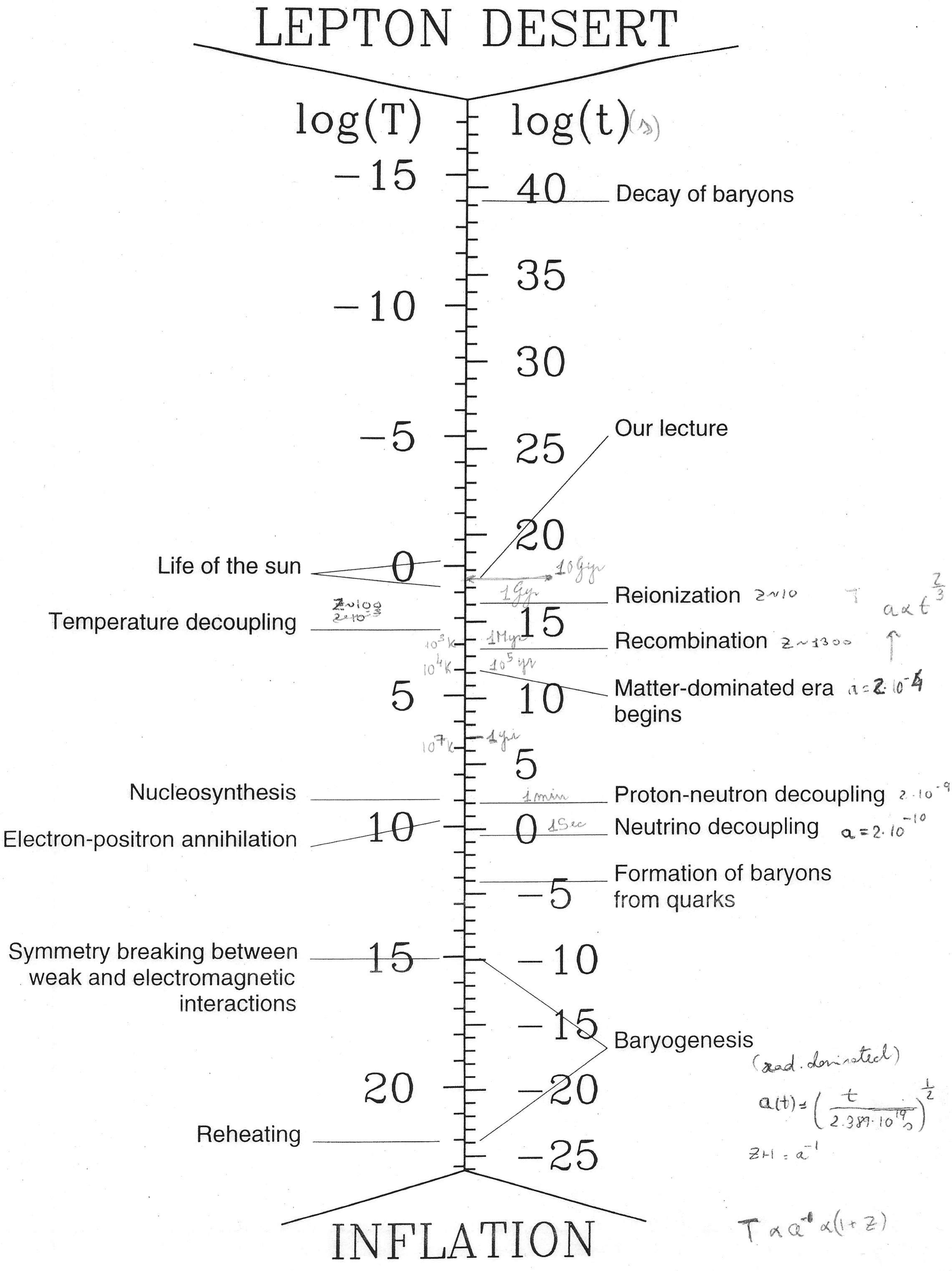


LEPTON DESERT
log(T)
log(t)
-15
-10
-5
0
5
10
15
20
40
35
30
25
20
15
10
5
0
-5
-10
-15
-20
-25
Decay of baryons
Our lecture
Life of the sun
Reionization
Temperature decoupling
Recombination
Matter-dominated era begins
Nucleosynthesis
Proton-neutron decoupling
Electron-positron annihilation
Neutrino decoupling
Formation of baryons from quarks
Symmetry breaking between weak and electromagnetic interactions
Baryogenesis
Reheating
INFLATION
10 Gyr
1 Gyr
1 Myr
10^5 yr
1 yr
1 min
1 sec
10^3 K
10^4 K
10^7 K
z~10
z~1300
$a \propto t^{\frac{2}{3}}$
$a = 2 \cdot 10^{-4}$
$2 \cdot 10^{-9}$
$a = 2 \cdot 10^{-10}$
(rad. dominated)
$a(t) = \left(\frac{t}{2.387 \cdot 10^{19} s}\right)^{\frac{1}{2}}$
$z+1 = a^{-1}$
$T \propto a^{-1} \propto (1+z)$

(1) Foundations of Cosmology. (Chapter 1 CL)

- Cosmological principles:

1) <u>Uniformity of physical laws</u>:
the laws of physics are invariant to space-time translations

2) <u>Copernican principle</u>:
we are not in a priviledged position in the Universe

+

Observed isotropicity (CMB)

||

Universe is HOMOGENEOUS on "large" scales.
"large" means $\gtrsim 100$ Mpc

($1\,\mathrm{Mpc} = 3 \cdot 10^{24}$ cm)

Perfect Cosmological principle: (Bondi & Gold 1948, Hoyle 1948, 1963-64)

$\Downarrow$

"Steady-state" Cosmology
(constant creation of matter in expanding Universe ≡ de Sitter)
Killed by CMBR discovery.

- Hubble expansion

Hot Big-Bang (GR + cosmological principle + Hubble exp.)

Justifications of the Cosmological principles:

- Introduced by Einstein without justification other than Mach principle.
- Simplify calculations.
- Physical ground:
  - Mix Master theory (viscous dissipation by neutrinos)
  - Inflation (Cosmological Horizon problem)
  -
- Most Important: Observational tests

. ~~~~ .

Alternative Cosmologies:

1) Anisotropic models (Bianchi models: vorticity, rotation; etc..) (homogeneous but anisotropic)

2) Alternatives to GR:
   - Brans-Dicke theory; Dirac arguments
   - Conformal gravity ≃ GR

   → Time varying physical constants: G; c; etc...

Observational Tests:

1) Isotropicity:

- CMB ($\Delta T / T \le 10^{-5}$)
- Galaxy distribution (SLOAN)
- Background radiation (x-ray; Optical; IR)

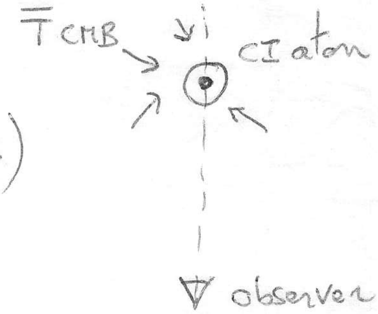
2) Homogeneity:

Homogeneous ≡ isotropic around distinct points

ex 1: Ref: Goodman, J., 1995, Phys. Rev. D, 52, 1821

fine structure lines of CI ($z = 1.776$)

$$\bar{T}_{CMB,z} = 7.4 \pm 0.8\,K$$



$$\bar{T}_{CMB,z} = (1+z)\, T_{CMB,z=0} \approx 2.776 \times (2.73 \pm 0.8\,K)$$

~ 7.57

$$\left| \frac{T_0}{T} - 1 \right| \le 11\%$$

CONCLUSIONS:

- Copernican principle is verified at about 10% level
- The universe is isotropic on large scales to better than 10 parts per milion.

GR:

Force of gravity $\equiv$ property of space-time.

interval $\quad ds^2 = g_{ij}\, dx^i dx^j$

$g_{ij} \rightarrow$ metric tensor

Einstein Equations relate geometry to energy-density

$$\underset{\text{Einstein Tensor}}{G_{ik}} \equiv \underset{\text{Ricci Tensor}}{R_{ik}} - \frac{1}{2} g_{ik} \underset{\text{Ricci scalar}}{R} \underbrace{- \cancel{\Lambda g_{ik}}}_{\text{cosmological constant}} = \frac{8\pi G}{c^4} \underset{\text{Energy-density tensor}}{T_{ik}}$$

The Robertson-Walker Metric:

(synchronous gauge) $\rightarrow$
$$ds^2 = (c\,dt)^2 - a(t)^2 \left[\frac{dr^2}{1-kr^2} + r^2 \underbrace{(d\theta^2 + \sin^2\theta\, d\varphi^2)}_{d\Omega^2} \right]$$

$$ds^2 = g_{00}(c\,dt)^2 + \cancel{2 g_{0i}\, c\,dt\, dx^i} + g_{ik}\, dx^i dx^k$$

$k = -1; 0; 1$ curvature parameter

$a(t)$ = scale factor

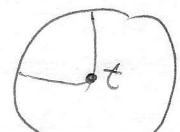
$r; \theta; \varphi \equiv$ comoving polar coordinates

The metric is determined by the cosmological principle (homogeneous + isotropic). No need to know or solve Einstein equations!

$$ds^2 = (c\,dt)^2 - dl^2 \qquad dl^2 = \gamma_{\alpha\beta}\, dx^\alpha dx^\beta \quad (\alpha, \beta = 1,2,3)$$

$\uparrow$

synchronous gauge ; dt = proper time



Understanding the metric in 2D:

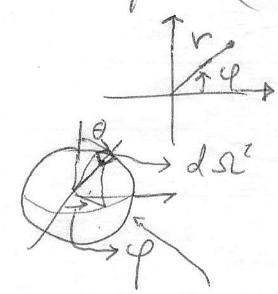
$k=0$ 1) Euclidean (Cartesian plane)

$k=1$ 2) Sphere of radius R (curvature $1/R^2$)

$k=-1$ 3) Hyperboloid (" $-1/R^2$)

1) $dl^2 = a^2 (dr^2 + r^2 d\varphi^2)$ $\qquad a \cdot r = \rho \quad (0 \le \rho \le \infty)$

2) $dl^2 = a^2 (d\theta^2 + \sin^2\theta\, d\varphi^2) =$
$= a^2 \left(\dfrac{dr^2}{1-r^2} + r^2 d\varphi^2 \right)$



$0 \le \theta \le \pi \qquad 0 \le \varphi \le 2\pi \qquad r^2 = \sin^2\theta \rightarrow 0 \le r \le 1 \qquad a$ = Radius

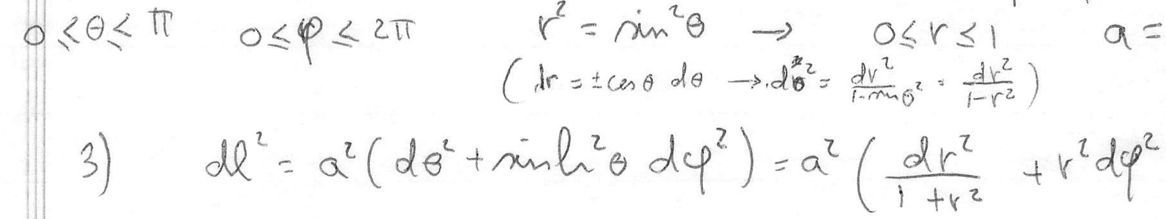
$\left(dr = \pm\cos\theta\, d\theta \rightarrow d\theta^2 = \dfrac{dr^2}{1-\sin^2\theta} = \dfrac{dr^2}{1-r^2} \right)$

3) $dl^2 = a^2 (d\theta^2 + \sinh^2\theta\, d\varphi^2) = a^2 \left(\dfrac{dr^2}{1+r^2} + r^2 d\varphi^2 \right)$

$\Downarrow$ 3D

$$\begin{cases} dl^2 = a^2 (dr^2 + r^2 d\Omega^2) \\ dl^2 = a^2 (d\chi^2 + \sin^2\chi\, d\Omega^2) = a^2 \left(\dfrac{dr^2}{1-r^2} + r^2 d\Omega^2 \right) \\ dl^2 = a^2 (d\chi^2 + \sinh^2\chi\, d\Omega^2) = a^2 \left(\dfrac{dr^2}{1+r^2} + r^2 d\Omega^2 \right) \end{cases}$$

$d\Omega^2 = d\theta^2 + \sin^2\theta\, d\varphi^2 \qquad 0 \le \chi \le \pi$



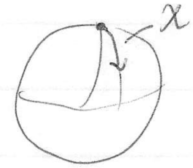
infinite Volume $\qquad$ infinite Volume $\qquad$ finite Volume;

(3) Hubble law, redshift, Friedman equations

Hubble Law:

Proper distance: $dt = 0 \quad (t = \text{const})$

$$d_p = a \int_0^r \underbrace{\frac{dr'}{(1-kr'^2)^{1/2}}}_{\to\, d\chi} = a\,\chi(r)$$



$$\begin{cases} \chi(r) = \arcsin r & k=1 \\ \chi(r) = r & k=0 \\ \chi(r) = \operatorname{arcsinh} r & k=-1 \end{cases}$$

Comoving distance: $d_p(t) = a(t)\,\chi$ $\quad \boxed{d_c \equiv d_p(t=t_0) = a_0\,\chi}$

$\downarrow$ comoving

$$\boxed{d_c = \frac{a_0}{a(t)}\, d_p(t)}$$

$$v_r = \frac{d\,d_p}{dt} = \dot{a}\,\chi = \frac{\dot{a}}{a}\, d_p = H(t)\cdot d_p$$

$$H(t) = \frac{\dot{a}}{a} \underbrace{=}_{\left(= \frac{d \ln a}{dt}\right)} \text{Hubble parameter}$$

$H_0 = H(t=t_0)$ $\qquad H_0 \simeq 72\ \mathrm{km\,s^{-1}\,Mpc^{-1}}$ (WMAP)

$$H_0 = \underbrace{h}_{(\simeq 0.72)} \cdot 100\ \mathrm{km\,s^{-1}\,Mpc^{-1}}$$

Redshift :

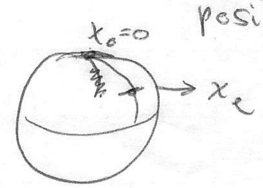
Photons Travel along null geodesic : $ds^2 = 0$

$t_0 \to$ t observed

$$\int_{t_e}^{t_0} \frac{c\,dt}{a(t)} = -\int_{\chi_e}^{\chi_0 = 0} d\chi' = \chi_e(r) - \cancel{\chi_0} = \text{const}(t) \quad \chi_0 = 0$$

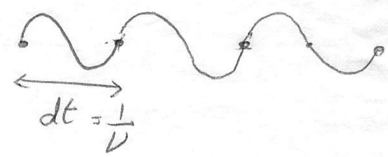
$t_e \to$ t emission

$\chi_0 = 0 \to$ observe position



$$t_e' = t_e + dt_e \qquad t_0' = t_0 + dt_0$$

$$dt_e = \frac{1}{\nu_e} \qquad dt_0 = \frac{1}{\nu_0}$$



$$\int_{t_e+dt_e}^{t_0+dt_0} \frac{c\,dt}{a(t)} = \chi_e(r) = \int_{t_e}^{t_0} \frac{c\,dt}{a(t)} \Rightarrow \int_{t_e+dt_e}^{t_e} + \cancel{\int_{t_e}^{t_0}} + \int_{t_0}^{t_0+dt_0} = \cancel{\int_{t_e}^{t_0}}$$

$\chi_e(r) \downarrow$ not moving

$$\frac{dt_e}{a_e} = \frac{dt_0}{a_0} \qquad \frac{dt_e}{dt_0} = \frac{\nu_0}{\nu_e} = \frac{a_e}{a_0} = \frac{\lambda_e}{\lambda_0}$$

$$\text{redshift} = z = \frac{\overbrace{\lambda_0 - \lambda_e}^{\Delta\lambda}}{\lambda_e} = \frac{\lambda_0}{\lambda_e} - 1 = \frac{a_0}{a_e(t)} - 1$$

$$\boxed{\frac{a_e(t)}{a_0} = (1+z)^{-1}} + (a_0 = 1) = \boxed{a(t) \propto (1+z)^{-1}}$$

$$\left(\nu_0 = \frac{\nu_e}{1+z} \right)$$

Question : where does the energy of photons go ?

Friedmann Equations:

$$\begin{cases} R_{ij} - \frac{1}{2} g_{ij} R \overset{\cancel{-\Lambda g_{ij}}}{=} \frac{8\pi G}{c^4} T_{ij} & \text{Einstein eq. + R-W metric} \\ T_{ij} = (p + \rho c^2)\, U_i U_j - p\, g_{ij} & \underline{\text{perfect fluid}} \end{cases}$$

4-velocity

pressure → energy-density

⇓ R-W (cosmological principle)

(typo on CL)

$$\boxed{\begin{aligned} \ddot{a} &= -\frac{4\pi G}{3}\left(\rho + 3\frac{p}{c^2}\right) a \\ \dot{a}^2 + kc^2 &= \frac{8\pi G}{3} \rho a^2 \end{aligned}}$$

time-time component (1)

space-space components (2)

$\left(\rho + \frac{3p}{c^2} > 0 \quad \ddot{a} < 0 \right.$

$\left. p > -\frac{1}{3}\rho c^2 \right)$

$k = \begin{matrix} -1 & \text{open} \\ 0 & \text{flat} \\ 1 & \text{closed} \end{matrix} \Big\}$ Universe

Adiabatic expansion: $d(\rho c^2 a^3) = -p\, da^3$

⇓

(3)

$$\Rightarrow \frac{d\rho}{dt} + 3\left(\rho + \frac{p}{c^2}\right)\frac{\dot{a}}{a} = 0$$

(1) and (2) not independent. $(1) + (3) \longrightarrow (2)$
if assume adiabatic expansion

Cosmological Constant:

1) Bring Einstein cosmological constant to the rhs $\rightarrow \begin{cases} \tilde{\rho} = \rho + \frac{\Lambda c^2}{8\pi G} \\ \tilde{p} = p - \frac{\Lambda c^4}{8\pi G} \end{cases}$

$$\ddot{a} = -\frac{4\pi}{3} G \left(\rho + \frac{3p}{c^2}\right) a + \frac{\Lambda c^2 a}{3}$$

$$\dot{a}^2 + kc^2 = \frac{8\pi G}{3} \rho a^2 + \frac{\Lambda c^2 a^2}{3}$$

→ Static solution for $\tilde{\rho} + \frac{3\tilde{p}}{c^2} = 0$ $\quad (\ddot{a} = 0)$
(unstable)

$$\tilde{\rho} = -\frac{3\tilde{p}}{c^2} = \frac{3kc^2}{8\pi G a^2} \quad (\dot{a} = 0)$$

* Derivation of Friedman equations

$$\begin{cases} T_{\hat t \hat t} = \rho c^2 \\ T_{\hat x \hat x} = T_{\hat\theta\hat\theta} = T_{\hat\phi\hat\phi} = P \end{cases} \qquad T = \begin{pmatrix} \rho c^2 & & & 0 \\ & P & & \\ & & P & \\ 0 & & & P \end{pmatrix}$$

$$\begin{cases} G_{\hat t \hat t} = \dfrac{3}{c^2}\left(\dfrac{\dot a}{a}\right)^2 + \dfrac{3k}{c^2 a^2} \\ G_{\hat x \hat x} = G_{\hat\theta\hat\theta} = G_{\hat\phi\hat\phi} = -\dfrac{2}{c^2}\dfrac{\ddot a}{a} - \dfrac{1}{c^2}\left(\dfrac{\dot a}{a}\right)^2 - \dfrac{k}{c^2 a^2} \end{cases}$$

$$\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \qquad (1)\ t\text{-}t \text{ component}$$

$$-\frac{2\ddot a}{\cancel{c^2} a} - \frac{1}{\cancel{c^2}}\left(\frac{\dot a}{a}\right)^2 - \frac{k}{\cancel{c^2} a^2} = \frac{8\pi G}{\cancel{c^2}}\frac{P}{c^2} \qquad (2)\ \text{space-space component}$$

$$\Downarrow \quad (1)+(2)$$

$$2\frac{\ddot a}{a} = -\frac{8\pi G}{3}\left(\rho + \frac{3P}{c^2}\right) \longrightarrow \frac{\ddot a}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right)$$

* Cosmological constant (cont)

For a dust universe $P=0 \rightarrow \Lambda = \frac{k}{a^2}$ $\quad \rho = \frac{kc^2}{4\pi G a^2}$

Since $\rho > 0 \Rightarrow k=1$ and $\Lambda > 0$ $\quad \Lambda_E = \frac{1}{a^2} = \frac{4\pi G\rho}{c^2}$

curvature $= \Lambda_E^{-\frac{1}{2}}$

* De Sitter universe: $(P = \rho = k = 0)$

$\tilde P = -\tilde\rho c^2 = -\frac{\Lambda c^4}{8\pi G}$

$\dot a^2 = \frac{1}{3}\Lambda c^2 a^2 \Rightarrow \left(\frac{\dot a}{a}\right)^2 = H^2 = \frac{\Lambda c^2}{3} = \text{const}$ $\quad \frac{d\ln a}{dt} = c\sqrt{\frac{\Lambda}{3}}$

$\ln\frac{a}{a_0} = \pm\sqrt{\frac{\Lambda}{3}}\,ct$ $\quad a(t) = a_0 \exp\left[\pm\sqrt{\frac{\Lambda}{3}}\,ct\right]$

(4) Cosmological models

$$H^2 - \frac{8\pi}{3} G\rho = H^2\left(1 - \rho \cdot \frac{8\pi G}{3H^2}\right) = -\frac{kc^2}{a^2} \qquad H^2\left(1 - \frac{\rho}{\rho_{cr}}\right) = -\frac{kc^2}{a^2}$$

$$\left(\frac{\dot a_0}{a_0}\right)^2 - \frac{8\pi}{3} G\rho \left(\frac{a_0}{a_0}\right)^2 = H_0^2\left(1 - \frac{\rho_0}{\rho_{0c}}\right) = H_0^2(1-\Omega_0) = -\frac{kc^2}{a_0^2}$$

$$\rho_{0c} = \frac{3H_0^2}{8\pi G} = \left(1.885 \cdot 10^{-29} h^2\right) gm^{-3}$$

$$\begin{cases} \Omega_0 = 1 & k=0 \text{ • flat} \\ \Omega_0 > 1 & k>0 \text{ • closed} \\ \Omega_0 < 1 & k<0 \text{ • open} \end{cases}$$

(Chapter 2 CL)

given $p = p(\rho)$ → solve (3) to get $\rho(t)$ and solve (2) to get $a(t)$.

$$\boxed{p = w\rho c^2} \qquad \underline{0 < w < 1}$$

a) "dust" $w=0$ (non-relativistic matter)

$p \ll \rho c^2$

$p=0$

$$\dot\rho + 3\frac{\dot a}{a}\rho = 0 \quad \Rightarrow \quad \rho a^3 = \text{const} \rightarrow \rho_m = \rho_{0m}(1+z)^3$$

b) "radiation" $w = \frac{1}{3}$ (ultra-relativistic matter)

$$p = \frac{1}{3}\rho c^2 \quad \left(w = \frac{1}{3}\right) \qquad \left(p = \frac{\varepsilon}{3}\right)$$

$$\dot\rho + 4\frac{\dot a}{a}\rho = 0 \qquad \rho a^4 = \text{const} \rightarrow \rho_r = \rho_{0r}(1+z)^4$$

in general ; $\rho a^{3(1+w)} = \text{const}$

$$\Omega_{0w} = \frac{\rho_{0w}}{\rho_{0c}}$$

$$\left(\frac{\dot{a}}{a_0}\right)^2 = H_0^2 \left[\Omega_{0w} \left(\frac{a_0}{a}\right)^{1+3w} + (1-\Omega_{0w}) \right] \quad \text{or}$$

$$H(t)^2 = H_0^2 \left(\frac{a_0}{a}\right)^2 \left[\Omega_{0w} \left(\frac{a_0}{a}\right)^{1+3w} + (1-\Omega_{0w}) \right]$$

where $H(t) = \dfrac{\dot{a}}{a}$

In general if we have more components:

$$\Rightarrow H^2 = \left(\frac{\dot{a}}{a}\right)^2 = H_0^2 \left[\Omega_0 \left(\frac{a}{a_0}\right)^{-3} + \Omega_K \left(\frac{a}{a_0}\right)^{-2} + \Omega_\Lambda + \Omega_R \left(\frac{a}{a_0}\right)^{-4} \right] \Leftarrow$$

IMPORTANT FORMULA

Ω_0 = dust $\boxed{w = 0}$

Ω_R = Radiation $\boxed{w = \frac{1}{3}}$

Ω_Λ = cosmological constant $\boxed{w = -1}$

Ω_K = curvature $\left(w = -\frac{1}{3}\right)$

<0 closed, 0 flat, >0 open

$\Omega_0 + \Omega_K + \Omega_\Lambda + \Omega_R = 1$

$\boxed{\Omega_K = 1 - \Omega_0 - \Omega_R - \Omega_\Lambda}$

At early times $\to$ radiation dominates $\xrightarrow{\text{then}}$ matter dominates $\to$ curvature (if $\Omega_K \neq 0$) $\to \Lambda$

Behaviour at early times :

Examples: set $\Omega_k = \Omega_\Lambda = \Omega_R = 0$ (matter dominated)

a) $H = H_0 \, a^{-3/2} \sqrt{\Omega_0}$ $a_0 = 1$

$$a(t) = \left(\frac{3}{2} H_0 \sqrt{\Omega_0}\, t \right)^{\frac{2}{3}}$$

b) set $\Omega_k = \Omega_\Lambda = \Omega_0 = 0$ (radiation dominated)

$$\frac{\dot a}{a} = H = H_0 \, a^{-2} \sqrt{\Omega_R} \quad \to \quad a(t) = \left(2 H_0 \sqrt{\Omega_R}\, t \right)^{\frac{1}{2}}$$

c) $\Omega_\Lambda = \Omega_0 = \Omega_R = 0$ (~~cosmological constant~~ curvature dominated)

$$\frac{\dot a}{a} = H_0 \sqrt{\Omega_K}\, a^{-1} \qquad a(t) = H_0 \sqrt{\Omega_K}\, t$$

for $a \to 0$ $\quad \frac{\Omega_R}{a^4} \gg \frac{\Omega_0}{a^3}, \frac{\Omega_k}{a^2}; \Omega_\Lambda$

At early times Universe was radiation dominated

$$\Rightarrow a(t) = \left(2 H_0 \sqrt{\Omega_R}\, t \right)^{\frac{1}{2}}$$

What is Ω_R? $\quad \Omega_R = \frac{\rho_R}{\rho_{crit}} \quad \rho_{crit} = \frac{3H_0^2}{8\pi G} = 1.885 \cdot 10^{-29}\ g\,cm^{-3}\,h^2$

$$\varepsilon_R = \rho_R c^2 = a_R T_0^4 + N \frac{7}{8} a_R T_\nu^4 \qquad N = 3$$

$$T_{0,\nu} = \left(\frac{4}{11} \right)^{\frac{1}{3}} T_0 \qquad T_0 = 2.728\ K$$

$$\varepsilon_r \simeq \Big[1 + \underbrace{\tfrac{21}{8}\left(\tfrac{4}{11}\right)^{\frac{4}{3}}}_{\simeq 0.68}\Big] a_R T_0^4 \qquad a_R = \frac{4\sigma_{SB}}{c} \;\; 7\cdot 10^{-15}\ \mathrm{erg/cm^3/K^4}$$

$$\varepsilon_R = 7.044\cdot 10^{-13}\ \mathrm{erg/cm^3}$$

$\downarrow$

$$\Omega_R = 4.158\cdot 10^{-5}\, h^{-2} \qquad H_0 = \frac{1}{3.085\cdot 10^{17}\,s}\cdot h$$

$\downarrow$

h cancels out in $\sqrt{\Omega_R}H_0$

$$a(t) = \left(\frac{t}{2.389\cdot 10^{19}\,s}\right)^{\frac{1}{2}}$$

The expansion rate in the radiation dominated phase is known to better than 0.1%.

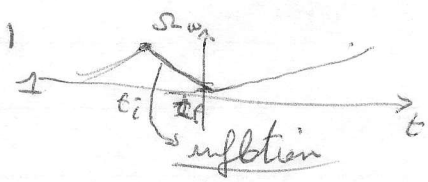
Evolution of Density parameter:

$$\Omega_w(z) = \frac{\rho_w(z)}{\rho_{crit}(z)} \qquad \rho_{crit}(z) = \frac{3H^2(t)}{8\pi G}$$

$$\rho_w = \rho_{0w}\,(1+z)^{3(1+w)}$$

$$H^2(z) = H_0^2 (1+z)^2 \left[\Omega_{0w}(1+z)^{1+3w} + (1-\Omega_{0w})\right]$$

$$\Longrightarrow (\Omega_w^{-1} - 1) = \frac{(\Omega_{0w}^{-1} - 1)}{(1+z)^{1+3w}} \xrightarrow[\substack{1+z\gg 1\\ 0<w<1}]{} 0 \qquad \Omega_w \to 1$$



Cosmological Horizon

$$R_H(t) = a(t)\int_0^t \frac{c\,dt'}{a(t')} \simeq 3\,\frac{1+w}{1+3w}\,ct \sim \frac{c}{H_w}$$

$\downarrow$

$$R_H = \frac{c}{H_0}\frac{a(t)}{a_0}\int_0^{a(t)} \frac{da'}{a'\left[\Omega_{0w}(a_0/a')^{1+3w} + (1-\Omega_{0w})\right]^{1/2}} \simeq \frac{c}{H_0\Omega_{0w}^{\frac{1}{2}}}\frac{2}{3w+1}\left(\frac{a}{a_0}\right)^{\frac{3}{2}(1+w)}$$

eg: $\underline{w=0}$ $\Omega_{0w}=1$ $R_H = \frac{2c}{H_0\sqrt{\Omega_{0w}}}(z+1)^{-\frac{3}{2}} = 3t_0 c\,(z+1)^{\frac{3}{2}}$ $\left(t_0 = \frac{2}{3}\frac{1}{H_0}\right)$