, $GmM/r > 1/2mv^2$
e.g. potential energy exceeds KE
So a critical radius is $r<r_s=2GM/v^2$

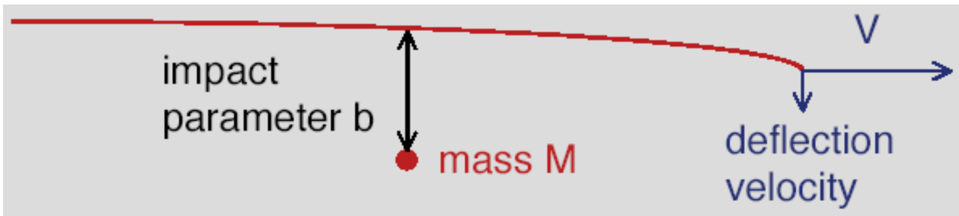


Putting in some typical numbers m~$1/2M_\odot$
v=30km/sec r_s=1AU
So how often do stars get that close?

consider a cylinder Vol=$\pi r^2_s vt$; if have n stars per unit volume than on average the encounter occurs when
$n\pi r^2_s vt=1$, $t_s=v^3/ 4\pi nG^2M^2$
Putting in typical numbers =$4\times10^{12}(v/10km/sec)^3(M/M_\odot)^{-2}(n/pc^3)^{-1}$ yr- a very long time (universe is only 10^{10}yrs old) - galaxies are essentially collisionless

What About Collective Effects ? sec 3.2.2

For a weak encounter $b >> r_s$
Need to sum over individual interactions- effects are also small

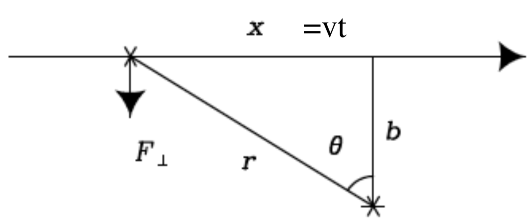


Relaxation Times

- Star passes by a system of N stars of mass m
- assume that the perturber is stationary during the encounter and that $\delta v/v<<1$- large $b >Gm/v^2$ (B&T pg 33-sec 1.2.1. sec 3.1 for exact calculation)
- Newton's Laws $m(dv/dt)=F$
  - $(b^2+x^2) =r^2$
- $F=Gm^2\cos\theta/(b^2+x^2)=Gbm^2/(b^2+x^2)^{3/2}=(Gm^2/b^2)(1+(vt/b)^2)^{-3/2}$ if v is constant
- Now integrate over time $\delta v =\int(F/m)dt=$

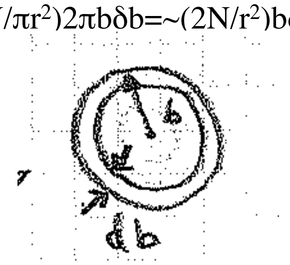
$(Gm/bv)\int(1+(vt/b)^2)^{-3/2}\, dt$ (change variables $s=(vt/b)$) ; $\delta v =2Gm/bv$

- **In words, δv is roughly equal to the acceleration at closest approach, Gm/b^2, times the duration of this acceleration $2b/v$.**



The surface density of stars is $\sim N/\pi r^2$
N is the number of stars

let δn be the number of interactions a star encounters with impact parameter between b and δb crossing the galaxy once $\sim(N/\pi r^2)2\pi b\delta b=\sim(2N/r^2)b\delta b$



Relaxation...continued (MBW pg

- The net vectoral velocity due to these encounters is zero, but the mean square change is not
  $\delta v^2 = (2Gm/bv)^2 (2N/r^2) b\delta b$ (see B&T pg 34 eq. 1.3.2) - now integrating this over all impact parameters from b_{min} to b_{max}
- one gets $\delta v^2 \sim 8N(Gm/rv)^2 \ln \Lambda$; where r is the galaxy radius
  $\ln \Lambda$ is the Coulomb logarithm = $\ln(b_{max}/b_{min})$
- For gravitationally bound systems the typical speed of a star is roughly $v^2 \sim GNm/r$ (from KE=PE) and thus $\delta v^2/v^2 \sim 8 \ln \Lambda/N$
- For each 'crossing' of a galaxy one gets the same δv so the number of crossing for a star to change its velocity by order of its own velocity is $n_{relax} \sim N/8 \ln \Lambda$
- So how long is this?? well $t_{cross} \sim r/v$; $v^2 \sim GNm/r$, $b_{max} \sim r$, $b_{min} \sim Gm/v^2$, so $\Lambda \sim rv^2/(Gm) \sim N$ and thus
- $\boxed{t_{relax} \sim (0.1N/\ln N) t_{cross}}$; if we use $N \sim 10^{11}$; t_{relax} is much much longer than t_{cross}
- In all of these systems the dynamics over timescales $t < t_{relax}$ is that of a collisionless system in which the constituent particles move under the influence of the gravitational field generated by a smooth mass distribution, rather than a collection of mass points

Relaxation

- Values for some representative systems

	<m>	N	r(pc)	t_{relax}(yr)	age(yrs)
Pleiades	1	120	4	1.7×10^7	$<10^7$
Hyades	1	100	5	2.2×10^7	4×10^8
Glob cluster	0.6	10^6	5	2.9×10^9	10^9-10^{10}
E galaxy	0.6	10^{11}	3×10^4	4×10^{17}	10^{10}
Cluster of gals	10^{11}	10^3	10^7	10^9	10^9-10^{10}

Scaling laws $t_{relax} \sim t_{cross} \sim R/v \sim R^{3/2} / (Nm)^{1/2} \sim \varrho^{-1/2}$

- However numerical experiments (Michele Trenti and Roeland van der Marel 2013 astro-ph 1302.2152) show that even globular clusters never reach energy **equipartition** (!) to quote from this paper 'Gravitational encounters within stellar systems in virial equilibrium, such as globular clusters, drive evolution over the two-body relaxation timescale. The evolution is toward a thermal velocity distribution, in which stars of different mass have the same energy (Spitzer 1987). This thermalization also induces mass segregation. As the system evolves toward energy equipartition, high mass stars lose energy, decrease their velocity dispersion and tend to sink toward the central regions. The opposite happens for low mass stars, which gain kinetic energy, tend to migrate toward the outer parts of the system, and preferentially escape the system in the presence of a tidal field"

So Why Are Stars in Rough Equilibrium?

- Another process, 'violent relaxation' (MBWsec 5.5), is crucial.
- This is due to rapid change in the gravitational potential (e.g., collapsing protogalaxy)
- Stellar dynamics describes in a statistical way the collective motions of stars subject to their mutual gravity-The essential difference from celestial mechanics is that each star contributes more or less equally to the total gravitational field, whereas in celestial mechanics the pull of a massive body dominates any satellite orbits
- The long range of gravity and the slow "relaxation" of stellar systems prevents the use of the methods of statistical physics as stellar dynamical orbits tend to be much more irregular and chaotic than celestial mechanical orbits-....woops.
- to quote from MBW pg 248
- Triaxial systems with realistic density distributions are therefore difficult to treat analytically, and one in general relies on numerical techniques to study their dynamical structure

How to Relax

- There are four different relaxation mechanisms at work in gravitational N-body systems: MBW sec 5.5.1-5.5.5
- phase mixing, chaotic mixing, Landau damping.
- Violent relaxation
  - time-dependent changes in the potential induce changes in the energies of the particles involved. Exactly how the energy of a particle changes depends in a complex way on the initial position and energy of the particle (effects are independent of the mass of the particles)
  - Time scale is very fast ~ free-fall time (e.g., collapsing protogalaxy)
- These processes are not well approximated by analytic calculations- need to resort to numerical simulations
  - simulations show that the final state depends strongly on the initial conditions, in particular on the initial virial ratio $|2T/W| \rightarrow$ collapse factor is inversely related to virial ratio.
  - Since $T \sim M\sigma^2$ and $W \sim GM^2/r = MV^2_c$ with V_c the circular velocity at r, smaller values for the initial virial ratio $|2T/W| \sim (\sigma/V_c)^2$ indicate cold initial conditions - these come naturally out of CDM models (MBW pg 257)

Collisionless Boltzmann Eq (= Vlasov eq)
S+G sec 3.4

- When considering the structure of galaxies, one cannot follow each individual star (10^{11} of them!),
- Consider instead stellar density and velocity distributions. However, a fluid model is not really appropriate since a fluid element has a single velocity, which is maintained by particle-particle collisions on a scale much smaller than the element.
- For stars in the galaxy, this is **not true** - stellar collisions are very rare, and even encounters where the gravitational field of an individual star is important in determining the motion of another are very infrequent
- So taking this to its limit, treat each particle as being collisionless, moving under the influence of the mean potential generated by all the other particles in the system $\phi(\mathbf{x},t)$

Collisionless Boltzmann Eq see MBW sec 5.4.2 S&G 3.4

- The distribution function is defined such that $f(\mathbf{r},\mathbf{v},t)d^3\mathbf{x}d^3\mathbf{v}$ specifies the number of stars inside the volume of phase space $d^3\mathbf{x}d^3\mathbf{v}$ centered on $(\mathbf{x},\mathbf{v})$ at time t-

At time t, a full description of the state of this system is given by specifying the number of stars

$f(x, v, t)d^3xd^3v$

Then $f(\mathbf{x}, \mathbf{v}, t)$ is called the "distribution function" (or "phase space number density") in 6 dimensions ($\mathbf{x}$ and $\mathbf{v}$) of the system.

$f \geq 0$ since no negative star densities

Since the potential is smooth, nearby particles in phase space move together-- fluid approx.

For a collisionless stellar system in dynamic equilibrium, the gravitational potential, ϕ , relates to the phase-space distribution of stellar tracers $f(\mathbf{x}, \mathbf{v}, t)$, via the collisionless Boltzmann Equation

number density of particles:

$n(\mathbf{x},t)=\int f(x, v, t)d^3\mathbf{v}$

average velocity:

$<v(\mathbf{x},t)>=\int f(x, v, t)\, \mathbf{v}d^3\mathbf{v}/\int f(x, v, t)d^3\mathbf{v}=$
$(1/n(\mathbf{x},t))\int f(x, v, t)\, \mathbf{v}d^3\mathbf{v}$

bold variables are vectors

See S&G sec 3.4

- The collisionless Boltzmann equation (CBE) is like the equation of continuity,

$$dn/dt = \partial n/\partial t + \partial(nv)\partial x = 0$$

but it allows for changes in velocity and relates the changes in f (x, v, t) to the forces acting on individual stars

- In one dimension, the CBE is

$$df/dt = \partial f/\partial t + v\partial f/\partial x - [\partial\phi(x, t)/\partial x]\, \partial f/\partial v = 0$$

Analogy with Gas- continuity eq see MBW sec 4.1.4

- $\partial\rho/\partial t + \nabla\bullet(\rho v) = 0$ which is equiv to
- $\partial\rho/\partial t + v\bullet\nabla\rho = 0$
- In the absence of encounters f satisfies the continuity eq, flow is smooth, stars do no jump discontinuously in phase space

- Continuity equation :

define w=(x,v) pair (generalize to 3-D)

$dw/dt=(v,-\nabla\phi)$ – 6-dimensional space

- $df/dt = 0$
- $\partial f/\partial t + \nabla_6(f\ \ dw/dt)=0$

Collisionless Boltzmann Eq

- This results in (S+G pg 143)

$$\boxed{\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f - \nabla \Phi \cdot \frac{\partial f}{\partial \mathbf{v}} = 0,}$$

- the flow of stellar phase points through phase space is incompressible – the phase-space density of points around a given star is always the same

- The distribution function f is a function of seven variables (t, **x**, **v**), so solving the collisionless Boltzmann equation in general is hard. So need either simplifying assumptions (usually symmetry), or try to get insights by taking moments of the equation.
- Take moments of an eq-- multipying f by powers of v

Collisionless Boltzmann Eq

astronomical structural and kinematic observations provide information only about the **projections of phase space distributions along lines of sight**,
limiting knowledge about f and hence also about ϕ.

Therefore all efforts to translate existing data sets into constraints on CBE involve simplifying assumptions.
• dynamic equilibrium,
• symmetry assumptions
• particular functional forms for the distribution function and/or the gravitational potential.

Collisionless Boltzmann Eq- Moments

define $n(x,t)$ as the number density of stars at position x

then the zeroth moment is:
$\partial n/\partial t+\partial/\partial x(nv)=0$; the same eq as continuity equation of a fluid

first moment:
$n\partial v/dt+nv\partial v/dx=-n\partial\phi/\partial x-\partial/\partial x(n\sigma^2)$

σ is the velocity dispersion
But unlike fluids, we do not have thermodynamics to help out....
nice math but not clear how useful

Jeans Equations MBW sec 5.4.3

- Since f is a function of 7 variables, obtaining a solution is challenging
- Take moments (e.g. integrate over all **v**)
- let n be the space density of 'stars'

$\partial n/\partial t + \partial(n\langle v_i\rangle)/\partial x_i = 0$; continuity eq. zeroth moment
first moment (multiply by v and integrate over all velocities)
$\partial(n\langle v_j\rangle/\partial t) + \partial(n\langle v_i v_j\rangle)/\partial x_i + n\partial\phi/\partial x_j = 0$
equivalently
$n\partial(\langle v_j\rangle/\partial t) + n\langle v_i\rangle\, \partial\langle v_j\rangle/\partial x_i = -n\partial\phi/\partial x_j - \partial(n\sigma^2_{ij})/\partial x_i$
where
n is the integral over velocity of f; $n=\int f\, d^3v$
$\langle v_i\rangle$ is the mean velocity in the i^{th} direction $= (1/n)\int f\, v_i\, d^3v$
$\sigma^2_{ij} = \langle (v_i - \langle v_i\rangle)(v_j - \langle v_j\rangle)\rangle$ "stress tensor"
$= \langle v_i v_j\rangle - \langle v_i\rangle\langle v_j\rangle$

Jeans Eq

- $n\partial(\langle v_j\rangle /\partial t) + n\langle v_i\rangle \partial\langle v_j\rangle/\partial x_i = -n\partial\phi/\partial x_j - \partial(n\sigma^2_{ij})/\partial x_i$
- So what are these terms??
- Gas analogy: Euler's eq of motion

  $\rho\, \partial\mathbf{v}/\partial t + \rho\, (\mathbf{v}\, .\, \boldsymbol{\nabla})v = -\boldsymbol{\nabla}P - \rho\boldsymbol{\nabla}\Phi$
- $n\partial\phi/\partial x_j$: gravitational pressure gradient
- $n\sigma^2_{ij}$ "stress tensor" is like a pressure, but may be anisotropic, allowing for different pressures in different directions - important in elliptical galaxies and bulges 'pressure supported' systems (with a bit of coordinate transform one can make this symmetric e.g. $\sigma^2_{ij}=\sigma^2_{ji}$)

Jeans Eq Cont

- $n\partial\mathbf{v}/dt+n\mathbf{v}\partial\mathbf{v}/d\mathbf{x}=-n\partial\phi/\partial\mathbf{x}-\partial/\partial\mathbf{x}(n\boldsymbol{\sigma}^2)$
- Simplifications: assume isotropy, steady state, non-rotating

  → terms on the left vanish
- Jean Eq becomes: $-n\, \nabla\phi = \nabla(n\sigma^2)$
- using Poisson eq: $\nabla^2\phi = 4\pi G\rho$
- Generally, solve for ρ (mass density)

Jeans Equations: Another Formulation

- Jeans equations follow from the collisionless Boltzmann equation; Binney & Tremaine (1987), MBW 5.4.2. S+G sec 3.4 .

cylindrical coordinates and assuming an axi-symmetric and steady-state system, the accelerations in the radial (R) and vertical (Z) directions can be expressed in terms of observable quantities:

the stellar number density distribution ν_*

And 5 velocity components

- a rotational velocity v_ϕ
- 4 components of random velocities (velocity dispersion components) $\sigma_{\phi\phi}$, σ_{RR}, σ_{ZZ}, σ_{RZ}

$$a_R = \sigma_{RR}^2 \times \frac{\partial(\ln\nu)}{\partial R} + \frac{\partial\sigma_{RR}^2}{\partial R} + \sigma_{RZ}^2 \times \frac{\partial(\ln\nu)}{\partial Z} + \frac{\partial\sigma_{RZ}^2}{\partial Z} + \frac{\sigma_{RR}^2}{R} - \frac{\sigma_{\phi\phi}^2}{R} - \frac{\overline{v_\phi}^2}{R},$$

$$a_Z = \sigma_{RZ}^2 \times \frac{\partial(\ln\nu)}{\partial R} + \frac{\partial\sigma_{RZ}^2}{\partial R} + \sigma_{ZZ}^2 \times \frac{\partial(\ln\nu)}{\partial Z} + \frac{\partial\sigma_{ZZ}^2}{\partial Z} + \frac{\sigma_{RZ}^2}{R}.$$

where a_Z, a_R are accelerations in the appropriate directions-
given these values (which are
the gradient of the gravitational potential),
the dark matter contribution can be estimated after accounting for the contribution from visible matter

Use of Jeans Eqs: Surface mass density near Sun

- Poissons eq $\nabla^2\phi = 4\pi\rho G = -\nabla\bullet\mathbf{F}$
- Use cylindrical coordinates, axisymmetry

$(1/R)\partial/\partial R(RF_R) + \partial F_z/\partial z = -4\pi\rho G$

- $F_R = -v_c^2/R$ v_c = circular velocity (roughly constant near Sun) –
  F_R = force in R direction

So $\rho = (-1/4\pi G)\partial F_z/\partial z$; **only vertical gradients count**

since the surface mass density $\Sigma = 2\int\rho dz$ (integrate 0 to $+\infty$ thru plane)

$\Sigma = -F_z/2\pi G$

Now use Jeans eq: $nF_z = -\partial(n\sigma^2_z)/\partial z + (1/R)\partial/\partial R(Rn\sigma^2_{zR})$; if R+z are separable, e.g $\phi(R,z) = \phi(R) + \phi(z)$ then $\sigma^2_{zR} \sim 0$ and voila! (eq 3.94 in S+G)

$\Sigma = -(1/2\pi Gn)\,\partial(n\sigma^2_z)/\partial z$; need to observe the number density distribution of some tracer of the potential above the plane [goes as $\exp(-z/z_0)$] and its velocity dispersion distribution perpendicular to the plane

Aside: Viral Theorem

- The quick way
- Consider for simplicity the one-dimensional analog of the Jeans Equation in steady state:

  $\partial/\partial x[\rho v^2]+\rho\partial\Phi/\partial x=o$
- After integrating over velocities, let 's now integrate over x :
- [one needs to use Gauss' theorem etc..]

- one gets $-2E_{kin}=E_{pot}$

Spherical systems- Elliptical Galaxies and Globular Clusters

- For a steady-state non-rotating spherical system, the Jeans equations simplify to

  $(1/n)d/dr\ (n\langle v^2_r\rangle)+2\beta\langle v^2_r\rangle/r=-GM(R)/r^2$
- where $GM(R)/r^2$ is the potential and $n(r)$, $\langle v^2_r\rangle$ and $\beta(r)$ describe the 3-dimensional density, radial velocity dispersion and orbital anisotropy of the tracer component (stars)

  $\beta(r)=1-\langle v^2_\theta\rangle/\langle v^2_r\rangle$; $\beta=0$ is isotropic, $\beta=1$ is radial
- We can then present the mass profile as
- $GM(r)=-r\langle v^2_r\rangle[(d\ln n/d\ln r)+(d\ln\langle v^2_r\rangle/d\ln r)+2\beta]$
- while apparently simple we have 3 sets of unknowns $\langle v^2_r\rangle$, $\beta(r)$, $n(r)$
- and 2 sets of observables I(r)- surface brightness of radiation (in some wavelength band) and the lines of sight projected velocity field (first moment is velocity dispersion)
- It turns out that one has to 'forward fit'- e.g. propose a particular form for the unknowns and fit for them.

This will become very important for elliptical galaxies

Motion Perpendicular to the Plane- Alternate Analysis-

(S+G pgs140-144, MBW pg 163)

For the motion of stars in the vertical direction only - stars whose motions carry them out of the equatorial plane of the system.

$d/dz[n_*(z)\sigma_z(z)^2]=-n_*(z)d\phi(z,R)/dz$; where **$\phi(z,R)$ is the vertical grav potential**

The study of such general orbits in axisymmetric galaxies can be reduced to a two-dimensional problem by exploiting the conservation of the z-component of angular momentum of any star

the first derivative of the potential is the grav force perpendicular to the plane - call it K(z)

$n_*(z)$ is the density of the tracer population and

$\sigma_z(z)$ is its velocity dispersion

then the 1-D Poisson's eq $4\pi G\rho_{tot}(z,R)=d^2\phi(z,R)/dz^2$

where ρ_{tot} is the total mass density - put it all together to get

$4\pi G\rho_{tot}(z,R)=-dK(z)/dz$ (S+G 3.93)

$d/dz[n_*(z)\sigma_z(z)^2]=n_*(z)K(z)$ - to get the data to solve this, we have to determine $n_*(z)$ and $\sigma_z(z)$ for the tracer populations(s)

Use of Jeans Eq For Galactic Dynamics

- Accelerations in the z direction from the Sloan digital sky survey for

1) all matter (top panel)

2) 'known' baryons only (middle panel)

3) ratio of the 2 (bottom panel)

Based on full-up numerical simulation from cosmological conditions of a MW like galaxy-this 'predicts' what a_Z should be near the Sun (Loebman et al 2012)

Compare with results from Jeans eq (ν is density of tracers, v_ϕ is the azimuthal velocity (rotation))

$$a_R = \sigma_{RR}^2 \times \frac{\partial(\ln\nu)}{\partial R} + \frac{\partial\sigma_{RR}^2}{\partial R} + \sigma_{RZ}^2 \times \frac{\partial(\ln\nu)}{\partial Z} + \frac{\partial\sigma_{RZ}^2}{\partial Z} + \frac{\sigma_{RR}^2}{R} - \frac{\sigma_{\phi\phi}^2}{R} - \frac{\overline{v_\phi}^2}{R}, \quad (1)$$

$$a_Z = \sigma_{RZ}^2 \times \frac{\partial(\ln\nu)}{\partial R} + \frac{\partial\sigma_{RZ}^2}{\partial R} + \sigma_{ZZ}^2 \times \frac{\partial(\ln\nu)}{\partial Z} + \frac{\partial\sigma_{ZZ}^2}{\partial Z} + \frac{\sigma_{RZ}^2}{R}. \quad (2)$$

Given accelerations $a_R(R,Z)$ and $a_Z(R,Z)$, *i.e.* the gradient of the gravitational potential, the dark matter contribution can be estimated after accounting for the contribution from visible matter.

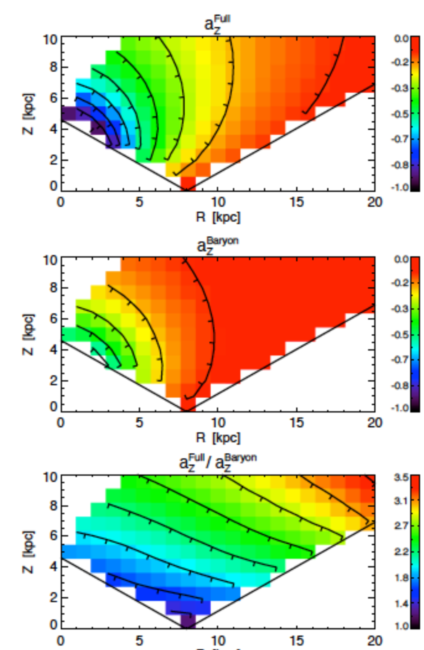


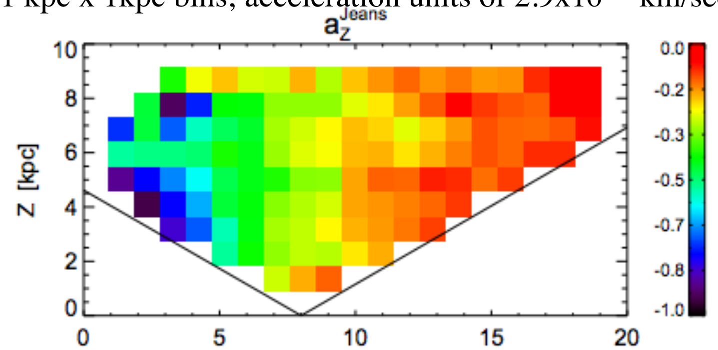
Figure 1. A comparison of the acceleration in the Z dir when all contributions are included (star, gas, and dark particles; top panel) to the result without dark matter (panel). The acceleration is expressed in units of 2.9×10^{-13} The ratio of the two maps is shown in the bottom panel. importance of the dark matter increases with the distance fr origin; at the edge of the volume probed by SDSS ($R \sim 2$

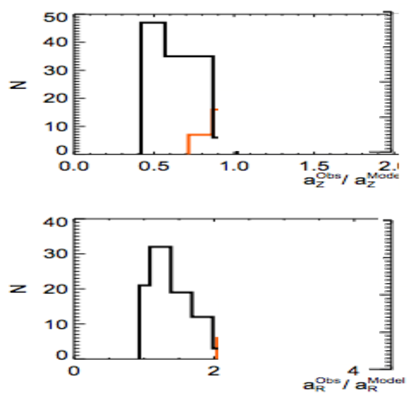
What Does One Expect The Data To Look Like

- Now using Jeans eq
- **Notice that it is not smooth or monotonic**

  and that the simulation is neither perfectly rotationally symmetric nor steady state..
- errors are on the order of 20-30%- figure shows comparison of true radial and z accelerations compared to Jeans model fits

1 kpc x 1kpc bins; acceleration units of 2.9×10^{-13} km/sec^2





Jeans (Continued)

- Using dynamical data and velocity data, get estimate of surface mass density in MW

$\Sigma_{total} \sim 70 \pm 6 M_\odot/pc^2$

$\Sigma_{disk} \sim 48 \pm 9 M_\odot/pc^2$

$\Sigma_{star} \sim 35 M_\odot/pc^2$

$\Sigma_{gas} \sim 13 M_\odot/pc^2$

we know that there is very little light in the halo so direct evidence for dark matter

Full Up Equations of Motion- Stars as an Ideal Fluid
(S+G pgs140-144, MBW pg 163)

Continuity equation (particles not created or destroyed)

$d\rho/dt+\rho\nabla\cdot v=0$; $d\rho/dt+d(\rho v)/dr=0$

Eq's of motion (Eulers eq)

$dv/dt = -\nabla P/\rho-\nabla\Phi$

Poissons eq

$\nabla^2\Phi(r) = -4\pi G\rho(r)$ (example potential)

Analogy of Stellar Systems to Gases
- Discussion due to Mark Whittle

- **Similarities :**

comprise many, interacting objects which act as points (separation >> size)

can be described by distributions in space and velocity eg Maxwellian velocity distributions; uniform density; spherically concentrated etc.

Stars or atoms are neither created nor destroyed -- they both obey continuity equations- not really true, galaxies are growing systems!

All interactions as well as the system as a whole obeys conservation laws (eg energy, momentum) **if isolated**

- **But** :
- The relative importance of short and long range forces is radically different :
  - atoms interact only with their neighbors
  - stars interact continuously with the entire ensemble via the long range attractive force of gravity
- eg uniform medium : $F \sim G (\rho\, dV)/r^2$; $dV \sim r^2 dr$; $F \sim \rho\, dr$

~ equal force from all distances

Analogy of Stellar Systems to Gases

- Discussion due to Mark Whittle

- The relative frequency of strong encounters is radically different :
  -- for atoms, encounters are frequent and all are strong (ie $\delta V \sim V$)
  -- for stars, pairwise encounters are very rare, and the stars move in the smooth global potential (e.g. S+G 3.2)

- Some parallels between gas (fluid) dynamics and stellar dynamics: many of the same equations can be used as well as :
  ---> concepts such as Temperature and Pressure can be applied to stellar systems
  ---> we use analogs to the equations of fluid dynamics and hydrostatics
- there are also some interesting differences
  ---> pressures in stellar systems can be anisotropic
  ---> self-gravitating stellar systems have negative specific heat
  $2K + U = 0 \rightarrow E = K + U = -K = -3NkT/2 \rightarrow C = dE/dT = -3Nk/2<0$
  and evolve away from uniform temperature.

- **Axisymmetric potentials:** separate into R and Z parts

$\phi(r,z)=J(R)Z(z)$

outside disk: $\nabla^2\phi=0$; find $Z(z)=A \exp(-k|z|)$

eq for R dependence of potential is

$(1/R)(d/dR(RdJ/dR)+k^2J(R)=0$ - the solutions of this are Bessel functions J(R); but it gets even messier (B & T 2.6)

Important result

- $Rd\phi/dR = v^2_c = GM(R)/R$ to within 10% for most 'reasonable' forms of mass distribution

see http://www.ast.cam.ac.uk/~ccrowe/Teaching/Handouts for lots of derivations/

Rotation Curve Mass Estimates sec 2.6 of B&T

- sec sec 11.1.2 in MBW
- Galaxy consists of a axisymmetric disk and spherical dark matter halo
- Balance centrifugal force and gravity
- $V^2(R)=R\mathbf{F}(R)$; $\mathbf{F}(R)$ is the acceleration in the disk
- Split rotation into 2 parts due to disk and halo
  $V^2(R)=V^2_d(R)+V^2_h(R)$
- for a **spherical** system
  $V^2(R)=R d\phi/dR=GM(R)/R$
- Few analytic solutions: point mass $V_c(R)\sim R^{-1/2}$
  singular isothermal sphere $V_c(R)$=constant (see S+G eq 3.14)
  uniform sphere $V_c(R)\sim R$

for a pseudo-isothermal sphere (S+G problem 2.20)
$\rho(r)=\rho(0)(R^2_c/R^2+R^2_c)$; $\rho(0) = V(\infty)^2/4\pi G R^2_c$ and the velocity profile is
$V(R)^2=V(\infty)^2(1-R_c/R \tan^{-1} R/R_c)$; for a **NFW** potential get a rather messy formula

Disks are Messy (MVW ch 11)

- Skipping the integrals of Bessel functions (eq 11.2 MBW) one gets
- $V^2_{c,d}(R)= 4\pi G\Sigma_0 R_d y^2[I_o(y)K_0(y)-I_1(y)K_1(y)]$

$y=R/(2R_d)$ and I_n and K_n are Bessel functions of the first and second kinds (n = order): which do not have simple asymptotic forms

Important bits: $V^2_{c,d}(R)$ depends only on radial scale length R_d and its central surface density Σ_0

Radial scale length of a spiral disk
$\Sigma(r)=\Sigma_0\exp(-R/R_d)$**;** integrate over r to get total mass $M_d=2\pi\Sigma_0 R^2_d$

Vertical density distribution is also an exponential $\exp(-z/z_0)$ so total distribution is product of the two
$\rho(R,z)=\rho_0 \exp(-R/R_d)\exp(-z/z_0)$
while we may know the scale length of the stars, that of the dark matter is not known.
Also the nature of the dark matter halo is not known:- disk/halo degeneracy

Nature is Cruel

$$v_c^2(R) = 2\pi G R \int_0^\infty \mathrm{d}k\, k J_1(kR) \int_0^\infty \mathrm{d}R'\, R'\Sigma(R') J_0(kR').$$

(BT 2.188)

- mathematics seems to be saying that it is as easy to determine a disk's surface density from measurements of its circular speed, as to obtain the circular speed from the surface density.

•Unfortunately, observational constraints destroy this symmetry.

•The key point is that the leftside of either equation (2.188) or (2.190) can be determined at any given value of R only if the variable on the right side can be measured out to radii at which its value becomes negligible.

•(BT 2.190) $$\Sigma(R) = \frac{1}{2\pi G} \int_0^\infty \mathrm{d}k\, k J_0(kR) \int_0^\infty \mathrm{d}R'\, v_c^2(R') J_1(kR').$$

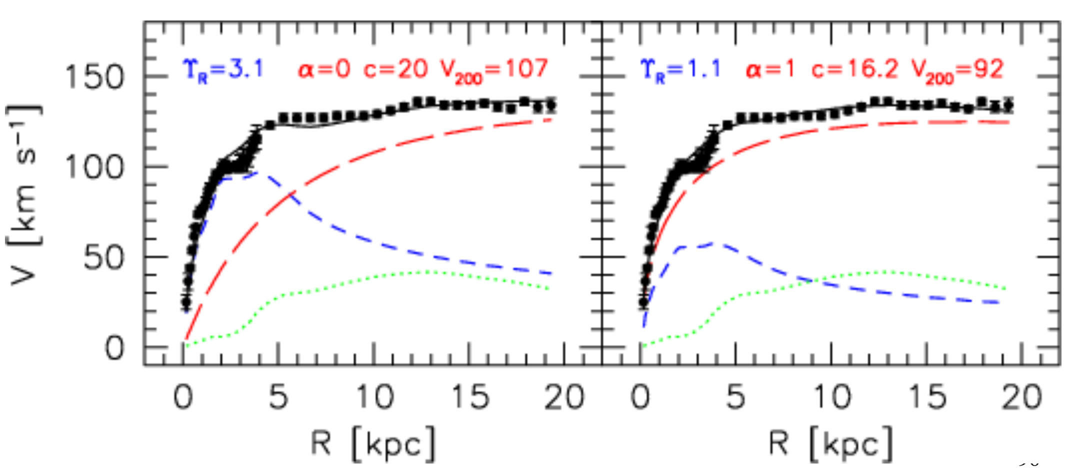
•The surface density declines rapidly with radius, so equation (2.188) can be used to derive accurate values of v_c.

•Circular speeds, by contrast, decline little if at all out to the largest observable radii. Consequently, in practice we cannot obtain the data needed to determine Σ accurately from equation (2.190)

Disk-Halo Degeneracy

- MBW fig 11.1: two solutions to rotation curve of NGC2403: stellar disk + bulge (blue lines), gas disk (green lines), dark matter halo (red lines).

- Left panel is a 'maximal' disk, using the highest reasonable mass to light ratio for the stars, the right panel a lower value of M/L



Potential of Spiral Galaxies B&T 2.7

- The potential of spirals is most often modeled as a 3-component system
  - -Bulge
  - -Dark halo
  - -Disk

as stressed by B&T usually one assumes that the potential has a certain form and is well traced by stars/gas

On pg 111 B&T give the observational constraints which models have to match.

Bulge; B&T assume $\rho(r)=\rho_B(0)(m/a_b)^{-\alpha_b}\exp(-m/r_b)^2{}_c)$; m=sqrt($R^2+z^2/q^2$)which for q<1 is an oblate spheroidal power-law model (no justification is given !)

They use IR star counts in the bulge (which is dominated by old stars) to get values for the parameters.

They use a similar form for the halo, but the parameters are much less well determined.

Disk: use a double exponential disk (thin and thick) for the stars and a somewhat more complex form for the gas (in the MW gas is ~25% of the mass of stars in the disk). The most important parameter is the disk scale length.

MW Mass Model (B&T)

- Notice that the mass of the inner halo and the surface mass density of the disk are highly uncertain (more later)

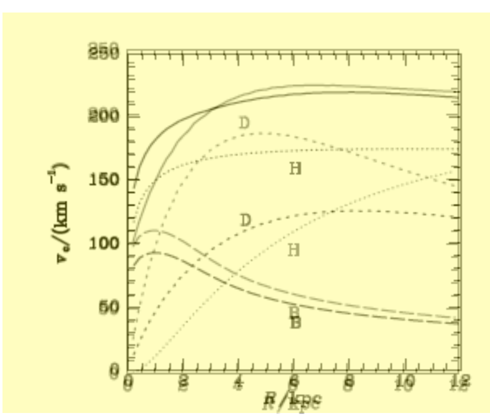


Table 2.3 Parameters of Galaxy models

Parameter	Model I	Model II
R_d/kpc	2	3.2
$(\Sigma_d+\Sigma_g)/\mathcal{M}_\odot\,\mathrm{pc}^{-2}$	1905	536
$\rho_{b0}/\mathcal{M}_\odot\,\mathrm{pc}^{-3}$	0.427	0.3
$\rho_{h0}/\mathcal{M}_\odot\,\mathrm{pc}^{-3}$	0.711	0.266
α_h	−2	1.63
β_h	2.96	2.17
a_h/kpc	3.83	1.90
$M_d/10^{10}\,\mathcal{M}_\odot$	5.13	4.16
$M_b/10^{10}\,\mathcal{M}_\odot$	0.518	0.364
$M_{h,<10\,\mathrm{kpc}}/10^{10}\,\mathcal{M}_\odot$	2.81	5.23
$M_{h,<100\,\mathrm{kpc}}/10^{10}\,\mathcal{M}_\odot$	60.0	55.9
$v_e(R_0)/\mathrm{km\,s^{-1}}$	520	494
f_b	0.05	0.04
f_d	0.60	0.33
f_h	0.35	0.63

f_b, f_d, f_h = fraction of the radial force supplied by the bulge, disk, and halo at R_0 = 8 kpc.

Virial Theorem B&T 4.8.3(a)

- This fundamental result describes how the total energy (E) of a self-gravitating system is shared between kinetic energy and potential energy .
- Go to one dimension and assume steady state

Integrate over velocity and space and one finds

$-2E_{kinetic}=PE_{\ potential}$ (W=PE)

see S&G pgs 120-121 for full derivation

This is important for finding the masses of systems whose orbital distribution is unknown or very complex and more or less in steady state (so assumptions in derivation are ok)

In general $<v^2>=W/M=GM/r_g$; r_g the gravitational radius (which depends on the form of the potential)

Many of the forms of the potential have their scale parameter $\sim 1/2r_g$ (pg 361 B&T)

Then if E is the total energy E=KE+PE=-KE=1/2PE (!); where does the energy go?? (radiation)

Use of Virial Theorem

- Consider a statistically steady state, spherical, self gravitating system of N objects with average mass m and velocity dispersion σ.
- Total $KE=(1/2)Nm\sigma^2$
- If average separation is r the PE of the system is $U=(-1/2)N(N-1)Gm^2/r$
- Virial theorem $E=-U/2$ so the total mass is $M=Nm=2\pi\sigma^2/G$ or using L as light and Σ as surface light density

  $\sigma^2\sim(M/L)\Sigma R$-picking a scale (e.g. half light radius R_e)

  $R_e\sim\sigma^\alpha\Sigma^\beta$ $\alpha=2,\beta=1$ from viral theorem

  value of proportionality constants depends on shape of potential

  For clusters of galaxies and globular clusters often the observables are the light distribution and velocity dispersion. then one measures the ratio of mass to light as

  $M/L\sim 9\sigma^2/2\pi GI(0)r_c$ for spherical <u>isothermal</u> systems

Jeans Again

- Jeans Mass $M_J=1/8(\pi k T/G\mu)^{3/2}\ \rho^{-1/2}$

- In astronomical units this is

$M_J=0.32M_\odot(T/10k)^{3/2}(m_H/\mu)^{3/2}(10^6cm^{-3}/n_H)^{1/2}$

So for star formation in the cold molecular medium with T~10k and n_H~10^5- M_J~2M

The growth time for the Jeans instability is

$\tau_J=1/sqrt(4\pi G\rho) = 2.3x10^4yr(n_H/10^6cm^{-3})^{-1/2}$

For pure free fall

$\tau_J= (3\pi/32G\rho)^{1/2}= 4.4x10^4yr(n_H/10^6cm^{-3})^{-1/2}$

Jeans growth rate about 1/2 the free fall time

τ_s time scale is the sound crossing time across the Jeans Length

$c_s= sqrt(kT/m_H\mu)$ $\lambda_J= (\pi c^2{}_s/G\rho)^{1/2}$ $\tau_s=\lambda_J/c_s$