Dynamics and how to use the orbits of stars to do interesting things

chapter 3 of S+G- parts of Ch 11 of MWB (Mo, van den Bosch, White)

READ S&G Ch 3 sec 3.1, 3.2, 3 .4

we are skipping over epicycles

A Guide to the Next Few Lectures

•The geometry of gravitational potentials : methods to derive gravitational potentials from mass distributions, and visa versa.

•Potentials define how stars move
consider stellar orbit shapes, and divide them into orbit classes.

•The gravitational field and stellar motion are interconnected :
the Virial Theorem relates the global potential energy and kinetic energy of the system.

• Collisions?

• The Distribution Function (DF) :
the DF specifies how stars are distributed throughout the system and with what velocities.

For collisionless systems, the DF is constrained by a continuity equation :
the Collisionless Boltzmann Equation

•This can be recast in more observational terms as the Jeans Equation.

The Jeans Theorem helps us choose DFs which are solutions to the continuity equations

A Reminder of Newtonian Physics sec 3.1 in S&G

Newtons law of gravity tells us that two masses attract each other with a force

eq 3.1
$$\frac{d}{dt}(m\boldsymbol{v}) = -\frac{GmM}{r^3}\boldsymbol{r}$$

ϕ (x) is the potential

If we have a collection of masses acting on a mass m_α the force is

$$\frac{d}{dt}(m_\alpha \boldsymbol{v}_\alpha) = -\sum_\beta \frac{Gm_\alpha M_\beta}{|\boldsymbol{x}_\alpha - \boldsymbol{x}_\beta|^3}(\boldsymbol{x}_\alpha - \boldsymbol{x}_\beta),\ \alpha \neq \beta$$
eq 3.2

eq 3.3
$$\frac{d}{dt}(m\boldsymbol{v}) = -m\nabla\Phi(\boldsymbol{x}),$$

with

eq 3.4
$$\Phi(\boldsymbol{x}) = -\sum_\alpha \frac{Gm_\alpha}{|\boldsymbol{x}-\boldsymbol{x}_\alpha|},\ for\ \boldsymbol{x} \neq \boldsymbol{x}_\alpha$$

Gauss's thm $\int \nabla\phi \bullet ds^2 = 4\pi GM$
the Integral of the normal component over a closed surface $=4\pi G$ x mass within that surface

the gravitational potential. If we can approximate the discrete stellar distribution with a continuous distribution ρ.

$$\boxed{\Phi(\boldsymbol{x}) = -\int \frac{G\rho(\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|} d^3\boldsymbol{x}'}$$

ρ(x) is the mass density distribution

Conservation of Energy and Angular Momentum

In the absence of external forces a star will conserve energy along its orbit

$$\boldsymbol{v}\cdot\frac{d}{dt}(m\boldsymbol{v}) = -m\,\boldsymbol{v}\cdot\nabla\Phi(\boldsymbol{x}),$$

$$\boldsymbol{v}\cdot\frac{d}{dt}(m\boldsymbol{v}) + m\,\boldsymbol{v}\cdot\nabla\Phi(\boldsymbol{x}) = 0$$

But since $\quad \frac{d\Phi}{dt} = \boldsymbol{v}\cdot\nabla\Phi(\boldsymbol{x})$

$$\nabla = \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}$$

where $\{\hat{x}, \hat{y}, \hat{z}\}$ are the unit vectors in their respective directions.

$$\frac{d}{dt}\left[\frac{m}{2}(\boldsymbol{v}^2) + m\Phi(\boldsymbol{x})\right] = 0$$

This is just the KE + PE

$$\frac{dL}{dt} = \boldsymbol{x}\times m\frac{d\boldsymbol{v}}{dt} = -m\,\boldsymbol{x}\times\nabla\Phi$$

Angular momentum L

Some Basics - M. Whittle

- The gravitational potential energy is a scalar field
- its gradient gives the net gravitational force (per unit mass) which is a vector field : see S&G pg 113

$$\boxed{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \nabla \cdot \nabla = \nabla^2}$$

$$\Phi(\mathbf{r}) = -G \int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r}' - \mathbf{r}|} \, d^3\mathbf{r}'$$

$$\boxed{\mathbf{F}(\mathbf{r}) = -\nabla\Phi(\mathbf{r})} = G \int_V \frac{\mathbf{r}' - \mathbf{r}}{|\mathbf{r}' - \mathbf{r}|^3} \rho(\mathbf{r}') \, d^3\mathbf{r}'$$

$$\nabla \cdot \mathbf{F}(\mathbf{r}) = -4\pi G\rho(\mathbf{r})$$

$$\boxed{\nabla^2\Phi(\mathbf{r}) = 4\pi G\rho(\mathbf{r})} \longleftrightarrow \text{Poissons eq inside the mass distribution}$$

$$\nabla^2\Phi(\mathbf{r}) = 0 \longleftrightarrow \text{Outside the mass dist}$$

Poisson's Eq+ Definition of Potential Energy (W)

So the force per unit mass is

$\rho(x)$ is the density dist

$$\boldsymbol{F}(\mathbf{x}) = -\nabla\Phi(\mathbf{x}) = \int G\rho(\mathbf{x}')\frac{(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}|^3} d^3\mathbf{x}'$$

To get the differential form we start with the definiti
of Φ and applying ∇^2 to both sides

S+G pg 112-113

$$\nabla^2\Phi(\mathbf{x}) = -\nabla^2 \int \frac{G\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} d^3\mathbf{x}'$$

$$= 4\pi G\rho(\mathbf{x})$$ Poisson's equation.

Potential energy W

$$W = \frac{1}{2}\int_V \rho(\mathbf{r})\,\Phi(\mathbf{r})\,d^3\mathbf{r} = -\frac{1}{8\pi G}\int_V |\nabla\Phi|^2\,d^3\mathbf{r}$$

Derivation of Poisson's Eq

So the force per unit mass is

$$F(x) = -\nabla\Phi(x) = \int G\rho(x')\frac{(x-x')}{|x-x|^3}d^3x'$$

To get the differential form we start with the definiti
of Φ and applying ∇^2 to both sides

$$\nabla^2\Phi(x) = -\nabla^2\int\frac{G\rho(x')}{|x-x'|}d^3x'$$

$$= 4\pi G\rho(x) \qquad \boxed{\text{Poisson's equation.}}$$

see S+G pg112 for detailed derivation or web page 'Poisson's equation'

More Newton-Spherical Systems

Newtons 1st theorem: a body inside a spherical shell has no net gravitational force from that shell; e.g. $\nabla\Phi(r)=0$

Newtons 2nd theorem: the gravitational force on a body outside a spherical shell is the same as if all the mass were at a point at the center of the shell.

Simple examples:
Point source of mass M; potential $\Phi(r) = -GM/r$;
definition of circular speed; speed of a test particle on a circular orbit at radius r
$\mathbf{v^2_{circular} = r\, d\Phi(r)/dt} = GM/r$; $v_{circular} = \mathrm{sqrt}(GM/r)$;Keplerian

escape speed $= \mathrm{sqrt}[2\Phi(r)] = \mathrm{sqrt}(2GM/r)$; from equating kinetic energy to potential energy $1/2mv^2 = |\Phi(r)|$

Characteristic Velocities

$v^2_{circular}$=r dΦ(r)/dt=GM/r; v=sqrt(GM/r) Keplerian

velocity dispersion $\sigma^2=(1/\rho)\int\rho\,(\partial\Phi(r,z)/\partial z)dz$
or alternatively $\sigma^2(R)=(4\pi G/3M(R)\int r\rho(r)\,M(R)\,dr$

escape speed =v_{esc}=sqrt(2Φ(r)) or $\Phi(r)=1/2v^2_{esc}$
so choosing r is crucial

Escape Speed

- As r goes to infinity $\phi(r)$ goes to zero
- so to escape $v^2>2\phi(r)$; e.q. v_{esc}=sqrt(-2ϕ(r))

- Alternate derivation using conservation of energy
- Kinetic + Gravitational Potential energy is constant
  - $KE_1+U_1=KE_2+U_2$
- Grav potential =-GMm/r; KE=1/2mv_{escape}^2
- Since final velocity=0 (just escapes) and U at infinity=0
- 1/2mv_{escape}^2-GMm/r=0

- The star's *energy E* is the sum of its *kinetic energy KE* = $m\mathbf{v}^2/2$ and the *potential energy PE* = $m\phi(\mathbf{x})$.
- The kinetic energy cannot be negative, and since far from an isolated galaxy or star cluster, $\phi(\mathbf{x}) \rightarrow 0$.
- So a star at position $\mathbf{x}$ can escape only if it has $E > 0$; it must be moving faster than the local

escape speed v_e, given by

$$v^2_e = -2\phi(\mathbf{x}).$$

Potential From Density Distribution MBW 5.4.2

For a spherical system, of radius r_0Newton's theorems allow a general solution for $\Phi(r)$ given $\varrho(r)$.
$\Phi(r)$ follows from splitting $\varrho(r)$ in
spherical shells, and adding the potentials of all these shells:

$$\Phi(r) = -4\pi G \left[1/r \int_0^r \varrho(r')\, r'^2\, \mathrm{d}r + \int_r^\infty \varrho(r')\, r'\, \mathrm{d}r' \right]$$

where the first intergral is from zero to r_0 and the second from r_0 to infinity

The first integral is just GM(<r)/r

Gravity and Dynamics-Spherical Systems- Repeat

- Newtons 1st theorm : a body inside a a spherical shell has no net force from that shell $\nabla\phi = 0$
- Newtons 2nd theorm ; a body outside the shell experiences forces as if they all came from a point at the center of the shell-Gravitational force at a point outside a closed **sphere** is the same as if all the mass were at the center
  - This does not work for a thin disk- cannot ignore what is outside of a given radius
- One of the prime observables (especially for spirals) is the **circular velocity;** in general it is $V_c^2(R)/R = G(M<R)/R^2$; more accurate estimates need to know shape of potential
- **so one can derive the mass of a flattened system from the rotation curve**

- point source has a potential $\phi = -GM/r$
- A body in orbit around this point mass has a circular speed $v_c^2 = r\, \phi d/dr = GM/r$
- $v_c = \mathrm{sqrt}(GM/r)$; Keplerian
- Escape speed from this potential $v_{escape} = \mathrm{sqrt}(2\phi) = \mathrm{sqrt}(2GM/r)$ (conservation of energy $KE = 1/2 m v^2_{escape}$)

Variety of "Simple" Potentials See problems 3.1-3.4,3.7 in S&G

- Point mass $\phi(r) = -GM/r$
- Plummer sphere : simple model for spherical systems
  - $\phi(r) = -GM/\mathrm{sqrt}(r^2 + a^2)$
- Singular isothermal sphere $\phi(r) = 4\pi G r^2_0 \rho(r_0) \ln(r/r_0)$
  some interesting properties- circular speed is constant at
  $\mathrm{sqrt}(4\pi G r^2_0 \rho(r_0))$
- A disk $\phi(R, z) = -GM/sqrt(R^2 + (a_K + |z|)^2$
- The *Navarro–Frenk–White* (NFW) potential
  $\phi(R) = 4\pi G a^2_0 \rho(r_0)\, [\ln(1 + r/a_0)/(r/a_0)]$
  - this form fits numerical simulations of structure growth

Homogenous Sphere B&T sec 2.2.2

- Constant density sphere of radius a and density ρ_0
- $M(r)=4\pi G r^3 \rho_0$; r<a
- $M(r)=4\pi G a^3 \rho_0$; r>a

$\phi(R)=-d/dr(M(R))$

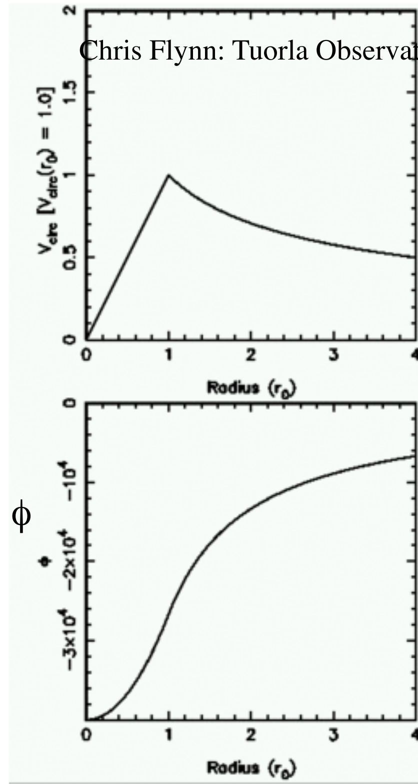
R>a: $\phi(r)=4\pi G a^3 \rho_0=-GM/r$

R<a: $\phi(r)=-2\pi G \rho_0 (a^2-1/3r^2))$;

$v^2_{circ}= (4\pi/3)G\rho_0 r^2$; solid body rotation R<a

V_c

ϕ



Some Simple Cases

- **Constant density sphere** of radius a and density ρ_0

Potential energy (B&T) eq 2.41, 2.32

$\phi(R)=-d/dr(M(R))$;

R>a: $\phi(r)=4\pi G a^3 \rho_0=-GM/r$

R<a : $\phi(r)=-2\pi G \rho_0 (a^2-1/3r^2))$;

$v^2_{circ}= (4\pi/3)G\rho_0 r^2$ solid body rotation

Potential is the same form as a harmonic oscillator

e.g. the eq of motion is $d^2r/dt^2=-GM(r)/r=4\pi/3Gr\rho$; solution to harmonic oscillator is

$r=A\cos(\omega t+\phi)$ with $\omega= \mathrm{sqrt}(4\pi/3G\rho)=2\pi/T$

$T=\mathrm{sqrt}(3\pi/G\rho_0)=2\pi r/v_{circ}$

Homogenous Sphere

- Potential energy of a self gravitating sphere of constant density r, mass M and radius R is obtained by integrating the gravitational potential over the whole sphere
- **Potential energy $U=1/2\int r\rho(r)\nabla\Phi d^3r$**

$U = \int_0^R -4\pi G\, M(r)\, \varrho(r)\, r\, dr = \int_0^R G[(4/3\pi\rho r^3) x\, (4\pi\rho r^2)dr]/r$
$=(16/3)\pi^2\rho^2 r^2)\int_0^R r^4 dr == (16/15)\pi^2\rho^2 R^5$

using the definition of total mass M (volumexdensity)$M=(4/3)\pi\rho R^3$

gives $\boxed{U=-\,(3/5)GM^2/R}$

Homogenous Sphere B&T sec 2.2.2

Orbital period $\mathbf{T=2\pi r/v_{circ}=sqrt(3\pi/G\rho_0)}$

Dynamical time=crossing time

$=T/4=\mathbf{sqrt(3\pi/16G\rho_0)}$

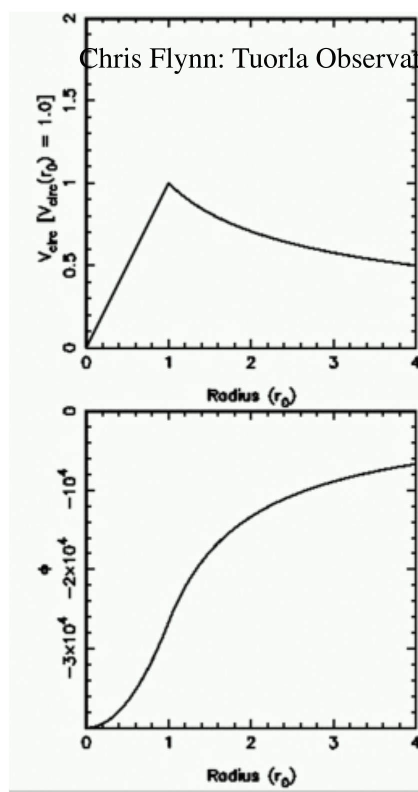
Potential is the same form as an harmonic oscillator with angular freq $2\pi/T$ (B&T 2.2.2(b))

Regardless of initial value of distance (r) a particle will reach r=0 (in free fall) in a time T=/4

Eq of motion of a test particle INSIDE the sphere is

$dr^2/dt^2=-GM(r)/r^2=-(4\pi/3)G\rho_0 r$

General result dynamical time ~sqrt(1/Gρ)



Spherical Systems: Homogenous sphere of radius a
Summary

- $M(r)=4/3\pi r^3\rho$ (r<a); r>a $M(r)=4/3\pi r^3 a$
- Inside body (r<a); $\phi(r)=-2\pi G\rho(a^2-1/3\ r^2)$ (from eq. 2.38 in B&T)

Outside (r>a);)$\phi(r)= -4\pi G\rho(a^3/3)$

Solid body rotation $v_c^2 = -4\pi G\rho(r^2/3)$

Orbital period $T=2\pi r/v_c$=sqrt($3\pi/G\rho$);

a crossing time (dynamical time) =T/4=sqrt($3\pi/16G\rho$)

potential energy U=$-3/5GM^2/a$

The motion of a test particle inside this sphere is that of a simple harmonic oscillator $d^2r/dt^2= -G(M(r)/\ r^2=4\pi G\rho r/3$ with angular freq $2\pi/T$

no matter the intial value of r, a particle will reach r=0 in the dynamical time T/4

In general the dynamical time t_{dyn}~1/sqrt(G<ρ>)

and its 'gravitational radius' $r_g= GM^2/U$

Star Motions in a Simple Potential

- if the density ϱ in a spherical galaxy is constant, then a star following a circular orbit moves so that its angular speed $\Omega(r) = V(r)/r$ is constant.
- a star moving on a radial orbit, i.e., in a straight line through the center, would oscillate harmonically in radius with period

P =sqrt[$3\pi/G\varrho$]~ $3t_{ff}$, where t_{ff} =sqrt[$1/G\varrho$]: S&G sec 3.1

Not so Simple - Plummer Potential (Problem 3.2S&G)

- Many astrophysical systems have a 'core'; e.g. the surface brightness flattens in the center (globular clusters, elliptical galaxies, clusters of galaxies, bulges of spirals) so they have a characteristic length
- so imagine a potential of the form $-\phi(r)=-GM/\mathrm{sqrt}(r^2+b^2)$; where b is a scale length

$\nabla^2\Phi(r)=(1/r^2)\, d/dr(r^2 d\phi/dr)=3GMb^2/(r^2+b^2)^{5/2}=4\pi G\rho(r)$ [Poissons eq]

and thus

$\rho(r) =(3M/4\pi b^3)[1+(r/b)^2]^{-5/2}$ which can also be written as

- **$\rho(r)=(3b^2M/4\pi)(r^2+b^2)^{-5/2}$.**

Not so Simple - Plummer Potential sec 2.2 in B&T

Now take limits $r<<b$ $\rho(r) =(3GM/4\pi b^3)$ constant

$r>>b$ $\rho(r) =(3GM/4\pi b^3)r^{-5}$ finite

Plummer potential was 'first' guess at modeling 'real' spherical systems; it is one of a more general form of 'polytropes' B&T (pg 300)

Potential energy $U=3\pi GM^2/32b$

Spherical systems- Plummer potential

- Another potential with an analytic solution is the Plummer potential - in which the density is constant near the center and drops to zero at large radii - this has been used for globular clusters, elliptical galaxies and clusters of galaxies.

- One such form- Plummer potential
  $\phi=-GM/(\mathrm{sqrt}(r^2+b^2)$; b is called a scale length

  The density law corresponding to this potential is
  (using the definition of $\nabla^2\phi$ in a spherical coordinates)

$$\boxed{\nabla^2 \equiv \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin^2\phi}\frac{\partial^2}{\partial\theta^2}+\frac{1}{r^2\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial}{\partial\phi}\right).}$$

$\nabla^2\phi = (1/r^2)d/dr(r^2 d\phi/dr)=(3GMb^2)/((r^2+b^2)^2)^{5/2}$

$\rho(r)=(3M/4\pi b^3)(1+(r/b)^2)^{-5/2}$

Potential energy $W=-3\pi GM^2/32b$

Many More Not So Simple Analytic Forms

- ; there are many more forms which are better and better approximations to the true potential of 'spherical' systems
- 2 others frequently used -are the modified Hubble law used for clusters of galaxies

- start with a measure quantity the surface brightness distribution (more later)

  $I(r)=2aj_o(1+(r/a)^2)^{-1}$

which gives a 3-D luminosity density

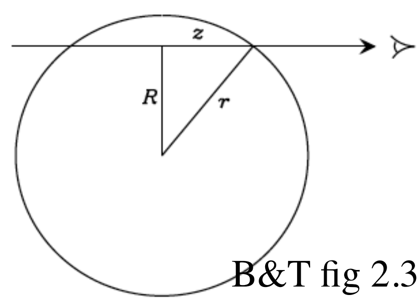
$j=j_o(1+(r/a)^2)^{-3/2}$

- at r=a ; I(a)=1/2I(0); a is the core radius
- <u>Now if light traces mass and the mass to light ratio is constant</u>

$M=\int j(r)d^3r=$

$4\pi a^3Gj_o[\ln[R/a+\mathrm{sqrt}(1+(r/a)^2)]-(r/a)(1+(r/a)^{-1/2}]$

- and the potential is also analytic



B&T fig 2.3

Problems: mass diverges logarithimically BUT potential is finite and at r>>a is almost GM/r

Spherical Systems

- A frequently used analytic form for the surface brightness of an elliptical galaxy is the Modified Hubble profile

- $I(R)=2j(o)a/[(1+(r/a)^2]$ which has a luminosity density distribution
  $j(r)=j(0)[(1+(r/a)^2]^{-3/2}$
- this is also called the 'pseudo-isothermal' sphere distribution

- the eq for ϕ is analytic and finite at large r even though the mass diverges

$\phi=-GM/r-(4\pi Gj_0a)^2/\mathrm{sqrt}[1+(r/a)^2]$

Last Spherical Potential S&G Prob 3.7

- In the last 15 years numerical simulations have shown that the density distribution of dark matter can be well described by a form called 'NFW' density distribution

$\rho(r)=\rho(0)/[(r/a)^\alpha(1+(r/a))^{\beta-\alpha}]$ with $(\alpha,\beta)= (1,3)$

Integrating to get the mass
$M(r)=4\pi G\rho(0)a^3\ln[1+(r/a)]-(r/a)/[1+(r/a)]$
and potential $\phi=[\ln(1+(r/a)]/(r/a)]$

See problem 3.7 in S&G

The NFW density distribution is an analytic approximation to numerical simulations of cold dark matter

Other Forms

- However all the forms so far have a Keplerian rotation $v \sim r^{-1/2}$ while real galaxies have flat rotation curves $v_c(R)=v_0$

- A potential with this property must have $d\phi/dr=v_0^2/R$; $\phi=v_0^2 \ln R+C$

- However this is a rather artificial form; real galaxies seem to be composed of 3 parts: disk (D), bulge (B), halo (H) and it is the <u>sum of the 3</u> that gives the flat rotation curve (very fine tuned and very flexible)

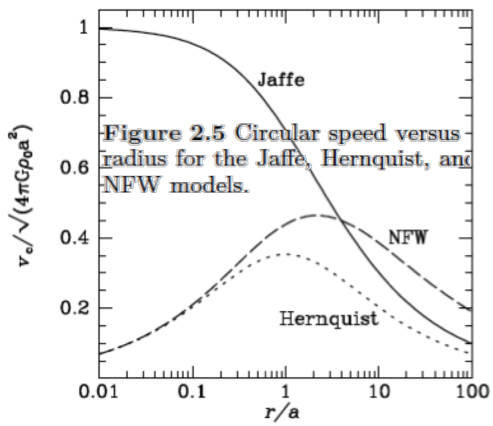


Figure 2.5 Circular speed versus radius for the Jaffe, Hernquist, and NFW models.