Astronaut on a bungee cord

Let $\mathbf{r}_1$ be position vector of astronaut from space station $S$. Then the full equation of motion for the astronaut is

$$m_1 \ddot{r}_1 = -\frac{Gm_1 m_2}{r_1^2} \hat{r}_1 + k (\mathbf{r}_2 - \mathbf{r}_1)$$

Following approach used in class, we split into radial and tangential components:

$$\ddot{r}_1 - \dot{r}_1 \dot{r}_1 = -\frac{GM}{R_1^2} + \frac{k}{m_1} (\mathbf{r}_2 - \mathbf{r}_1) \cdot \hat{r} \quad \ldots (1)$$

$$R_1 \ddot{\theta}_1 + 2 \dot{r}_1 \dot{\theta}_1 = \frac{k}{m_1} (\mathbf{r}_2 - \mathbf{r}_1) \cdot \hat{\theta} \quad \ldots (2)$$

As in class, we set:

$$r_1 = R_0 + x_1$$

$$\dot{r}_1 = \Delta R_0 + y_1/R_0$$

where $\Delta R_0 = (\Gamma m / R_0^3)^{1/2}$ is any velocity of reference circular orbit.
Thus means that
\[ \begin{align*}
\vec{r}_1 &= (R_0 + x_1)^2 + y_1 \hat{y} \\
\vec{r}_2 &= (R_0 + x_2)^2 + y_2 \hat{y}
\end{align*} \]

So equation 1 becomes
\[ \ddot{x}_1 - (R_0 + x_1) (\dot{\omega}_0 + \dot{y}_1/R_0)^2 = -\frac{GM}{(R_0 + x_1)^2} + \frac{k_2}{m_1} (x_2 - x_1) \]
\[ \Rightarrow \ddot{x}_1 - R_0 \dot{\omega}_0^2 - \omega_0^2 x_1 - 2R_0 \dot{y}_1 \dot{\omega}_0/R_0 = \frac{GM}{R_0^2} - 2x_1 \frac{GM}{R_0^3} + \frac{k_2}{m_1} (x_2 - x_1) \]
(dropping terms like \( \dot{y}_1^2 \) and \( x_1 \dot{y}_1 \) since they are small, and Taylor expanding \( \frac{GM}{(R_0 + x_1)^2} \) to first order)
\[ \Rightarrow \ddot{x}_1 - R_0 \dot{\omega}_0^2 - \omega_0^2 x_1 - 2R_0 \dot{y}_1 \dot{\omega}_0/R_0 = -R_0 \dot{\omega}_0^2 + 2xc_1 \dot{\omega}_0^2 + \frac{k_2}{m_1} (x_2 - x_1) \]
where we have used \( \omega_0^2 = \frac{GM}{R_0^3} \)
\[ \Rightarrow \ddot{x}_1 - 3R_0 \dot{\omega}_0^2 x_1 - 2R_0 \dot{y}_1 - \frac{k_2}{m_1} (x_2 - x_1) = 0 \] ... ③

Equation 2 becomes
\[ (R_0 + x_1) \frac{\dot{y}_1/R_0}{R_0} + 2x_1 (\dot{\omega}_0 + \dot{y}_1/R_0) = \frac{k_2}{m_1} (y_2 - y_1) \]
\[ \Rightarrow \dot{y}_1 + 2x_1 \dot{\omega}_0 - \frac{k_2}{m_1} (y_2 - y_1) = 0 \] ... ④
where we have neglected terms like \( x_1 \dot{y}_1 \) since they are small in comparison.

Equations ③ & ④ are the required equations of motion.
The derivation of the equation of motion for the space station will look exactly the same except that the labels "1" and "2" will be transposed. So,

\[
\begin{align*}
\ddot{x}_2 - 3\alpha_0^2 x_2 - 2\alpha_0 y_2 &= -\frac{k}{m_2} (x_1-x_2) = 0 \\
\ddot{y}_2 + 2\dot{x}_2\alpha_0 - \frac{k}{m_2} (y_1-y_2) &= 0
\end{align*}
\]

Subtract (5) from (3) gives

\[
\begin{align*}
\ddot{x}_1 - \ddot{x}_2 - 3\alpha_0^2 x_1 + 3\alpha_0^2 x_2 - 2\alpha_0 y_1 + 2\alpha_0 y_2 \\
- \frac{k}{m_1} (x_2-x_1) + \frac{k}{m_2} (x_1-x_2) &= 0 \\
\Rightarrow (\ddot{x}_1 - \ddot{x}_2) - 3\alpha_0^2 (x_1-x_2) - 2\alpha_0 (y_1-y_2) + k (x_1-x_2) \left( \frac{1}{m_2} + \frac{1}{m_1} \right) &= 0
\end{align*}
\]

Let \( \Delta x = x_1-x_2 \) and \( \Delta y = y_1-y_2 \). So, their last eq. read

\[
\ddot{x} - 3\alpha_0^2 \Delta x - 2\alpha_0 \Delta y + k \Delta x \left( \frac{1}{m_2} + \frac{1}{m_1} \right) = 0
\]

But \( m_2 \gg m_1 \) so \( \frac{1}{m_2} \ll \frac{1}{m_1} \). Thus, neglecting \( \frac{1}{m_2} \) term gives

\[
\ddot{x} - (3\alpha_0 - \frac{k}{m_1}) \Delta x - 2\alpha_0 \Delta y = 0 \quad \text{\ldots (7)}
\]

Subtract (6) from (4)

\[
\begin{align*}
\ddot{y}_1 - \ddot{y}_2 + 2\dot{x}_1\alpha_0 - 2\dot{x}_2\alpha_0 - \frac{k}{m_2} (y_2-y_1) + \frac{k}{m_2} (y_1-y_2) &= 0 \\
\Rightarrow \Delta y + 2\alpha_0 \Delta x + k \Delta y \left( \frac{1}{m_1} + \frac{1}{m_2} \right) &= 0 \\
\Rightarrow \Delta \ddot{y} + 2\alpha_0 \Delta \ddot{x} + \frac{k}{m_1} \Delta y &= 0 \quad \text{\ldots (8)}
\end{align*}
\]

As required
K/m, has units of frequency-squared, so it should be compared with the only other frequency in the problem, \( \omega_0 \). Suppose that
\[
\frac{k}{m} \gg \omega_0^2
\]
Then, we can essentially neglect the term containing \( \omega_0 \) in the equation of motion. So, we have
\[
\ddot{\Delta x} + \frac{k}{m} \Delta x = 0
\]
\[
\ddot{\Delta y} + \frac{k}{m} \Delta y = 0
\]
This is the simple harmonic oscillator, e.g., both \( \Delta x \) and \( \Delta y \) execute SHO with frequency \( \omega = \sqrt{k/m} \), i.e.
\[
\Delta x = A \cos(\omega t + \phi_x)
\]
\[
\Delta y = B \cos(\omega t + \phi_y)
\]
\[
\omega = \sqrt{k/m}, \text{ and } A, B, \phi_x, \phi_y \text{ constants}
\]
So, the astronaut executes "orbits" around the space-station.
The space-station is at the center of the elliptical orbit and the period of the orbit is \( T = \sqrt{k/m} \).
(2) Need to combine eqs. to eliminate \( \Delta y \). Start with
\[
\dddot{\Delta x} - (3\omega_0^2 - \frac{k}{m})\Delta x - 2i\omega_0 \Delta y = 0
\]
\[
\Rightarrow \dddot{\Delta x} - (3\omega_0^2 - \frac{k}{m})\Delta x - 2i\omega_0 \Delta y = 0 \quad \text{differentiating twice}
\]
But, we know that
\[
\ddot{\Delta y} = -2i\omega_0 \Delta x - \frac{k}{m} \Delta y
\]
Subst. our into eq. for $\ddot{\Delta x}$ gives

$$\ddot{\Delta x} - (3\Omega_0^2 - k/m_1) \Delta x - 2\Omega_0 (-2\Omega_0 \Delta x - \frac{k}{m_1} \Delta y) = 0$$

$$\Rightarrow \ddot{\Delta x} + (\Omega_0^2 + k/m_1) \Delta x + 2\Omega_0 k/m_1 \Delta y = 0$$

But, from $\theta$ we have

$$2\Omega_0 \Delta y = \dot{\Delta x} - (3\Omega_0^2 - k/m_1) \Delta x$$

Subst. our into eq. for $\ddot{\Delta x}$ gives

$$\ddot{\Delta x} + (\Omega_0^2 + k/m_1) \Delta x + 2\Omega_0 k/m_1 (\dot{\Delta x} - [3\Omega_0^2 - \frac{k}{m_1}] \Delta x) = 0$$

$$\Rightarrow \ddot{\Delta x} + (\Omega_0^2 + \frac{2k}{m_1}) \Delta x - \frac{k}{m_1} (3\Omega_0^2 - \frac{k}{m_1}) \Delta x = 0$$

This is the required differential equation.

To find characteristic equation, substitute in $\Delta x = A e^{nt}$

Then

$$\Delta x = A e^{nt} = \Delta x$$

$$\ddot{\Delta x} = A \Delta x = A^2 \Delta x$$

$$\dddot{\Delta x} = A^3 \Delta x$$

$$\Delta x = A^4 \Delta x$$

So, we get

$$A^4 \Delta x + (\Omega_0^2 + \frac{2k}{m_1}) A^2 \Delta x - \frac{k}{m_1} (3\Omega_0^2 - \frac{k}{m_1}) \Delta x = 0$$

$$\Rightarrow A^4 + (\Omega_0^2 + \frac{2k}{m_1}) A^2 - \frac{k}{m_1} (3\Omega_0^2 - \frac{k}{m_1}) = 0$$

As required.
2) Suppose that the characteristic equation has a positive real root $\alpha$. Then the general solution of the differential equation will have a term that looks like

$$Ae^{\alpha t}$$

But $e^{\alpha t} \to \infty$ as $t \to \infty$ and so solution blows up!!!

View characteristic equation as a quadratic in $\beta \equiv \alpha^2$. Then there will be a positive real root if and only if the quadratic for $\beta$ has a positive real root.

3) Look at characteristic equation as quadratic in $\beta \equiv \alpha^2$.

$$\beta^2 + (\omega_0^2 + \frac{2k}{m_1}) \beta - \frac{k}{m_1} (3\omega_0^2 - \frac{k}{m_1}) = 0$$

i.e. $f(\beta) = 0$, $f(\beta) = \beta^2 + (\omega_0^2 + \frac{2k}{m_1}) \beta - \frac{k}{m_1} (3\omega_0^2 - \frac{k}{m_1})$

Let's graph this:

![Graph of f(\beta)](image)

The equation $f(\beta) = 0$ has real roots if and only if

$$\frac{k}{m_1} (3\omega_0^2 - \frac{k}{m_1}) \geq 0$$

$$\Rightarrow k \leq 3\omega_0^2 m_1$$
Thus, if the bungee cord is weak \((k < 3.02^{2} \text{m})\), then the general equation of the differential equation will have an exponentially diverging piece.

The astronaut will drift away from the station at an exponential rate!!