Stellar Dynamics

- Stellar systems vs. gases
- Gravitational potential
- Spherical and disk potentials
- Orbits in the stellar neighborhood
- Orbits of single stars
- Orbits of stars in clusters
- The virial theorem
- Measuring masses from motions
- Effective potentials and epicycles
- Relaxation of orbits and encounters
- The Boltzmann equation

Stellar Systems vs. Gases

Similarities

- Comprise many interacting point-like objects
- Can be described by distribution functions of position and velocity
- Obey continuity equations (are not created or destroyed)
- Interactions and the systems as a whole obey conservation laws of energy and momentum
- Concepts like pressure and temperature apply

Differences

- Relative importance of short (gas) and long-range (stellar systems) forces
- Stars interact continuously with entire ensemble via long-range force of gravity
- Gases interct continuously via frequent, short-range, strong, elastic, repulsive collisions
- Stellar pairwise encounters are very rare
- Pressures in stellar systems can be anisotropic
- Stellar systems have negative specific heat and evolve away from uniform temperature
- Gases evolve toward uniform temperature and have positive specific heats

Potential Theory

Gravitational potential is a scalar field whose gradient gives the net gravitational force (per unit mass), a **vector field**.

$$-\Phi(\mathbf{r}) = G\int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r}'-\mathbf{r}|}d^3\mathbf{r}' = \frac{G\mathcal{M}(r)}{r} + 4\pi G\int_r^\infty \rho(r')r'dr' = \int_r^\infty \frac{G\mathcal{M}(r')}{r'^2}dr'$$

$$\frac{\mathbf{F}(\mathbf{r})}{m} = \frac{d\mathbf{v}}{dt} = -\nabla\Phi(\mathbf{r}) = G\int_V \rho(\mathbf{r}')\frac{\mathbf{r}'-\mathbf{r}}{|\mathbf{r}'-\mathbf{r}|^3}d^3\mathbf{r}' = -\frac{G\mathcal{M}(r)}{r^2} = -\frac{V_c(r)^2}{r}$$

By convention, $\Phi(\mathbf{r}) \to 0$ as $\mathbf{r} \to \infty$. Outside a spherically symmetric object, $\Phi(r) = -G\mathcal{M}/r$. Inside a spherically symmetric uniform density shell, $\Phi(r) = 0$. The divergence of $\mathbf{F}$ gives Poisson's equation:

$$-\frac{1}{m}\nabla\cdot\mathbf{F}(\mathbf{r}) = \nabla^2\Phi(\mathbf{r}) = 4\pi G\rho(\mathbf{r}).$$

Using Gauss' Theorem,

$$4\pi G\mathcal{M} = 4\pi G\int_V \rho(\mathbf{r})d^3\mathbf{r} = -\frac{1}{m}\int_V \nabla\cdot\mathbf{F}(\mathbf{r})d^3\mathbf{r} = -\frac{1}{m}\int_A \mathbf{F}(\mathbf{r})\cdot d^2\mathbf{S}$$

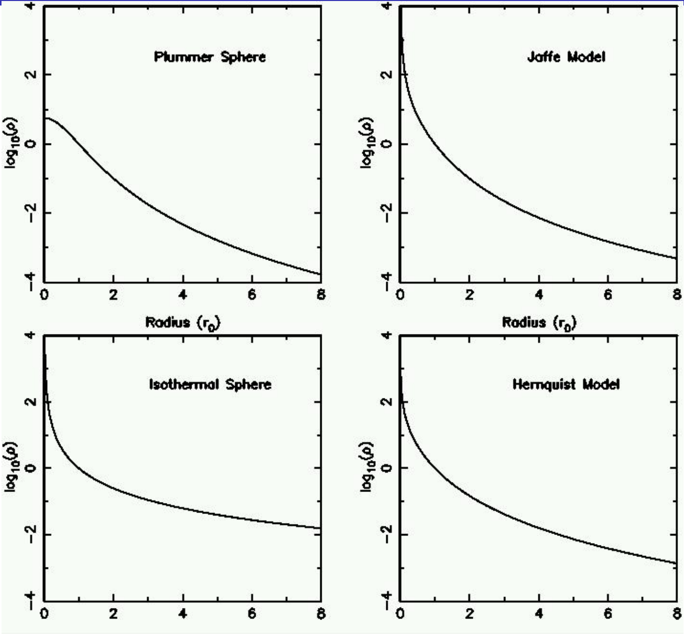
Gravitational potential energy (last equality for spherical symmetry)

$$W = \frac{1}{2}\int_V \rho(\mathbf{r})\Phi(\mathbf{r})d^3\mathbf{r} = -\frac{1}{8\pi G}\int_V |\nabla\Phi|^2 d^3\mathbf{r} = -G\int_{\mathcal{M}} \frac{\mathcal{M}(r)}{r}d\mathcal{M}.$$

Analytic Density-Potential Pairs in Spherical Symmetry

- Homogenous sphere (radius R, $\rho(r<R)=C$)
  Inside: $\Phi(r) = -2\pi G(R^2 - r^2/3)$, $\quad F(r) = -G\mathcal{M}(r)/r^2$
  $V_c^2 = G\mathcal{M}(r)/r = 4\pi G\rho r^2/3$ ($\omega(r) =$ constant).
- Singular isothermal sphere ($\rho(r) = \rho_o r_o^2/r^2$)
  $\Phi(r) = 4\pi G\rho_o r_o^2 \ln r + C$, $\quad \mathcal{M}(r) = 4\pi\rho_o r_o^2 r$, $\quad V_c^2 = 4\pi G\rho_0 r_o^2$.
- Power law ($\rho(r) = \rho_o(r/r_o)^{-\alpha}, 2<\alpha<3$)
  $\Phi(r) = -4\pi G\rho_o r_o^2 (r/r_o)^{2-\alpha}/[(3-\alpha)(\alpha-2)]$,
  $\mathcal{M}(r) = 4\pi G\rho_o r_o^3 (r/r_o)^{3-\alpha}/(3-\alpha)$.
- Hernquist ($\rho(r) = \mathcal{M}a/[2\pi r(r+a)^3]$)
  $\Phi(r) = -G\mathcal{M}/(r+a)$, $\quad \mathcal{M}(r) = \mathcal{M}r^2/(r+a)^2$.
- Jaffe ($\rho(r) = \mathcal{M}a/[4\pi r^2(r+a)^2]$)
  $\Phi(r) = -(G\mathcal{M}/a)\ln(1+a/r)$, $\quad \mathcal{M}(r) = \mathcal{M}r/(r+a)$.
- Plummer ($\rho(r) = 3a^2\mathcal{M}/[4\pi(r^2+a^2)^{5/2}]$)
  $\Phi(r) = -G\mathcal{M}/\sqrt{r^2+a^2}$, $\quad \mathcal{M}(r) = \mathcal{M}r^3/(r^2+a^2)^{3/2}$.
- Navarro-Frenk-White ($\rho(r) = \rho_N a^3/[r(r+a)^2]$)
  $\Phi(r) = -4\pi G\rho_N a^3 r^{-1}\ln(1+r/a)$,
  $\mathcal{M}(r) = 4\pi\rho_N a^3[\ln(1+r/a) - r/(r+a)]$.

Density Laws



Orbits of Single Stars

$$\mathbf{v}\cdot\left[\frac{d\mathbf{v}}{dt}+\nabla\Phi(\mathbf{r})\right]=0=\frac{d}{dt}\left[\frac{1}{2}\mathbf{v}^2+\Phi(\mathbf{r})\right].$$

Star's energy $\mathcal{E}$ is therefore constant, where

$$\mathcal{E}=\frac{m}{2}\mathbf{v}^2+m\Phi(\mathbf{r})=\mathrm{~KE~}+\mathrm{~PE~}.$$

A star can escape only with $\mathcal{E}>0$ since KE>0, thus

$$V_{esc}^2(\mathbf{r})=-2\Phi(\mathbf{r}).$$

Changes to the star's angular momentum $\mathcal{L}=m\mathbf{r}\times\mathbf{v}$:

$$\frac{d\mathcal{L}}{dt}=\frac{d}{dt}m\mathbf{r}\times\mathbf{v}=m\mathbf{r}\times\frac{d\mathbf{v}}{dt}=-m\mathbf{r}\times\nabla\Phi(\mathbf{r}).$$

For spherical symmetry, $\mathcal{L}$ is therefore conserved.
In a stellar system, individual stellar energies or angular momenta are not conserved, but their sums are.

The Virial Theorem

Newton's Law of Gravity $d(m\mathbf{v})/dt = -Gm\mathcal{M}\mathbf{r}/r^3$.

$$\frac{d}{dt}(m_i\mathbf{v}_i) = -\sum_{j\neq i}\frac{Gm_im_j}{|\mathbf{r}_i-\mathbf{r}_j|^3}(\mathbf{r}_i-\mathbf{r}_j).$$

$$\sum_i\frac{d}{dt}(m_i\mathbf{v}_i\cdot\mathbf{r}_j) = -\sum_i\sum_{j\neq i}\frac{Gm_im_j}{|\mathbf{r}_i-\mathbf{r}_j|^3}(\mathbf{r}_i-\mathbf{r}_j)\cdot\mathbf{r}_i+\sum_i\mathbf{F}_i\cdot\mathbf{r}_i.$$

$$\sum_j\frac{d}{dt}(m_j\mathbf{v}_j\cdot\mathbf{r}_j) = -\sum_i\sum_{j\neq i}\frac{Gm_im_j}{|\mathbf{r}_i-\mathbf{r}_j|^3}(\mathbf{r}_j-\mathbf{r}_i)\cdot\mathbf{r}_i+\sum_j\mathbf{F}_j\cdot\mathbf{r}_j.$$

$$\sum_i\frac{d}{dt}(m_i\mathbf{v_i}\cdot\mathbf{r}_j) = \frac{1}{2}\sum_i\frac{d^2}{dt^2}(m_i\mathbf{r}_i\cdot\mathbf{r}_i)-\sum_i m_i\mathbf{v}_i\cdot\mathbf{v}_i = \frac{1}{2}\frac{d^2I}{dt^2}-2\ \mathrm{KE}.$$

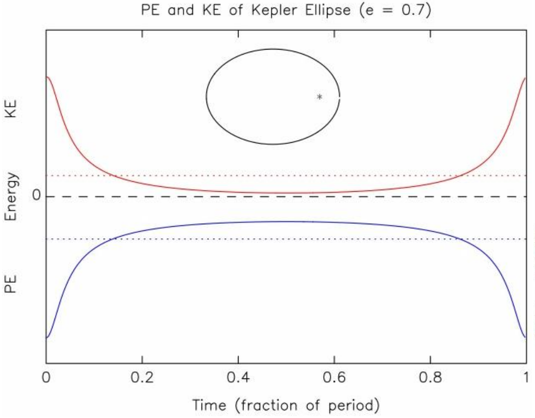
Add, divide by 2: $\frac{1}{2}\frac{d^2I}{dt^2}-2\ \mathrm{KE}\ =\ \mathrm{PE}\ +\sum\mathbf{F}_i\cdot\mathbf{r}_i$

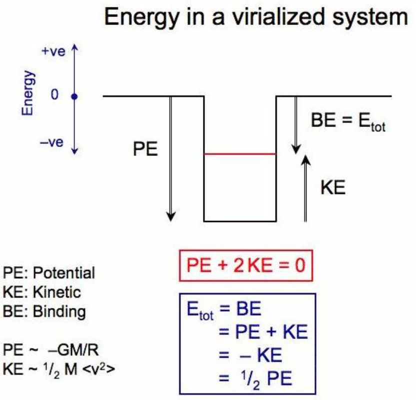
Average: $\frac{1}{2\tau}\left[\frac{dI}{dt}(\tau)-\frac{dI}{dt}(0)\right] = 2\overline{\mathrm{KE}}+\overline{\mathrm{PE}}+\sum_i\overline{\mathbf{F}_i\cdot\mathbf{r}_i}\ \underset{\tau\to\infty}{\longrightarrow}\ 0$

Compare to 2KE + PE $-3P_oV = 0$.

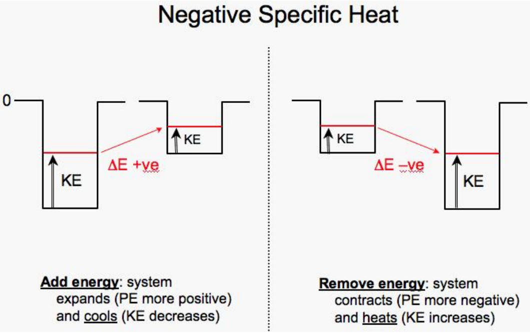
Validity:

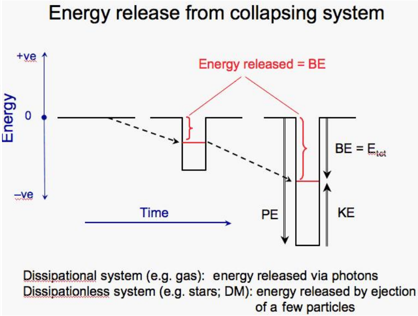
- Self-gravitating
- Steady state
- Time-averaged (or many objects)
- Isolated (or slowly varying potential)





Virial Theorem and Energy Changes





Measuring Masses

Assume uniformity of the mass-to-light ratio $\mathcal{M}/L$ in the system. The surface brightness $I(\mathbf{x}) = L/D^2$ is the surface luminosity density.
The surface mass density is the projection of ρ along the line-of-sight $z = \sqrt{r^2 - R^2}$, with R the impact parameter. Using the Plummer model,

$$\Sigma(R) = \int_{-\infty}^{\infty} \rho(r(z))dz = 2\int_0^{\infty} \frac{3a^2\mathcal{M}dz}{4\pi(a^2+z^2+R^2)^{5/2}} = \frac{\mathcal{M}a^2}{\pi(a^2+R^2)^2}.$$

The parameter a can be inferred from $I(x)$:

$$\frac{I(r_c)}{I(0)} = \frac{1}{2} = \frac{\Sigma(r_c)}{\Sigma(0)} = a^4/(a^2+r_c^2)^2, \quad a = \frac{r_c}{\sqrt{\sqrt{2}-1}} \simeq 1.55 r_c.$$

Kinetic energy $$\mathrm{KE} = \frac{1}{2}\sigma^2\mathcal{M} = -\frac{1}{2}\mathrm{PE} = \frac{3G\mathcal{M}^2}{64\pi a}$$

for the Plummer model. One measures velocity dispersion σ^2 averaging radial velocities v_r relative to the system's mean motion : $\sigma_r^2 =< v_r^2 >$. Tangential motions are undetectable. Typically $\sigma_r \sim 10$ km s^{-1} and v_r errors are about 0.5 km s^{-1}. For isotropy, $\sigma^2 = \sigma_{\mathbf{i}} \cdot \sigma_{\mathbf{i}} = 3\sigma_r^2$. Thus

$$\mathcal{M} = \frac{32a\sigma_r^2}{\pi G}.$$

Measuring Masses

An alternate method makes use of a measurement of the total luminosity

$$L_{tot} = 2\pi \int_0^\infty I(R) R dR = 2\pi \frac{I(0)}{\Sigma(0)} \int_0^\infty \Sigma(R) R dR$$
$$= 2\pi I(0) a^4 \int_0^\infty \frac{R dR}{(a^2 + R^2)^2} = \pi I(0) a^2,$$

giving $a = \sqrt{L_{tot}/(\pi I(0))}$.
To find the average motion of one star within $\Phi(\mathbf{x})$, the gravity of all other stars gives a net external force. Then

$$\langle \mathbf{v}^2 \rangle = \langle \nabla \Phi(\mathbf{x}) \cdot \mathbf{x} \rangle.$$

Assuming the Galaxy's mass is spherically symmetrically distributed within the location of an object located far from the Galactic center and the Sun, so $r \simeq d$, one may compute $\mathcal{M}_G = v^2 r/G$ where $v = v_{r,\odot} + V_0 \sin\ell \cos b$ converts the radial velocity relative to the Sun to the velocity relative to the Galactic center.

Circular Motion – Reprise

Oort's A (shear) and B (vorticity) constants are defined in terms of circular rotation:

$$A = \frac{1}{2}\left(\frac{V_c}{R} - \frac{dV_c}{dR}\right)_{R_0} = -\frac{R}{2}\left(\frac{d\Omega}{dR}\right)_{R_0}$$

$$B = -\frac{1}{2}\left(\frac{V_c}{R} + \frac{dV_c}{dR}\right)_{R_0} = -\frac{R}{2}\left(\frac{d\Omega}{dR} + \frac{2\Omega}{R}\right)_{R_0}$$

$A \simeq 15$ km s^{-1} kpc^{-1}, $B \simeq -12$ km s^{-1} kpc^{-1}. Note that

$$A + B = -\left(\frac{dV_c}{dR}\right)_{R_0}, \qquad A - B = \left(\frac{V_c}{dR}\right)_{R_0} = \Omega_0$$

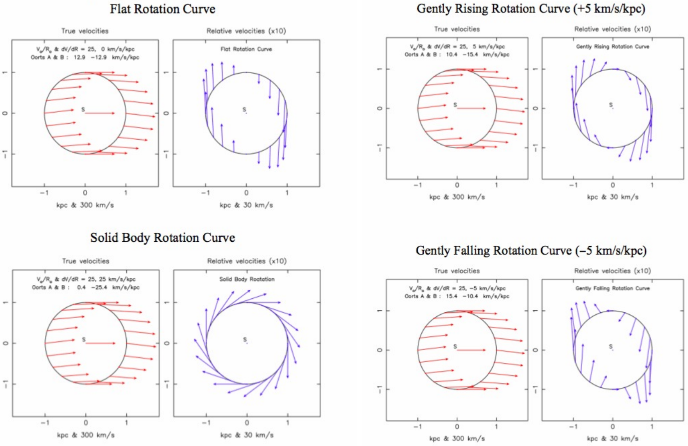
Rotation curve is fairly flat, the Sun's orbital period is $P_0 = 2\pi/\Omega_0 = 230$ Myr, and it's circular velocity is $V_0 = \Omega_0 R_0 = 220$ km s^{-1}. These results assume circular orbits, which is not actually the case in detail.

Circular Motion – Reprise

Objects close to the Sun have radial and tangential velocities

$$v_r = Ar\sin 2\ell, \qquad v_t = r(B + A\cos 2\ell).$$

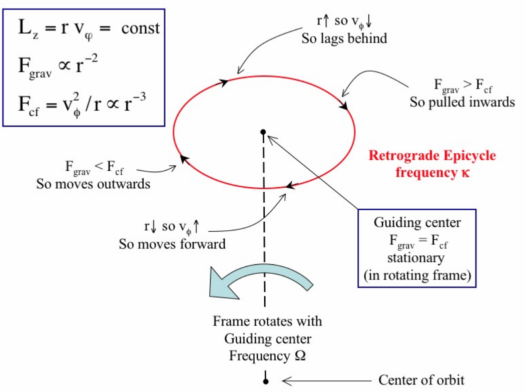
Here are some examples of circular disk rotation, with full velocities shown on the left (red vectors), and the differential velocity on the right (blue vectors). Each pair is for a particular rotation curve gradient near the sun (dV/dR). The galactic center is at y = –8.5 kpc, and V_{sun} = 220 km/s. The circle is 1 kpc in radius.



Epicycles

Stellar orbits in a rotating galaxy can be described by superposition of a background circular motion (guiding center at R_g with Ω_g) and an elliptical epicycle with angular velocity κ_g.

Consider the motion in a rotating frame. For a Keplerian potential ($\Omega_g \propto R_g^{-3/2}$), the orbit and epcicyclic frequencies are the same, $\kappa_g = \Omega_g$. The orbit is closed, an off-centered ellipse. In general $\kappa_g \neq \Omega_g$ so orbits don't close unless viewed from a frame rotating at $\Omega_g - \kappa_g/2$.



phases same

phases advance with radius

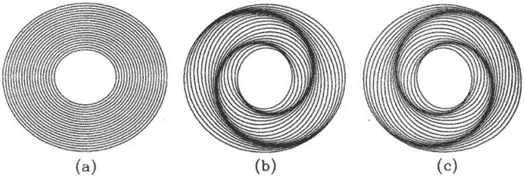


Figure 6-11. Arrangement of closed orbits in a galaxy with $\Omega - \frac{1}{2}\kappa$ independent of radius, to create bars and spiral patterns (after Kalnajs 1973).

Axisymmetric Geometry

Equations of Motion:

$$\ddot{\mathbf{r}} = -\nabla\Phi(R,z)$$

$$L_z = R^2\dot{\phi} = \text{ constant}$$

$$\ddot{R} = R\dot{\phi}^2 - \partial\Phi/\partial r, \quad \ddot{z} = -\partial\Phi/\partial z$$

Vertical (z) motions:

$$(\partial\Phi/\partial z)_{z=0} = 0$$

$$\ddot{z} = -\left(\frac{\partial\Phi}{\partial z}\right)_{z=0} - z\left(\frac{\partial^2\Phi}{\partial z^2}\right)_{z=0}$$

$$= -z\left(\frac{\partial^2\Phi}{\partial z^2}\right)_{z=0} = -\nu^2 z$$

$$z(t) = Z\cos(\nu t + \psi_0)$$

From Poisson's equation:

$$4\pi G\rho(R,0) \simeq (dV_c^2/dR)/R + \nu^2 \simeq \nu^2$$

Radial (R) motions:

$$\left(\frac{\partial\Phi}{\partial R}\right)_{R_g} = \frac{V_c^2}{R_g} = R_g\Omega_g^2$$

$$\ddot{R} = -\frac{\partial}{\partial R}\left[\Phi + \frac{L_z^2}{2R^2}\right] \equiv -\frac{\partial\Phi_{eff}}{\partial r}$$

$$(\partial\Phi_{eff}/\partial R)_{R_g} = (\partial\Phi/\partial R)_{R_g} - L_z^2/R_g^3$$
$$= R_g\Omega_g^2 - V_c^2/R_g = 0$$

$$\ddot{\mathbf{x}} = -\left(\frac{\partial\Phi_{eff}}{\partial R}\right)_{R_g} - z\left(\frac{\partial^2\Phi_{eff}}{\partial R^2}\right)_{R_g} = -\kappa_g^2 z$$

$$\kappa_g^2 = \left(\frac{\partial^2\Phi_{eff}}{\partial R^2}\right)_{R_g} = \left(\frac{\partial^2\Phi}{\partial R^2}\right)_{R_g} + \frac{3L_z^2}{R_g^4}$$

$$= \left(R\frac{d\Omega^2}{dR} + 4\Omega^2\right)_{R_g}$$

$$R = R_g + x, \quad x(t) = X\cos(\kappa_g t + \phi_0)$$

Axisymmetric Geometry

Azimuthal motions:

$$\dot{\phi} = \frac{L_z}{R^2} = \frac{L_z}{(R_g + x)^2}$$

$$\simeq \frac{L_z}{R_g^2}\left(1 - \frac{2x}{R_g}\right) = \Omega_g\left(1 - \frac{2x}{R_g}\right)$$

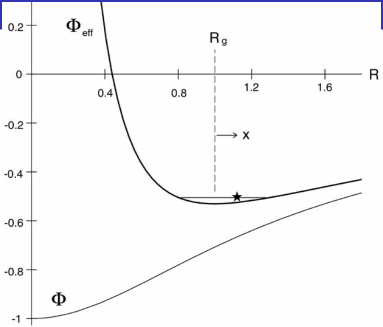
$$\phi(t) = \Omega_g t - \frac{2\Omega_g X}{\kappa_g R_g}\sin(\kappa_g t + \phi_0)$$

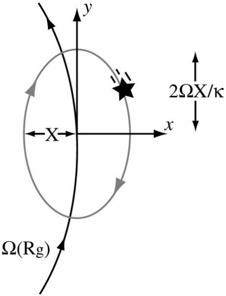
$$y(t) = -\frac{2\Omega_g}{\kappa_g} X \sin(\kappa_g t + \phi_0)$$

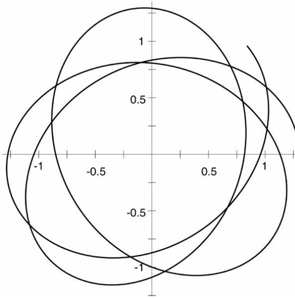
$$x(t) = X\cos(\kappa_g t)$$

$$y(t) = -\frac{2\Omega_g}{\kappa_g} X \sin(\kappa_g t)$$

Motion is retrograde. For Keplerian potential, $\kappa_g = \Omega_g$. For flat rotation, $\Omega_g \propto R_g^{-1}$, $\kappa_g = \sqrt{2}\Omega_g$. For solid rotation, Ω_g constant, $\kappa_g = 2\Omega_g$ (circular and closed).







Values in the Solar Neighborhood

In terms of Oort's constants:

$\kappa_0^2 = -4B(A-B) = -4B\Omega_0$

$\kappa_0 \simeq 37 \text{ km s}^{-1} \text{ kpc}^{-1} = 0.037\ Myr^{-1}$

$\nu_0 \simeq 96 \text{ km s}^{-1} \text{ kpc}^{-1} = 0.096\ Myr^{-1}$

$\Omega_0 = A - B \simeq 27 \text{ km s}^{-1} \text{ kpc}^{-1}$

Since $\kappa_0/\Omega_0 \approx 1.4$, solar neighborhood stars make 1.4 epicyclic rotations per orbit; the orbit appears to regress.

The azimuthal/radial extent of epicycles is $2\Omega_0/\kappa_0 \approx 1.46$.

The mean-square azimuthal/radial velocities at R_g: $\overline{\dot{y}^2}/\overline{\dot{x}^2} = 4\Omega_0^2/\kappa_0^2$.

But the azimuthal/radial velocity dispersion near the Sun is actually $\sigma_{\phi,0}^2/\sigma_{R,0}^2 = \kappa_0^2/4\Omega_0^2 \approx 0.47$ because this is measured at R_0.

Epicycle size $X \approx \sigma_R/\kappa, Z \approx \sigma_z/\nu$.

$\sigma_R \sim 30$ km s^{-1} implies $X \sim 1$ kpc.
$\sigma_z \sim 30$ km s^{-1} and
$\nu = \sqrt{4\pi G\rho_0} \sim 0.1$ Myr^{-1} $\sim 3\Omega_0$
implies $Z \sim 300$ pc.

The Sun is at $z_\odot = 40$ pc with $v_{z,\odot} = 7$ km s^{-1}, suggesting that $Z_\odot \simeq 80$ pc.

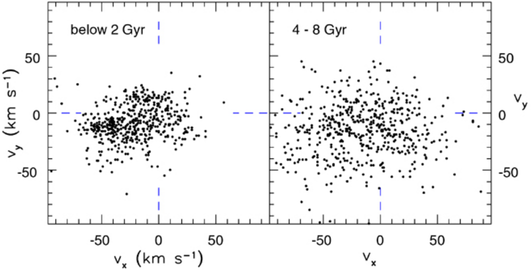


Fig 3.11 'Galaxies in the Universe' Sparke/Gallagher CUP 2007

Velocity Dispersion Near the Sun

Epicyclic trajectories in rest frame at R_g:

$$x(t) = X\cos(\kappa_g t) \qquad y(t) = -\frac{2\Omega_g}{\kappa_g} X \sin(\kappa_g t)$$

At R_0, the azimuth obeys $\phi = \Omega_g t + y(t)/R_g$. Relative to circular motion at R_0:

$$v_y = v_\phi - v_c = R_0(\dot\phi - \Omega_0) = R_0\left[\dot\phi - \Omega_g - x(t)\left(\frac{d\Omega}{dR}\right)_{R_g}\right]$$

$$\simeq -R_0 x(t)\left(\frac{2\Omega}{R} + \frac{d\Omega}{dR}\right)_{R_0} = 2Bx(t) = -\frac{\kappa_0^2}{2\Omega_0}x(t)$$

We also have $v_x = v_R = \dot x(t)$. Then

$$\frac{\langle v_y^2\rangle}{\langle v_x^2\rangle} = \frac{\sigma_y^2}{\sigma_x^2} \simeq \frac{\kappa_0^2}{4\Omega_0^2}$$

We ignored that the density of stars decreases with R. There should be more stars in the solar neighborhood on the outer parts of their epicycles, with $x > 0$, than the inner, with $x < 0$. Therefore $\langle v_y\rangle < 0$, which is called asymmetric drift. The effect is enhanced in older stars, those with velocities further removed from circular motion.

Axisymmetric, Flattened Potentials

Kuzmin disk An infinitely thin sheet of mass $\mathcal{M}$.

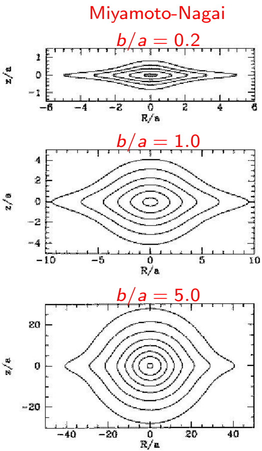
$$\Phi(R,z) = -\frac{G\mathcal{M}}{\sqrt{R^2+(a+|z|)^2}},$$

$$\Sigma(R) = \frac{1}{2\pi G}\left(\frac{\partial\Phi}{\partial z}\right)_{z=0} = \frac{a\mathcal{M}}{2\pi(R^2+a^2)^{3/2}}$$

Miyamoto-Nagai disk $b=0$ is a Kuzmin disk,
$a=0$ is a Plummer sphere.

$$\Phi(R,z) = -\frac{G\mathcal{M}}{\sqrt{R^2+(a+\sqrt{z^2+b^2})^2}}$$

$$\rho(R,z) = \frac{b^2\mathcal{M}}{4\pi}\frac{aR^2+(a+3\sqrt{z^2+b^2})(a+\sqrt{z^2+b^2})^2}{(z^2+b^2)^{3/2}[R^2+(a+\sqrt{z^2+b^2})^2]^{5/2}}$$



Satoh disk

$$\Phi(R,z) = \frac{-G\mathcal{M}}{S} \equiv \frac{-G\mathcal{M}}{\sqrt{R^2+(a+\sqrt{z^2+b^2})^2-b^2}}$$

$$\rho(R,z) = \frac{b^2\mathcal{M}}{4\pi S^3(z^2+b^2)}\left[\frac{a}{\sqrt{z^2+b^2}}+3-3\frac{R^2+z^2}{S^2}\right].$$

Stellar Encounters

Although the overall galactic potential Φ is smooth, on small scales it has deep potential wells around each star. Encounters aren't as catastrophic as collisions, and don't affect the overall motion of a star as much as the overall smoot potential, but are extremely important in changing an individual star's motion and randomizing the overall velocity distribution.
We distinguish between tidal capture ($b < 3r_{star}$), strong encounters ($b < r_s, \Delta V \simeq V$), in which the potential energy at closest approach is larger than the initial kinetic energy, and weak encounters ($b >> r_s, \Delta V << V$), when it is less. The strong encounter radius is

$$r_s = 2Gm/V^2 \simeq 1 \text{ AU}$$

where $m \sim 0.5 \mathcal{M}_\odot$ is a stellar mass and $V \sim 30$ km s^{-1} is the initial relative velocity.
Had this happened to the Sun since its formation, the orbits of the planets would have been disrupted. The time between close encounters is

$$\begin{aligned} t_s &\simeq (\pi r_s^2 V n)^{-1} = V^3/(4\pi G^2 m^2 n) \\ &\simeq 4 \times 10^{12} \text{ yr} \left(\frac{V}{10 \text{ km s}^{-1}}\right)^3 \left(\frac{m}{\mathcal{M}_\odot}\right)^{-2} \left(\frac{n}{\text{pc}^{-3}}\right)^{-1}. \end{aligned}$$

Encounter Geometry

Distant weak encounters

Use the impulse approximation, ignoring the deviation in the stellar paths. The impact parameter is b. The perpindicular pull of star m on star $\mathcal{M}$ is $Gm\mathcal{M}/r^2$ times b/r, with $r^2 = b^2 + V^2t^2$:

$$F_\perp(t) = \frac{Gm\mathcal{M}b}{(b^2 + V^2t^2)^{3/2}} = \mathcal{M}\frac{dV_\perp}{dt}$$

Deflection angle:

$$\theta = \frac{\Delta V_\perp}{V} = \frac{1}{\mathcal{M}V}\int_{-\infty}^{+\infty} F_\perp dt = \frac{2Gm}{bV^2}$$

After many encounters

$$\langle \Delta V_\perp^2 \rangle = \int_{b_{min}}^{b_{max}} nVt\left(\frac{2Gm}{bV}\right)^2 2\pi b db$$

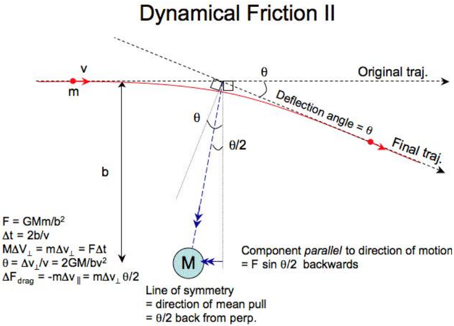
$$= \frac{8\pi G^2 m^2 nt}{V}\ln\frac{b_{max}}{b_{min}} = \frac{8\pi G^2 m^2 nt}{V}\ln\Lambda$$

$\langle \Delta V_\perp^2 \rangle = V^2$ when $t = t_{relax}$:

$$t_{relax} = \frac{V^3}{8\pi nG^2m^2\ln\Lambda} = \frac{t_s}{2\ln\Lambda}$$

$$\Lambda = \frac{b_{max}}{b_{min}} \sim \frac{R}{r_s} = \frac{0.3 - 30\ \mathrm{kpc}}{1\ \mathrm{AU}}$$

$\ln\Lambda = 18 - 22$



Relaxation Applications

If integration is instead performed over a Maxwellian velocity distribution, t_{relax} increases by a factor of 8 (replace $1/8\pi$ by 0.34).

$$t_{relax} \simeq \frac{2\times10^{10}\ \mathrm{yr}}{\ln\Lambda}\left(\frac{V}{10\ \mathrm{km\ s^{-1}}}\right)^3\left(\frac{\mathcal{M}_\odot}{m}\right)^2\left(\frac{10^3\ \mathrm{pc}^{-3}}{n}\right)$$

- For the Sun, $t_{relax} \sim 10^{12}$ yr.
- ω Cen has $N = 10^5$, $t_{relax} \sim 0.5$ Gyr and $t_{cross} \sim 0.5$ Myr. On crossing times, stars are little affected by encounters. But over its lifetime, ω Cen has been modified by relaxation.
- Open clusters, have lower densities and random velocities: $N = 100$, $t_{relax} \sim 10$ Myr, $t_{cross} \sim 1$ Myr. Have to include effects of stellar evolution and mass loss to simulate evolution of open clusters.
- Elliptical Galaxy: $N = 10^{11}$, $t_{relax} = 4\times10^{16}$ yr, $t_{cross} = 10^8$ yr.

For a virialized system of size R with N stars moving with an average V:

$$\frac{N}{2}mV^2 = \frac{G(Nm)^2}{2R}, \quad \Lambda = \frac{R}{r_s} = \frac{GmN}{V^2}\cdot\frac{V^2}{2Gm} = \frac{N}{2}$$

With $t_{cross} = R/V$ and $4\pi n = 3N/R^3$

$$\frac{t_{relax}}{t_{cross}} = \frac{V^3}{8\pi nG^2m^2\ln\Lambda}\cdot\frac{V}{R} = \frac{V^4R^2}{6NG^2m^2\ln\Lambda} = \frac{N}{6\ln(N/2)}$$

Evaporation

Without collisions, $\Phi(\mathbf{x})$ does not change. But encounters alter the energies of individual stars, preferentially removing energy from massive stars. On average, encounters shuffle velocities toward a Maxwellian distribution

$$f(\mathcal{E}) \propto \exp\left(-\left[m\Phi(\mathbf{X}) + \frac{mv^2}{2}\right]/kT\right)$$

for equal mass stars. The effective temperature is

$$m\langle v^2(\mathbf{x})\rangle/2 = 3kT/2.$$

More massive stars move less rapidly. At the upper end of the velocity distribution, stars achieve escape velocity:

$$\langle\frac{1}{2}mv_e^2(\mathbf{x})\rangle = -\frac{1}{N}\sum_i m_i\Phi(\mathbf{x}_i) = -\frac{2}{N}\,\mathrm{PE} = \frac{4}{N}\,\mathrm{KE}.$$

This means escaping stars satisfy $v_e^2 \geq 12kT/m$. Note that the fraction, at any given time, of stars capable of escaping is

$$\frac{\int_{v_e}^{\infty} f(\mathcal{E})v^2dv}{\int_0^{\infty} f(\mathcal{E})v^2dv} = 0.0074 \approx \frac{1}{136}.$$

Thus $t_{evap} = 136t_{relax}$.

Mass Segregation

As massive stars (and binaries) lose energy, they sink to the center; light stars migrate outwards. In addition, the stars near the center gain velocity, so stars near the center tend to lose energy even faster.

Mass segregation is a runaway process, leading to *core collapse* after $12-20t_{relax}$. Note the too-small-to-see dense core in M15.

Encounters with binaries lead to energy losses from binaries; they become tighter. Release of energy from binaries ("binary burning") can halt or reverse core contraction.

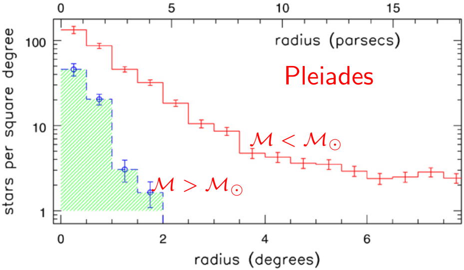


Fig 3.6 (J. Adams) Galaxies in the Universe' Sparke/Gallagher CUP 2007

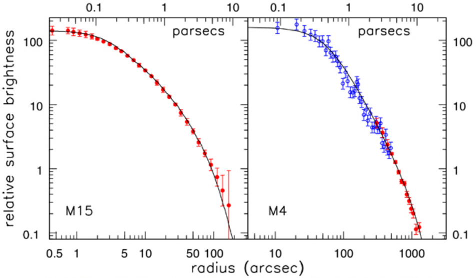
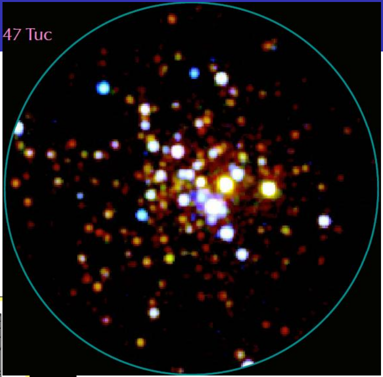
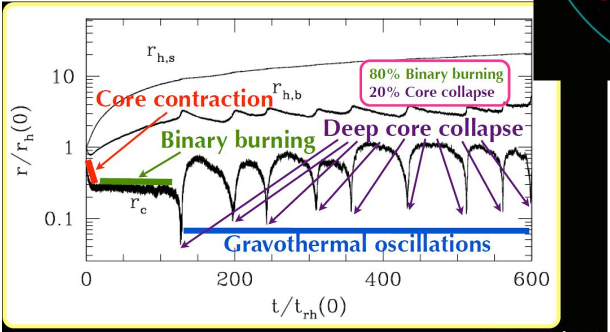


Fig 3.7 (Pasquali, Fahlman, Pryor) 'Galaxies in the Universe' Sparke/Gallagher CUP 2007

X-ray Sources





Fregeau et al. 2003

Collisionless Flows

Assume all stars have the same mass m and ignore encounters (collisions). The distribution function $f(\mathbf{x}, \mathbf{v}, t)$ is the probability density in phase space, so that the number density at position $\mathbf{x}$ and time t is

$$n(\mathbf{x}, t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\mathbf{x}, \mathbf{v}, t) dv_x dv_y dv_z.$$

Begin with 1-D, and the concepts that no stars are created or destroyed in the flow and stars don't jump across phase space (no deflective encounters). The net flow in x:

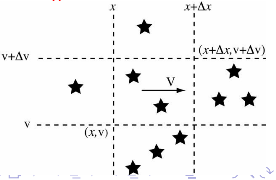
$$\frac{dx}{dt} dt dv_x [f(x, v_x, t) - f(x + dx, v_x, t)] = -dt dv_x v_x \frac{\partial f}{\partial x} dx.$$

The net flow due to the velocity gradient:

$$dx dt \frac{dv_x}{dt} [f(x, v_x, t) - f(x, v_x + dv_x, t)] = -dt dx \frac{dv_x}{dt} \frac{\partial f}{\partial} dv_x.$$

Adding:

$$dx dv_x dt \frac{\partial f}{\partial t} = -dt dx dv_x \left[v_x \frac{\partial f}{\partial x} + \frac{dv_x}{dt} \frac{\partial f}{\partial v_x} \right].$$



$$0 = \frac{\partial f}{\partial t} + v_x \frac{\partial f}{\partial x} + \frac{dv_x}{dt} \frac{\partial f}{\partial v_x} = \frac{\partial f}{\partial t} + v_x \frac{\partial f}{\partial x} - \frac{\partial \Phi}{\partial x} \frac{\partial f}{\partial v_x}$$

Collisionless Boltzmann Equation

Extending this to 3-D (other dimensions are independent) gives the CBE

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f - \nabla \Phi \cdot \frac{\partial f}{\partial \mathbf{v}} = 0.$$

This has followed from: 1. conservation of stars; 2. smooth orbits; 3. flow through $\mathbf{r}$ implicitly defines $\mathbf{v}$; 4. flow through $\mathbf{v}$ given by -$\nabla\Phi$. It can also be written with a convective (or total or Lagrangian) derivative instead of an Eulerian one:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial \mathbf{x}} \cdot \frac{d\mathbf{x}}{dt} + \frac{\partial f}{\partial \mathbf{v}} \cdot \frac{d\mathbf{v}}{dt} = 0.$$

This is incompressible flow. Think of a traffic jam: in a dense region, σ increases; in a rarefied region, σ decreases. It also applies to all sub-populations of stars (e.g., spectral classes) even though no one class determines Φ. A self-consistent field can be introduced which itself generates Φ.

Jeans Equations

The CBE is of limited use; what we observe are averages (e.g., $\langle v^2 \rangle$). These can be extracted using moments. The number density is the zeroth moment, the mean velocity is the first moment:

$$n(\mathbf{r}, t) = \int f(\mathbf{r}, \mathbf{v}, t) d^3 v, \qquad \langle v_i(\mathbf{r}, t) \rangle = \frac{1}{n} \int v_i f(\mathbf{r}, \mathbf{v}, t) d^3 v.$$

0th moment CBE in 1-D:

$$\frac{\partial n}{\partial t} + \frac{\partial (n \langle v_x \rangle)}{\partial x} = 0.$$

1st moment CBE in 1-D:

$$\frac{\partial \langle v_x \rangle}{\partial t} + \langle v_x \rangle \frac{\partial \langle v_x \rangle}{\partial x} = -\frac{\partial \Phi}{\partial x} - \frac{1}{n} \frac{\partial (n \sigma_x^2)}{\partial x}$$

where $\sigma_x^2 = \langle v_x^2 \rangle - \langle v_x \rangle^2$. You can show in 3-D ($\sigma_{i,j}$ is the stress tensor, representing an anisotropic pressure):

$$\frac{\partial \langle v_j \rangle}{\partial t} + \langle v_i \rangle \frac{\partial \langle v_j \rangle}{\partial i} = -\frac{\partial \Phi}{\partial x_j} - \frac{1}{n} \frac{\partial (n \sigma_{i,j}^2)}{\partial x_i}.$$

Compare to the Euler Equation for fluid flow, which has, however, $p(\rho)$:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla \Phi - \frac{1}{\rho} \nabla p$$

Applications of the Jeans Equations

- Deriving $\mathcal{M}/L$ profiles in spherical galaxies
- Determining of the surface and volume densities of the Galactic disc
- Deriving the flattening of a rotating spheroid with isotropic velocity dispersion
- Analysis of asymmetric drift
- Analysis of the local velocity ellipsoid in terms of Oort's constants

In spherical symmetry, $\langle v_r \rangle = \langle v_\theta \rangle = 0$, $\langle v_i^2 \rangle = \sigma_i^2 (i, j, k = r, \theta, \phi)$.

$$\frac{1}{n}\frac{d(n\sigma_r^2)}{dr} + \frac{1}{r}\left[2\sigma_r^2 - \sigma_\theta^2 - \sigma_\phi^2\right] - \frac{\langle v_\phi \rangle^2}{r} = -\frac{d\Phi}{dt}$$

Define $\beta = 1 - (\sigma_\theta^2 + \sigma_\phi^2)/(2\sigma_r^2)$, $V_{rot} = \langle v_\phi \rangle$,

$$\frac{1}{n}\frac{d(n\sigma_r^2)}{dr} + 2\beta\frac{\sigma_r^2}{r} - \frac{V_{rot}^2}{r} = -\frac{d\Phi}{dr}$$

$$\frac{d(n\sigma_r^2)}{dr} + 2\beta\frac{n\sigma_r^2}{r} = -\frac{G\mathcal{M}(r)n}{r^2} + \frac{n}{r}V_{rot}^2 = \frac{n}{r}(V_{rot}^2 - V_c^2)$$

σ_r^2 looks like T, $n\sigma_r^2$ looks like p: equation of hydrostatic equilibrium.
Measuring $I(\mathbf{x}), \sigma_r, V_{rot}$, and assuming β, can find $\mathcal{M}(r)$ and $\mathcal{M}/L\ (r)$.

Mass of the Galactic disc

Select a tracer population of stars (e.g., K dwarfs) and measure $n(z)$ and $\sigma_z(z)$. Assuming Φ is time-independent and stars are well-mixed, then f and n are also time-independent. At large heights, $\langle v_z\rangle n(z) \to 0$, so $\langle v_z\rangle = 0$. The CBE for z is

$$\frac{1}{n(z)}\frac{d}{dz}[n(z)\sigma_z^2(z)] = -\frac{\partial\Phi}{\partial z}.$$

Take a derivative:

$$\frac{d}{dz}\left(\frac{1}{n(z)}\frac{d}{dz}[n(z)\sigma_z^2(z)]\right) = -\frac{\partial^2\Phi}{\partial z^2}.$$

The Poisson equation in cylindrical coordinates with axisymmetry is

$$4\pi G\rho(R,z) = \nabla^2\Phi(R,z) = \frac{\partial^2\Phi}{\partial z^2}+\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial\Phi}{\partial R}\right) = \frac{\partial^2\Phi}{\partial z^2}+\frac{1}{R}\frac{d}{dR}[V^2(R)].$$

For uniform rotation, the last term is small. Integrating along z:

$$\int_{-z}^{z} 2\pi G\rho(R,z)dz \equiv 2\pi G\Sigma(<z) = -\frac{1}{2}\int_{-z}^{z} d\left(\frac{1}{n(z)}\frac{d}{dz}[n(z)\sigma_z^2(z)]\right)$$

$$= -\frac{1}{n(z)}\frac{d}{dz}[n(z)\sigma_z^2(z)]$$

Integrals of Motion

Functions $\mathcal{I}(\mathbf{x}, \mathbf{v})$ that remain constant along an orbit are *integrals of motion*.

- The energy per mass $E(\mathbf{x}, v) = \mathbf{v}^2/2 + \Phi(\mathbf{x})$ if Φ is independent of time.
- L_z in an axisymmetric potential $\Phi(R, z, t)$.
- $\mathbf{L}$ in a spherically symmetric potential $\Phi(r, t)$.

An integral of motion satisfies

$$\frac{d}{dt}\mathcal{I}(\mathbf{x}, \mathbf{v}) = \mathbf{v} \cdot \nabla\mathcal{I} + \frac{d\mathbf{v}}{dt} \cdot \frac{\partial \mathcal{I}}{\partial \mathbf{v}} = 0.$$

Any function $f(\mathbf{x}, \mathbf{v})$ which is a time-independent solution of the CBE is an integral of motion. Conversely, the function $f(\mathcal{I}_1, \mathcal{I}_2, \dots)$ is a steady-state solution of the equations of motion: the **Jeans Theorem**.

The strong Jeans Theorem states that steady state distribution functions are functions only of 3 (or less) independent integrals of motion.

- For spherical systems, $f = f(E, |L|)$
- If $f = f(E)$, velocity dispersions are isotropic $\sigma_r = \sigma_\theta = \sigma_\phi$.
- If $f = f(E, |L|)$, velocity dispersions are anisotropic $\sigma_r \neq \sigma_\theta = \sigma_\phi$

Integrals of Motion

Motion of disk stars on circular orbits perpindicular to the plane is independent of motion in the plane, so the energy of vertical motion E_z is an integral of motion. Select a tracer population of stars that are easy to find and measure and which are well-mixed (f is time-independent). Then

$$f(z, v_z) = f(E_z) = f(\Phi(R_0, z) + v_z^2/2).$$

- If we knew $f(E_z)$ and $\Phi(R_0, z)$ we could integrate $f(v_z)$ to find $n(z)$ and σ_z.
- If we measured $n(z)$ and guessed $f(E_z)$ we could determine $\Phi(R_0, z)$. Suppose stars with $E_z > 0$ escape:

$$f(E_z) = \frac{n_0}{\sqrt{2\pi\sigma_z^2}} e^{-E_z/\sigma_z^2}, \qquad E_z < 0; \qquad f(E_z) = 0, \qquad E_z > 0.$$

$$n(z) = n_0 e^{-\Phi(R_0,z)/\sigma^2}, \sigma_z = \sigma \qquad \text{if } v_e = -2\Phi(R_0, z) >> \sigma$$

  However, note that $v_e \sim 2\sigma$.
- If $n(z)$ and σ_z^2 measured, $\Phi(R_0, z)$ can be found from

$$\frac{d}{dz}\left[n(z)\sigma_z^2\right] = -n(z)\frac{\partial\Phi(R_0, z)}{\partial z}.$$

Consistency

If the stars described by f provide all the gravitational force, then the density $n(\mathbf{x}, \mathbf{v})$ found by integrating $f(\mathbf{x}, \mathbf{v}, t)$ over $\mathbf{v}$ is equivalent to the density $\rho(\mathbf{x}, t)$ in Poisson's Equation. Many forms of f can give rise to the same $\Phi(\mathbf{x}, t)$: all give the same $n(\mathbf{x}, t)$ but different $\mathbf{v}(\mathbf{x}, t)$.
In a spherically symmetric potential, any function $f(E, \mathbf{L})$ not including unbound stars will be a solution. If $f = f(E)$, velocity dispersions are isotropic.
Example: $f(E) = k(-E)^{N-3/2}$ for $E < 0, \quad N > 3/2$.

$$n(r) = 4\pi \int_0^{v_e} k \left[-\Phi(r) - \frac{v^2}{2} \right]^{N-3/2} v^2 dv$$

$$= 4\pi k 2^{3/2} (-\Phi(r))^N \int_0^{\pi/2} \sin^{2N-2}\theta \cos^2\theta d\theta = k c_N (-\Phi(r))^N,$$

after substituting $\cos(\theta) = v/\sqrt{-2\Phi(r)}$. Compare to Plummer sphere

$$\rho(r) = -\frac{3a^2}{4\pi G^5 \mathcal{M}^4} \Phi^5(r) = \frac{3a^2}{4\pi} \frac{\mathcal{M}}{(r^2 + a^2)^{5/2}}$$

suggesting $N = 5$ and $f(E) = k(-E)^{7/2}$.
Total mass $\mathcal{M} \propto k, \quad a = [G\mathcal{M}/\Phi(0)]^2$.

Isothermal Models

Consider a Boltzmann-like distribution function:

$$f(E) = \frac{n_o}{(2\pi\sigma^2)^{3/2}} e^{-E/\sigma^2} = \frac{n_o}{(2\pi\sigma^2)^{3/2}} e^{-(\Phi+v^2/2)/\sigma^2}$$

$$n(r) = 4\pi \int_0^\infty f(E(v)) v^2 dv = n_o e^{-\Phi/\sigma^2}$$

Poisson's equation

$$\frac{d}{dr}\left(r^2 \frac{d\ln n}{dr}\right) = -\frac{4\pi G}{\sigma^2} r^2 n$$

which is the isothermal spherical solution.

(i) Singular isothermal sphere

$$n(r) = \frac{\sigma^2}{2\pi G r^2}, \quad V_c = \sqrt{2}\sigma, \quad \langle v^2 \rangle = 3\sigma^2$$

But has infinite central density and $\mathcal{M} \to \infty$ as $r \to \infty$.

Isothermal Models

General isothermal sphere
$n(0) = n_0, \quad (dn/dr)_{r=0} = 0.$

The density varies slowly near the center, out to $r_o = 3\sigma/\sqrt{4\pi G\rho_0}$.

r_o is the core (King) radius, and is also the scale length of the envelope.

$I(r_o) = 0.5013 I(0)$.

$V_c^2 = \sigma^2 d\ln n/d\ln r$.

At small radii,
$n(r) = n_0(1 + (r/r_o)^2)^{-3/2}$.

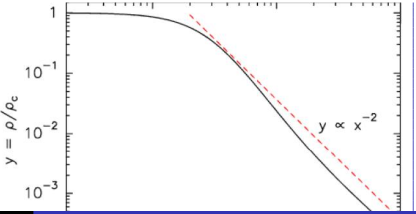
At large radii, $n(r) \propto (r/r_o)^{-2}$

$\sigma^2 = 4\pi G n_o r_o/9$

A good fit to the centers of elliptical galaxies can be used to estimate central M/L.

- Measure $I(R)$ and determine r_o and $I(0)$
- Also measure σ^2.
- Then $\mathcal{M}/L = 9\sigma^2/(2\pi G I(0) r_o)$.

But this still has an infinite total mass. The problem is $f(-E) > 0$ even when E is positive, i.e., the model includes unbound stars.



Isothermal Models

Lowered isothermal sphere

Suppress stars at large radii; $f(-E) \to 0$ when $E \to 0, v \to v_e$.

It is convenient to define $\Psi = -\Phi$ and $E_r = -E = \Psi - v^2/2$.

$$f(E_r) = \frac{n_o}{(2\pi\sigma_o)^{3/2}}\left[e^{E_r/\sigma_o^2} - 1\right]$$

$$\frac{d}{dr}\left(r^2\frac{d\Psi}{dr}\right) = -4\pi G n_o r^2 \times$$
$$\left[e^{\Psi/\sigma_o^2}\,\mathrm{erf}\left(\frac{\sqrt{\Psi}}{\sigma_o}\right) - \sqrt{\frac{4\Psi}{\pi\sigma_o^2}}\left(1 + \frac{2\Psi}{3\sigma_o^2}\right)\right]$$

Inner regions: core radius $\sim r_o$, $\sigma^2 \simeq \sigma_o^2$

Outer regions: truncated at r_t, $\sigma^2 << \sigma_o^2$

If $\Psi(0) = q\sigma_o^2$, $r_t \simeq r_o 10^{q/4}$.

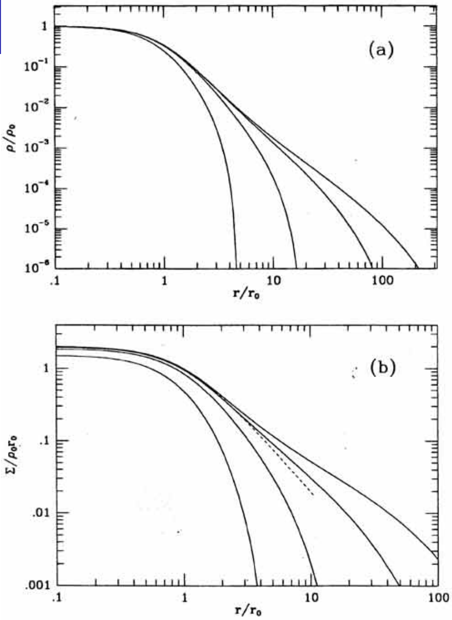


Figure 4-9. (a) Density profiles of four King models: from top to bottom the central potentials of these models satisfy $\Psi(0)/\sigma^2 = 12, 9, 6, 3$. (b) The projected mass densities of these models (full curves), and the projected modified Hubble law (dashed curve).

Consistency

In general, we find a single equation to be satisfied for consistency with the steady state CBE and Poisson's equation:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Psi}{dr}\right) = -16\pi^2 G\int_0^{\sqrt{2\Psi}} f(\Psi - v^2/2)v^2 dv$$

$$-4\pi G\rho(\Psi) = -16\pi^2 G\int_0^{\Psi} f(E_r)\sqrt{2(\Psi - E_r)}dE_r$$

$$\frac{\rho(\Psi)}{\sqrt{8}\pi} = 2\int_0^{\Psi} f(E_r)\sqrt{\Psi - E_r}dE_r, \quad \frac{1}{\sqrt{8}\pi}\frac{d\rho}{d\Psi} = \int_0^{\Psi}\frac{f(E_r)dE_r}{\sqrt{\Psi - E_r}}.$$

This is an Abel integral equation with solution

$$f(E_r) = \frac{1}{\sqrt{8}\pi}\frac{d}{dE_r}\int_0^{E_r}\frac{d\rho}{d\Psi}\frac{d\Psi}{\sqrt{E_r - \Psi}}$$

$$= \frac{1}{\pi^2\sqrt{8}}\left[\int_0^{E_r}\frac{d^2\rho}{d\Psi^2}\frac{d\Psi}{\sqrt{E_r - \Psi}} + \frac{1}{\sqrt{E_r}}\left(\frac{d\rho}{d\Psi}\right)_{\Psi=0}\right]$$

This is an alternate method, begininning with measuring $\rho(r)$ from surface photometry. Find $\Psi(r) = -\Phi(r) = G\mathcal{M}(< r)/r$ from $\rho(r)$, then eliminate r to find $\rho(\Psi)$.