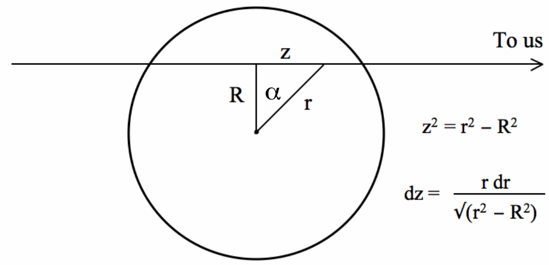
Projection Effects
from M. Whittle

http://www.astro.virginia.edu/class/whittle/astr553/Topic07/t7_projection.ht

- Observed luminosity density I(R)=integral over true density distribution j(r) (in some wavelength band)
- Same sort of projection for velocity field but weighted by the density distribution of tracers
- $I(R)\sigma(r)^2 = 2\int [(v_r \cos\alpha - v_\theta \sin\alpha)^2 nr]/\mathrm{sqrt}(r^2-R^2)$
- Density distribution solution is an Abel integral (see appendix B.2 in B&T) with solution of the form
- while the velocity field solution is also an Abel integral
- There are a few useful I(R) & j(r) pairs that can both be expressed algebraically

$$I(R) = \int_{-\infty}^{\infty} j(r)\, dz = 2\int_R^{\infty} \frac{j(r)\, r\, dr}{\sqrt{(r^2 - R^2)}}$$



$$j(r) = \frac{-1}{\pi}\int_r^{\infty} \frac{dI}{dR} \frac{dR}{\sqrt{(R^2 - r^2)}}$$

Orbits in a **static** spherical potential B& T sec 3.1

angular momentum (**L)** is conserved

- $d^2r/dt^2 = \phi(r)\boldsymbol{e}_r$ $\boldsymbol{e}_r$ is the unit vector in radial direction; the radial acceleration $\phi = d^2r/dt^2$
- d/dt(r x dr/dt)=(dr/dt x dr/dt)+r x d^2r/dt^2=g(r)r x$\boldsymbol{e}_r$ **=**0;
- conservation of angular momentum L=rxdr/dt (eqs. 3.1-3.5)
- Define **L**=**r**xd**r**/dt; d**L**/drt=0

- Since this vector is constant, we conclude that the star moves in a plane, the orbital plane.
- This simplifies the determination of the star's orbit, for since the star moves in a plane, we may use plane polar coordinates
- for which the center is at r = 0 and φ is the azimuthal angle in the orbital plane

Stellar Dynamics B&T ch 3, S&G 3

- Orbits in a static spherical potential:

  angular momentum is conserved

  – $d^2r/dt^2=g(r)\boldsymbol{e}_r$ $\boldsymbol{e}_r$ is the unit vector in radial direction; the radial acceleration $g=d^2r/dt^2$

  – $d/dt(r \times dr/dt)=(dr/dt \times dr/dt)+r \times d^2r/dt^2=g(r)r \times \boldsymbol{e}_r =0$; conservation of angular momentum $L=r\times dr/dt$ (eqs. 3.1-3.5)around the z axis

Conservation of energy:

total energy =PE+KE or in above formalism

  – stars move in a plane (orbital plane) $\theta=0$

  – Use plane polar coordinates (R,φ,z) (Appendix B)

  – eqs of orbits

  R coordinate: $d^2R/dt^2-R(d^2\varphi/dt^2)=\phi(R)$

  ψ coordinate :$2(dR/dt)(d\varphi/dt)+R(d^2\varphi/dt^2)=0$

  equation of motion is $(d^2R/dt^2)-(L^2/R^3)=\phi(R)$

Stellar Dynamics B&T ch 3

- Orbits in a static spherical potential:

  angular momentum is conserved

  – $d^2r/dt^2=g(r)\boldsymbol{e}_r$ $\boldsymbol{e}_r$ is the unit vector in radial direction; the radial acceleration $g=d^2r/dt^2$

  – $d/dt(r \times dr/dt)=(dr/dt \times dr/dt)+r \times d^2r/dt^2=g(r)r \times \boldsymbol{e}_r=0$; conservation of angular momentum $L=r\times dr/dt$ (eqs. 3.1-3.5)

Conservation of energy:

total energy =PE+KE or in above formalism

- if substitute $u = 1/r$, the energy equation takes the form :
- $2E/L^2 = 2\phi/ L^2 + u^2 + (du/d\phi)^2$.
- bound orbits are those in which the radius r is always finite , Thus, for bound orbits $u = 1/r$ is finite while for unbound orbits u tends to zero.
- In a bound orbit the condition $du/d\phi = 0$, when this occurs
- $u^2 +2 [\phi(1/ u)- E] / L^2 = 0$.
- This equation has 2 roots, u_1 and u_2 . And thus a " bound " star orbiting in a conservative potential will thus move in an orbit twixt 2 radii $r_1 =1/u_1$ and $r_2 =1/u_2$; the pericenter and apocenter

Read B&T 143-147

- Since $L=R^2 d\varphi/dt$
- $(L^2/R^2)d/d\varphi(1/R^2 dR/d\varphi)-L^2/R^3=\phi(r)$;
- using $u=1/R$

$\boxed{\mathbf{d^2u/d\varphi^2+u=-\phi(1/u)/L^2u^2}}$ eq. 3.11 in B&T

now putting in a spherical potential

$\phi=-GM/R^2$ and substituting

$d^2u/d\varphi^2+u=GM/L^2$

two general solutions, bound and unbound

bound orbits $du/d\varphi=0$ and orbit is confined between pericenter and apocenter

- For a halo with outer radius r_h a flat rotation curve, and circular velocity V_c the escape velocity at R is $V_{esc}(R)^2 = 2V^2_c \ln(1+r_h/R)$ (Binney & Tremaine)

Some Simple Cases

- Point source potential (pg 147 B&T; eq 3.23 and following)

$\phi(R)=-GM/R^2$; $\phi(u)=-GMu^2$

using

$\mathbf{d^2u/d\varphi^2+u=-\phi(1/u)/L^2u^2}$

$\mathbf{d^2u/d\varphi^2+u= -GM/u^2}$

general solution: $\mathbf{u(\varphi)=C\cos(\varphi-\varphi_0)=GM/L^2}$

C and φ_0 are constants

Nature of solutions; v_{circ}=sqrt(GM/r)

C=0 circular orbits (B&T define the eccentricity as CL^2/GM so if C=0, eccentricity=0

if C> GM/L^2 unbound orbit; e.g r can go to infinity if u=0

C<GM/L^2 bound orbit; we know this solution(!); ellipse with pt source at one focus and complete a radial period in $\Delta\psi=2\pi$

From before

Potential energy (B&T) eq 2.41, 2.32

$\phi(R)=-d/dr(M(R))$;

$R>a$ $\phi(r)=4\pi G a^3\rho_0=-GM/r$

$R<a$ $\phi(r)=-2\pi G\rho_0(a^2-1/3r^2)$);

Use Cartesian coordinates $x=r\cos\varphi$, $y=r\sin\varphi$

$F_x=-4\pi GR\rho_0\cos(\varphi)\boldsymbol{e}_x=-4\pi Gx\rho_0\boldsymbol{e}_x$

$F_y=-4\pi Gy\rho_0\boldsymbol{e}_y$

need to transform $d^2\mathbf{r}/dt^2=\boldsymbol{e}_x d^2\mathbf{x}/dt^2+\boldsymbol{e}_y d^2\mathbf{y}/dt^2$

define $\Omega^2=4\pi/3G\rho_0$;$d^2\mathbf{x}/dt^2=-\Omega^2\boldsymbol{x}$; $d^2\mathbf{y}/dt^2=-\Omega^2\boldsymbol{y}$

this the harmonic oscillator general solution

$x=A\cos(\Omega t+k_x)$; $y=B\cos(\Omega t+k_y)$;

A,B are amplitudes and k's the initial phase

going backwards to polar coordinates

$R=\mathrm{sqrt}[A^2\cos^2(\Omega t+k_x)+B^2\cos^2(\Omega t+k_y)]$

$\psi=\tan^{-1}[B\cos(\Omega t+k_y)/A\cos(\Omega t+k_x)$;]

Constant Density Sphere

The R and define a closed ellipse on the center of the sphere; A and B are the major and semi-major axis.

Complete radial period in $\Delta\varphi=\pi$

Most mass distributions will lie between a pt mass and a uniform sphere so radial and azimuthal periods not the same ; rosette pattern for orbits

'Real' Orbits

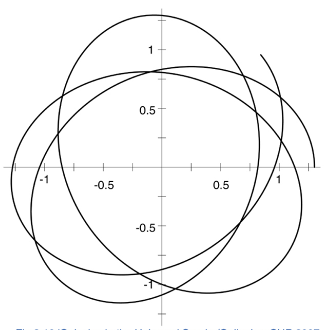
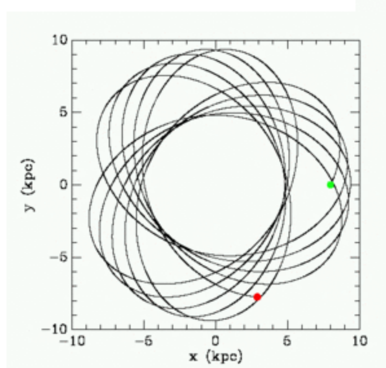
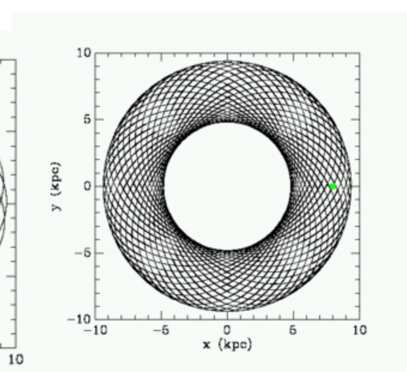


Fig 3.10 'Galaxies in the Universe' Sparke/Gallagher CUP 2007





- A few orbits, ~2 Gyr of orbits- 20 Gyrs from C. Flynn

Stellar Dynamics B&T ch 3; S&G 3.3

- Orbits of disk stars

only the component of angular momentum parallel to symmetry axis is constant.

  - Since **L** is conserved, stars move in a plane - can use polar coordinates (R,φ) (do not need z, appendix B B&T B.24)
  - R eq of motion $dR^2/dt^2 - Rd\varphi^2/dt^2 = \phi(r)$
  - φ eq of motion $(2dR/dt*d\varphi/dt)+Rd\varphi^2/dt^2=0$; $\mathbf{L}=R^2 d\varphi/dt$ is a constant
  - total equation of motion $dR^2/dt^2 - \mathbf{L}/R^2 = \phi(r)$

- Stars whose motions are confined to the equatorial plane of an axisymmetric galaxy 'feel' only an effectively spherically symmetric potential
- Therefore their orbits will be identical with those discussed previously
- ;the radial coordinate R of a star on such an orbit oscillates between the peri and apo-galacticon as the star revolves around the center, and the orbit forms a rosette figure.

Orbits in Axisymmetric Potentials- B&T 3.2, S&G 3.3

cylindrical coordinate system (R; φ; z) with origin at the galactic center, the z axis is the galaxy's symmetry axis.

- Stars in a axisymmetric galaxy 'see' a potential which is spherically symmetric. orbits will be identical to those in such a potential
- The situation is much more complex for stars whose motions carry them out of the equatorial plane of the system.
- orbits in axisymmetric galaxies can be reduced to a two-dimensional problem by exploiting the conservation of the z-component of angular momentum
- **S&G give nice physical description**

- $d^2\mathbf{r}/dt^2 = -\nabla\Phi$ (R,z); which can be written in cylindrical coordinates as
- $d^2R/dt^2 - Rd\varphi^{\mathbf{2}}/dt^2 = -\partial\Phi/\partial R$
- Motion in the φ direction : $d/dt\,(R^{\mathbf{2}}\, d\varphi/dt)=0$; $L_{\mathbf{z}}= R^{\mathbf{2}}(d\varphi/dt)= 0$ constant
- z direction : $d^2z/dt^2 = -\partial\Phi/\partial z$

Orbits in Axisymmetric Potentials- B&T 3.2

- Eliminating $d\varphi/dt$ and putting in angular momentum
- $d^2R/dt^2 - L^2_z/R^3 = -\partial\Phi/\partial R$ - if we define an effective potential $\Phi_{eff} = \Phi\,(R,z) + L^2_z/2R^2$
- $d^2R/dt^2 = -\partial\Phi_{eff}/\partial R$ (see B&T **eq 3.67-3.68**)

- Unless it has enough energy to escape from the Galaxy, each star must remain within some apogalactic outer limit.

Orbits in Axisymmetric Potentials- B&T 3.2

The three-dimensional motion of a star in an axisymmetric potential (R; z) can be reduced to the two dimensional motion of the star in the (R; z) plane (the meridional plane

- Since the change in ang mom in the z direction is zero (planar orbits)

$\partial/\partial z(L^2_z/2R^2) = 0$; $d^2z/dt^2 = -\partial\Phi_{eff}/\partial z$;

The effective potential is the sum of gravitational potential and KE in the φ direction.

and rises very steeply near the z axis

The minimum in Φ_{eff} has a "simple" physical meaning (see next page)

$0 = \partial\Phi_{eff}/\partial R = \partial\Phi/\partial R - L^2_z/R^3$; which is satisfied at a particular radius - the guiding center radius R_G where

$(\partial\Phi/\partial R)|_{R_G} = L^2_z/R^3 = R_G\,(d\varphi/dt)^2$

and $0 = \partial\Phi_{eff}/\partial z$ which is satisfied in the equatorial plane

these are the conditions for a circular orbit with angular speed $d\varphi/dt$

Orbits in Axisymmetric Potentials- B&T 3.2

- the minimum of Φ_{eff} occurs at the radius at which a circular orbit has angular momentum L_z, and the value of Φ_{eff} at the minimum is the energy of this circular orbit
- Unless Φ has a special form these eq's cannot be solved analytically

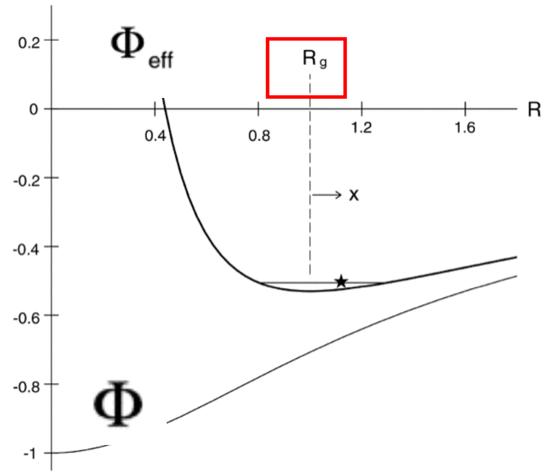


Fig 3.8 'Galaxies in the Universe' Sparke/Gallagher CUP 2007

Orbits in Axisymmetric Potentials- B&T 3.2.3

If assume in disk galaxies that the orbits are *nearly* circular
What approx can we make to the orbits??

let $x = R - R_g$; where $R_g(L_z)$ is the guiding-center radius for an orbit of angular momentum L_z (eq. 3.72).

Expand Φ_{eff} around x (see B&T eq 3.76) ; the epicycling approx ignores all terms of xz^2 or higher

Then define 2 new quantities:
$\kappa^2(R_G) = (\partial^2\Phi_{eff}/\partial R^2)$; $\nu^2(R_G) = (\partial^2\Phi_{eff}/\partial z^2)$; then keeping the lower orders $d^2x/dt^2 = -\kappa^2 x$; $d^2z/dz^2 = -\nu^2 z$; these are the harmonic oscillator eq's around x and z with frequencies κ and ν.
κ is the epicycle freq and ν the vertical frequency
this gives a vertical period $T = 2\pi/\nu \sim 6\times10^7$ yrs for the MW

EpiCycles B&T,S&G 3.3

- Remember the Oort constants??
- Well in the same limit (remember $v_{circle} = R\Omega(R)$)

Ω=A-B; κ^2 =-4B(A-B)=4BΩ ~ 2Ω^2 (eq 3.84); using the measured values of these constants one finds that near the sun κ_0^2= 37km/sec/kpc and the ratio of the freq of the suns orbit around the GC and the radial freq κ_0/Ω_0=sqrt(−B/(A−B))= 1.35

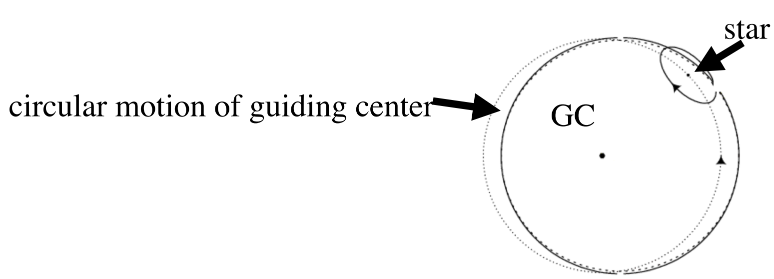
- Stellar orbits do not close on themselves in an inertial frame, but form a rosette figure like those discussed above for stars in spherically symmetric potentials

- The ratio ν^2/κ^2 ~3/2 $\rho/\langle\rho\rangle$ <u>a measure of how concentrated the mass is near the plane</u>
- The value of this approximation is in its ability to describe the motions of stars in the disk plane (does not work well for motion perpendicular to the plane) .
- The angular momentum on a circular orbit is $R^2\Omega(R)$;

  if it increases outward at radius R, the circular orbit is stable. This condition always holds for circular orbits in galaxy-like potentials.

Motion in Both Coordinates B&T 3.91-3.94

- $d^2x/dt^2=-\kappa^2x$; $d^2z/dz^2=-\nu^2z$; these are the harmonic oscillator eq's around x and z with frequencies κ and ν.
- and the general solution is
- $x(t)=C\cos(\kappa t+A)$; C>0 and A are arbitrary constants
- the solution for the φ direction is a bit messier and is
- $\varphi=(L_z/R^2_g)t-(\kappa/2B)(C/R_g)\sin(\kappa t+A)+\varphi_0$
- B&T go back to Cartesian coordinates (argh!) and define
- $y= -(\kappa/2B)C\sin(\kappa t+A)=Y\sin(\kappa t+A)$
- In the (x; y) plane the star moves on an ellipse called the epicycle around the guiding center



Epicycles

- Why did we go thru all that??
- Want to understand how to use stellar motions determine where the mass is.
- the orbits of stars take them through different regions of the galaxies -their motions at the time we observe them have been affected by the gravitational fields through which they have travelled earlier.
- use the equations for motion under gravity to infer from observed motions how mass is distributed in those parts of galaxies that we cannot see directly.

- The motions we have considered so far are the simplest !
- Using epicycles, we can explain the observed motions of disk stars near the Sun.

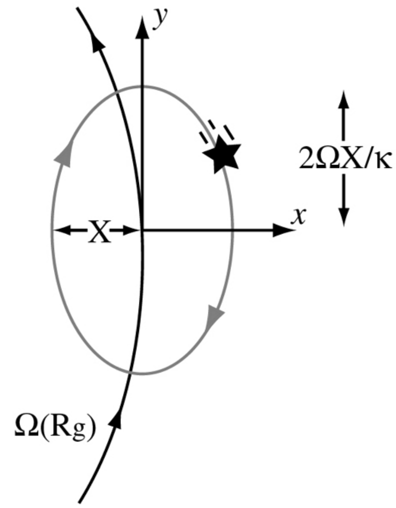


Fig 3.9 'Galaxies in the Universe' Sparke/Gallagher CUP 2007

Virial Theorem

- S+G pg 120-121, MBW 5.4.4, B&T pg 360
- A rather different derivation (due to H Rix)
- Consider (for simplicity) the 1-D Jeans eq in steady state (more later)
- $\partial/\partial x[\rho v^2]+\rho\partial\phi/\partial x=o$
- Integrate over velocities and then over positions...
- $-2E_{kin}=E_{pot}$

- or restating in terms of forces
- if T= total KE of system of N particles < >= time average
- $2\langle T\rangle=-\Sigma(F_k \bullet r_k)$; summation over all particles k=1,N

call the 'virial 'Q

$$Q=\frac{1}{2}\frac{dI}{dt}=m\sum r.\frac{dr}{dt}=\sum p.r$$

$$dQ/dt=\sum F.r+2T$$

Virial Theorem - Simple Cases

- Circular orbit: $mV^2/r=GmM/r^2$
- Multiply both sides by r $mV^2=GmM/r$
- $mV^2=2KE$; $GmM/=-W$ so $2KE+W=0$

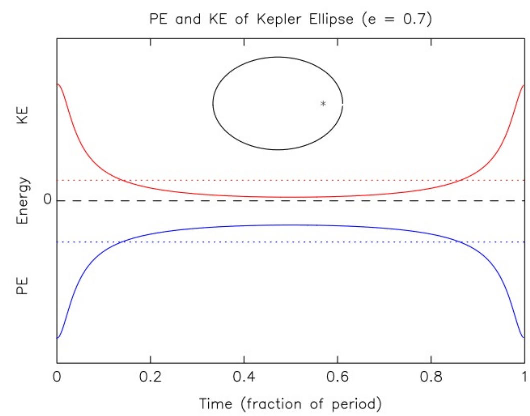
- Time averaged Keplerian orbit

define $U=KE/|W|$; as show in figure it clearly changes over the orbit; but

take averages

$,-W.=<GM/r>=GM<1/r>=GM(1/a)$

$KE=<1/2mV^2>=GM<1/r-1/2a>$
$=1/2GM(1/a)$ and again $2KE+W=0$



Red: kinetic energy (positive) starting at perigee
Blue: potential energy (negative)

Virial Theorem

- Another derivation following Bothun http://ned.ipac.caltech.edu/level5/Bothun2/Bothun4_1_1.html
- Moment of inertia, I, of a system of N particles
- $I=\Sigma m_i r_i^2$ sum over i=1,N (express r_i^2 as $(x_i^2+y_i^2+z_i^2)$)
- take the first and second time derivatives ; let dx^2/dt^2 be symbolized by ***x,y,z***
- $dI^2/dt^2 =\Sigma m_i (dx_i^2/dt+dy_i^2/dt+dz_i^2/dt)+\Sigma m_i(x_i \boldsymbol{x}+y_i\boldsymbol{y}+z_i\boldsymbol{z})$

mv^2 (KE)+Potential energy (W) r •(m**a**)

after a few dynamical times, if unperturbed a system will come into Virial equilibrium–time averaged inertia will not change so 2<T>+W=0

For self gravitating systems $W=-GM^2/2R_H$; is the harmonic radius- the sum of the distribution of particles appropriately weighted

$1/R_H =1/N \Sigma_i 1/r_i$

The virial mass estimator is $M=2\sigma^2 R_H/G$; for many mass distributions $R_H\sim1.25\ R_{eff}$

where R_{eff} is the half light radius σ is the 3-d velocity dispersion

Virial Thm MBW 5.4.4

- If I is the moment of inertia
- $1/2 d^2I/t^2 = 2KE+W+\Sigma$
  - where Σ is the work done by external pressure
  - KE is the kinetic energy of the system
  - w is the potential energy (only if the mass outside some surface S can be ignored)
- For a static system ($d^2I/t^2 = 0$)
  $2KE+W+\Sigma = 0$

Time Scales for Collisions

- N particles of radius $r_{p;}$ Cross section for a direction collision $\sigma_d = \pi r^2_p$
- Definition of mean free path; if V is the volume of a particle $4/3\pi r^3_p$
  $\lambda = V/n\sigma_d$ where n is the number density of particles (particles per unit volume)
  $n = 3N/4\pi r^3_p$

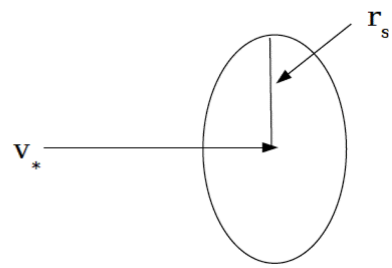
  and the characteristic time between collisions (Dim analysis)is
  $t_{collision} = \lambda/v \sim (\ell/r_p)^2 t_{cros}/N$ where v is the velocity of the particle.
  for a body of size ℓ, $t_{cross} = \ell/v$

  So lets consider a galaxy with $\ell \sim 10$kpc, $N=10^{10}$ stars and v~200km/sec
  if $r_p = R_{sun}$, $t_{collision} \sim 10^{21}$ yrs

- For indirect collisions the argument is more complex (see S+G sec 3.2.2, MWB pg 231) but the answer is the same - it takes a very long time for star interactions to exchange energy (relaxation).
- $t_{relax} \sim N t_{cross}/10 \ln N$
- Its only in the centers of the densest globular clusters and galactic nuclei that this is important

How Often Do Stars Encounter Each Other

For a 'strong' encounter $GmM/r>1/2mv^2$ e.g. potential energy exceeds KE
So a critical radius is $r<r_s=2GM/v^2$



Putting in some typical numbers $m\sim 1/2M_\odot$ $v=30$km/sec $r_s=1$AU
So how often do stars get that close?

consider a cylinder $Vol=\pi r^2_s vt$; if have n stars per unit volume than on average the encounter occurs when
$n\pi r^2_s vt=1$, $t_s=v^3/ 4\pi nG^2m^3$
Putting in typical numbers
$=4\times10^{12}(v/10km/sec)^3(m/M_\odot)^{-2}(n/pc3)^{-1}$ yr- a very long time (universe is only 10^{10}yrs old- galaxies are essentially collisionless

What About Collective Effects ? sec 3.2.2

For a weak encounter $b >> r_s$
Need to sum over individual interactions- effects are also small