Gravitational Waves from Inflation

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1 What are gravitational waves?

Gravitational waves (GW) are tensor perturbations of the metric of curved spacetime that propagate. The metric g is a matrix (or differential 2-form, to be specific) that fully describes the 4-dimensional manifold of a spacetime in general relativity. The components of the full metric $g_{\mu\nu}$ are the sum of an unperturbed metric and the perturbation $h_{\mu\nu}$, which varies on a smaller scale (separated in Fourier transform) than the unperturbed metric, poetically like ripples moving on a pond surface. For example, on a sufficiently local scale or away from massive bodies, spacetime is not significantly curved, and the spacetime is described by the flat or Minkowski metric $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$ of special relativity. In the linearized gravity approximation, a small perturbation to flat spacetime can be considered, by taking the metric:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \text{ , where } |h_{\mu\nu}| \ll 1. \tag{1}$$

The Einstein equation of general relativ6ity is necessary to find the equation governing the evolution of the tensor perturbation h in flat space:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}. \tag{2}$$

Here, $R_{\mu\nu}$ (Ricci tensor) and R (Ricci scalar) are contractions of the Riemann curvature tensor, which is a function of the metric g, while $T_{\mu\nu}$ is the stress-energy tensor, which describes the physical characteristics of massive sources of spacetime curvature. If terms quadratic in h are ignored (since h itself is small) and the Lorentz gauge condition $\partial^\nu \bar{h}_{\mu\nu}$ is chosen (by close analogy to solutions of the Maxwell equation that describe electromagnetic waves), then the Einstein equation becomes

$$\partial_\alpha \partial^\alpha \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}, \tag{3}$$

where $\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - 1/2\, \eta_{\mu\nu} h^\alpha_\alpha$ is defined. Since $\partial_\alpha \partial^\alpha = -c^{-2}\partial_t^2 + \nabla^2$, this is a wave equation with a source, and suggests that the gravitational waves move at the speed of light c. There were 10 parts in the Einstein equation (4x4 symmetric matrices), minus 4 constraints in the gauge condition, and minus 4 superfluous coordinate degres of freedom, which can be eliminated by imposing that h be traceless, $h^\alpha_\alpha = 0$ (so that $\bar{h}_{\mu\nu} = h_{\mu\nu}$) and transverse, $h_{0i} = 0$. This leaves 2 actual degrees of freedom, which are manifested in two independent GW polarizations, called plus(+) and cross($\times$), in the so-called transverse-traceless(TT) gauge. So, for instance, the spacetime solution in a vacuum ($T_{\mu\nu} = 0$) is a plane wave in the TT gauge:

$$\begin{aligned} ds^2 &= -c^2dt^2 + (1 + h_+ e^{i\omega(t-\frac{z}{c})})dx^2 + h_\times e^{i\omega(t-\frac{z}{c})}dxdy \\ &+ (1 - h_+ e^{i\omega(t-\frac{z}{c})})dy^2 + dz^2, \end{aligned} \tag{4}$$

where h_+ and $h_\times$ are real amplitudes.

But what are the physical properties of GW, and how are the produced? Since the gravitons, or carriers of the gravitational force, are spin-2 bosons, the GW are quadrupolar (multipole $\ell = 2$) or "tensor" waves; whereas electromagnetic waves are dipolar ($\ell = 1$) or ""vector" waves since photons are spin-1 bosons, while acoustic or sound waves for example are "scalar". This means that gravitational waves periodically deform space perpendicular to propagation direction, so that one transverse direction is squeezed as the orthogonal transverse direction is stretched, as suggested by the solution metric above. The amplitudes h_{ij} of the GW are dimensionless and a measure of the stress or strain in a particular direction, meaning ratio of change in length ΔL to that length L in a particular direction. In practice, h should be *extremely* small, has never been measured directly, and is expected to be $h \sim 10^{-20}$ for the strongest astrophysically imaginable source. GW cannot be observed "locally" in a small laboratory, because of the general relativistic property that spacetime appears locally flat for a single observer, and this is why detectors like LIGO or LISA have to observe variations in length on kilometer scales.

The physical reason for the quadrupole nature of GW is just energy-momentum conservation: monopole modes cannot exist by mass-energy conservation ($\dot{m} \sim \dot{M} = 0$), and dipole modes cannot exist by linear and angular momentum conservation ($\ddot{d} \sim \dot{P} = 0$ and $\ddot{\mu} \sim \dot{J} = 0$). Then, a source of GW must be a system with accelerating masses such that the mass-quadrupole moment I_{ij} is changing in time ($\ddot{I} \neq 0$). In fact, the (retarded) solution to the wave equation for $\bar{h}_{\mu\nu}$ becomes approximately proportional to the second time derivative of the mass-quadrupole moment sufficiently far (at distance $r \to \infty$) from a non-relativistic source:

$$h_{\mu\nu}(t, \vec{x}) = \frac{4G}{c^4} \int dV' \frac{T_{\mu\nu}(t - |\vec{x} - \vec{x}'|/c, \vec{x}')}{|\vec{x} - \vec{x}'|} \to \frac{2}{r}\frac{G}{c^4}\ddot{I}_{\mu\nu}(t - r/c), \tag{5}$$

$$\text{where } I_{ij}(t) \equiv \int dV \rho(t, \vec{x})(x_i x_j - \frac{1}{3}\delta_{ij} r^2) \text{ or zero if } i = 0 \text{ or } j = 0, \tag{6}$$

with ρ specifying the mass-density of the source.

2 Sources of gravitational waves

The strongest GW signals are expected to come from coalescing binaries of compact objects - typically black holes or neutron stars. Their frequency ranges from about 10 Hz to about 1000 Hz, scales as the square root of the total mass, and will eventually be detectable by Earth-based interferometric detectors such as advanced LIGO. Indirect observation of gravitational radiation was first observed by Hulse and Taylor in PSR B1913+16, a binary system of a pulsar and a black hole, where the orbital frequency was measured to evolve over many years in a way consistent with the predicted rate of energy loss via GW radiation. Other astrophysical sources include asymmetric pulsars, which should have a periodic signal, and supernovae and other transient or burst sources. Astrophysical sources are all localized (nearly from one point) and “modern” in that they are mostly produced later in cosmological history.

We will henceforth consider “cosmological” sources, which emit a stochastic (or noise-like) background of primordial or relic gravitational radiation, produced early in cosmological history, much like the cosmic microwave radiation. These are ”random” and not point sources, because universal expansion causes primordial radiation to appear as originating in large patches of the sky (again like CMB). The earliest GW signals are expected to originate about 10^{-22} seconds after the Big Bang, during inflation, when gravitons decouple at grand unified theory (GUT) energy scales. Because gravitational radiation does not scatter significally and interacts relatively weakly with matter in the universe, GW signals are expected to carry information from earlier in cosmological history than CMB signals (at recombination). There are many theories that hypothesize the mechanisms of production of primordial GW, and of these the most prevalent are production by inflation and by primordial phase transitions.

One theory (Kosowsky and Kamionkowski, 1992) of primordial GW generation involves a first-order phase transition in the early universe, where bubbles of a new, low energy density phase form within a medium consisting of the old, higher energy density phase. The latent energy released during the transition contributes to the kinetic energy of the bubble walls as the bubbles expand. The bubble walls collide with one another at relativistic speeds and emit gravitational radiation. The GW spectrum should be peaked at a frequency of roughly $10^{-2}Hz$ (as observed today, and in LISA’s range) characteristic of the cosmological time at which this phase transition occured. Another theory (Vilenkin, 1985) involves a second-order phase transition where 1-dimensional defects termed cosmic strings (like vortex lines in the superfluid He transition), which have a mass per unit length $\lambda \sim 10^{22}g/cm$ (GUT scale, where $E_{GUT} \sim 10^{16}GeV$). These cosmic strings are formed in loops which have a tension $T \sim \lambda c^2$, oscillate and collapse in a time $\tau \sim R_H/c$, and then decay slowly and quasi-periodically after breaking, as they would be stable if not for GW emission. This hypothesized process has the strongest predicted GW power spectrum $\Omega_{GW}(f)$, which may even be detectable by advanced LIGO. Of course, these processes, among many others, are pretty speculative and meant to be illustrative rather than serious. A power spectrum vs frequency plot appears in Allen’s review [2].

The emission of GW during inflation is typically simpler, better understood, and more accepted as a background gravitational radiation theory than the aforementioned phase transition processes. The predicted power spectrum $\Omega_{GW}(f)h^2$ (where h is the Hubble constant factor) assuming slow-roll inflation is approximately constant with frequency (for a wide range, from about 10^4 Hz to 10^{-12} Hz), like white noise, and the amplitude upper bound is about 10^{-14}. Basically, during the exponentially rapid expansion ($a \sim e^{Ht}$, like a deSitter universe, where H is constant) in inflation, perturbations arise in post-inflation cosmology from an adiabatically amplified initial, minimal “zero-point” quantum fluctuation (due to uncertainty principle) about the vacuum or ground state. This process is essentially related to the lack of a unique vacuum state in curved spacetime quantum field theory, where vacuum states shift during inflation and are related by Bogoliubov transformations (?). Thus, vacuum fluctuations during inflation generate an adiabatic density (scalar) perturbation as well as GW (tensor perturbation). Conceptually, during inflation, matter-energy moves apart due to expansion and thus emits GW, and this contributes to large-scale anisotropy of the CMB.

3 Inflation models with slow-roll approximation

Consider a field theory in flat spacetime with a single scalar field $\phi(x^\alpha)$, called the inflaton field. Take a Lagrangian density ($c = 1, \hbar = 1$ units),

$$\mathcal{L} = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\nabla\phi \cdot \nabla\phi - V(\phi) = -\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi) = \tag{7}$$

$$= -\frac{1}{2}\partial^\nu\phi\partial_\nu\phi - V(\phi) \text{ , with potential } V(\phi) = \frac{1}{2}m^2\phi^2 + \frac{1}{2}\lambda\phi^4 + \ldots . \tag{8}$$

Minimizing the resulting action S over arbitrary variations $\delta\phi$ which vanish at the integration boundary (or "infinity") gives the field equation:

$$\delta S = \int dt \int d^3x \delta\mathcal{L} = \int d^4x \left[\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu)}\right]\delta\phi = 0 \tag{9}$$

$$\Rightarrow -\partial^\nu\partial_\nu\phi + \frac{dV}{d\phi} = \ddot{\phi} - \nabla^2\phi + V'(\phi) = 0. \tag{10}$$

Without non-linear interaction terms in the potential ($\lambda = 0$), we get $V'(\phi) = m^2\phi$, and in this case the field equation is just the Klein-Gordon equation for a free particle with mass m. Next, for a universe that is expanding in curved spacetime, with Hubble constant $H \equiv \dot{a}/a$, this field equation gets an additional term, $+3H\dot{\phi}$, and the Laplacian gets a factor a^{-2}. Consider the spatial perturbation to the scalar field and its Fourier components, defined by

$$\phi(x^\mu) \equiv \phi_0(t) + \delta\phi(t, \vec{x}) \equiv \phi_0(t) + \sum_{\vec{k}} \delta\phi_{\vec{k}}(t)e^{i\vec{k}\cdot\vec{x}} \text{ . Then} \tag{11}$$

$$(\delta\ddot{\phi}_{\vec{k}}) + 3H(\delta\dot{\phi}_{\vec{k}}) + (k/a)^2(\delta\phi_{\vec{k}}) + m^2(\delta\phi_{\vec{k}}) + o[(\delta\phi_{\vec{k}})^3] = 0, \tag{12}$$

is the field equation for this spatial Fourier component $\delta\phi_{\vec{k}}(t)$ of the scalar field. This perturbation component exits the horizon when its covariant wavelength $a\lambda$ reaches the Hubble "circumference" $2\pi R_H = 2\pi c/H$, or equivalently k drops below aH (in $c = 1$ units).

The slow-roll approximation (described in Liddle&Lyth textbook[1]) is a set of conditions that apply before ($k \geq aH$) the perturbation component has exited the horizon that ensures that inflation proceeds and does so in an adiabatic (or quasi-static) way. With the simple model potential described above, assuming $\lambda < 0$, a second-order phase transition occurs (like spontaneous symmetry breaking or magnetization below Curie temperature) in which the inflaton state $\phi(t)$ will shift adiabatically (or "roll slowly" down the potential) from the initial $\phi = 0$ vacuum state which becomes unstable after the critical time, to either of the newly created potential minima (vacuum states) at $\phi = \pm m/(-\lambda)$ after the critical time. In the other models of inflation, a qualitatively similar phenomenon occurs. Let $M_P \equiv (8\pi G)^{-1/2}$ be the reduced Planck mass (in $c = 1, \hbar = 1$ units), and define two slow-roll parameters (which differ for different models),

$$\epsilon(\phi) \equiv \frac{M_P^2}{2}\left(\frac{V'(\phi)}{V(\phi)}\right)^2 \simeq -\frac{\dot{H}}{H^2} \text{ , } \eta(\phi) \equiv M_P^2\left(\frac{V''(\phi)}{V(\phi)}\right)^2 . \tag{13}$$

The four slow-roll conditions are as follows:

$$\text{(i) } V'(\phi) \simeq -3H\dot{\phi} \text{ , (ii) } V(\phi) \simeq 3H^2 M_P^2 = \rho_{crit} \text{ , (iii) } \epsilon \ll 1 \text{ , (iv) } |\eta| \ll 1. \tag{14}$$

Here, (i) means that the evolution is approximately harmonic oscillator-like and curvature is locally flat during inflation, because the Fourier transformed field equation before horizon exit becomes

$$(\delta\ddot{\phi}_{\vec{k}}) + E_k^2(\delta\phi_{\vec{k}}) \simeq 0 \text{ , where } E_k^2 = (k/a)^2 + m^2 \simeq (k/a)^2 \tag{15}$$

is total energy. However, soon after horizon exit, because ($k \ll a \sim e^{Ht}$) the third term in this field equation disappears and we have the "modern", k-independent evolution evolution $(\delta\ddot{\phi}_{\vec{k}}) + 3H(\delta\dot{\phi}_{\vec{k}}) + m^2(\delta\phi_{\vec{k}}) \simeq 0$. Next, (ii) is the critical density condition and suggests that kinetic energy is much less than potential energy ($\dot{\phi}^2 \ll V(\phi)$), since energy density in field theory is given by $\rho = \dot{\phi}^2 + V(\phi) \simeq V(\phi)$, and again that the universe is nearly flat during inflation ($\rho \simeq \rho_{crit}$). Condition (iii) can be simply interpreted to mean that the slope of the potential down which the inflaton state rolls is not steep. Also, (iv) constrains the curvature (second derivative) of the smooth potential curve, suggests that the fourth term in the Fourier component field equation is negligible relative to the third (and also second) term before horizon exit, since together with (ii), it states that $|\eta| \ll 1$ or $M_P^2 V''(\phi) \ll V(\phi)$ or $m^2 \ll 3H^2 \leq 3(k/a)^2$.

The power spectrum $P_\phi(k)$ of the Gaussian vacuum fluctuation about the initial ground state $|0\rangle$ (such that $\mathbf{a}_k|0\rangle = 0$) during inflation can be found quantum mechanically using the harmonic oscillator equation above. The solution well before horizon exit, in terms of initial creation $\mathbf{a}_k^\dagger$ and annihilation $\mathbf{a}_k$ operators, is

$$\begin{aligned}
\delta\phi_{\vec{k}}(t) &= w_k(t)\mathbf{a}_k + w_k^*(t)\mathbf{a}_{-k}^\dagger \text{ , where} && (16)\\
w_k(t) &= \frac{1}{L^{3/2}}\frac{H}{(2k^3)^{1/2}}\left(i + \frac{k}{aH}\right)\exp\left(\frac{ik}{aH}\right) && (17)\\
&\simeq \frac{1}{(aL)^{3/2}}\frac{1}{(2E_k)^{1/2}}e^{-iE_k t} \text{ , when } \frac{k}{aH} \simeq \left.\frac{k}{aH}\right|_T - \frac{k}{a}(t-T),
\end{aligned}$$

where the more recognizable approximated expression uses that $E_k \simeq k/a \gg H$ and ignores a slowly-varying phase factor, and T is some time well before the horizon exit, so that $|t-T| \ll H^{-1}$. Then, the expectation of the mean-square vacuum state fluctuation amplitude is

$$\langle|\delta\phi_{\vec{k}}(t)|^2\rangle = \langle 0|\delta\phi_{\vec{k}}^\dagger(t)\delta\phi_{\vec{k}}(t)|0\rangle = w_k(t)w_k^*(t)\langle 0|\mathbf{a}_k\mathbf{a}_{-k}^\dagger|0\rangle = |w_k(t)|^2, \tag{18}$$

and it allows one to find the power density, which remains approximately static after horizon exit time t_*:

$$\begin{aligned}
\langle|\delta\phi_{\vec{x}}(t_*)|^2\rangle &\equiv \int_0^\infty P_\phi(k;t_*)\frac{dk}{k} = \frac{1}{(2\pi)^3}\int d^3x\langle|\delta\phi_{\vec{k}}(t_*)|^2\rangle && (19)\\
\Rightarrow P_\phi(k;t>t_*) &= k\cdot\frac{1}{(2\pi)^3}\int d^3x\int_{|\vec{k}'|=k} d^2k'\langle|\delta\phi_{\vec{k}}(t_*)|^2\rangle\\
&= \frac{2k^3L^3}{(2\pi)^2}|w_k(t_*)|^2 = \frac{2k^3L^3}{(2\pi)^2}\frac{H^2(t_*)}{2k^3L^3} = \left(\frac{H}{2\pi}\right)^2|_{k=aH} && (20)
\end{aligned}$$

This inflaton vacuum fluctuation power spectrum at horizon exit gives rise to both the scalar adiabatic density (or curvature) perturbation and the tensor GW perturbation. Because the power density scales as $(\delta\phi_{\vec{k}})^2$, the power spectrum of another perturbation generated by the inflaton vacuum fluctuation, say $\delta\psi = \kappa(\delta\phi_{\vec{k}})$ for constant κ, would have a power spectrum $P_\psi(k) = \kappa^2 P_\phi$.

4 Density perturbations produced during slow-roll inflation

We use the relation between the adiabatic energy density perturbation $\delta_{\vec{k}}(t)$ and the inflaton vacuum fluctuation $(\delta\phi_{\vec{k}})$ that causes it. On very large scales that enter the horizon after matter-domination, and ignoring the cosmological constant, it is found in equations (4.6) and (5.2) of [1] to be

$$\delta_{\vec{k}}(t) = -\frac{2}{5}\left(\frac{k}{aH}\right)^2\left(\frac{H}{\dot{\phi}}\delta\phi_{\vec{k}}\right)|_{t=t_*} = -\frac{2}{5}\left(\frac{H}{\dot{\phi}}\right)|_{k=aH}\cdot\delta\phi_{\vec{k}}. \tag{21}$$

Then, the power density of adiabatic energy density perturbations, in the slow roll approximation of inflation and using the slow-roll conditions and parameter ϵ, is

$$\begin{aligned} P_\delta(k) &= \frac{4}{25}\left(\frac{H}{\dot{\phi}}\right)^2\left(\frac{H}{2\pi}\right)^2|_{k=aH} = \frac{1}{25\pi^2}\frac{(H^2)^3}{H^2\dot{\phi}^2} & (22)\\ &\simeq \frac{1}{25\pi^2}\left(-\frac{3}{V'(\phi)}\right)^2\left(\frac{V}{3M_P^2}\right)^3 = \frac{1}{150\pi^2 M_P^4}\frac{V}{\epsilon}. \end{aligned}$$

Suppose that $P_\delta(k)$ has a power law dependence on k, with a spectral index n defined by $P_\delta(k) \simeq P_{\delta 0}k^{n-1}$ by convention. We can find this spectral index for adiabatic density perturbations by algebraic manipulation of the above expression using the slow-roll conditions. At horizon exit, since H varies much slower than scale parameter a in time, we have

$$\begin{aligned} \frac{d[log(k)]}{dt} &= \frac{1}{k}\frac{dk}{dt} = \frac{\dot{a}H + a\dot{H}}{k} \simeq \dot{a}\frac{H}{k} = \frac{\dot{a}}{a} = H & (23)\\ \Rightarrow d[log(k)] &\simeq Hdt = \frac{H}{\dot{\phi}}d\phi \simeq -\frac{3H^2}{V'(\phi)}d\phi \simeq -\frac{1}{M_P^2}\frac{V(\phi)}{V'(\phi)}d\phi & \\ \Rightarrow \frac{d\epsilon}{d[log(k)]} &\simeq -M_P^2\frac{V'(\phi)}{V(\phi)}\frac{d}{d\phi}\left[\frac{M_P^2}{2}\left(\frac{V'(\phi)}{V(\phi)}\right)^2\right] = -2\epsilon\eta + 4\epsilon^2 & (24)\\ \Rightarrow n-1 &\equiv \frac{d[log(P_\delta(k))]}{d[log(k)]} = \frac{1}{P_\delta(k)}\frac{dP_\delta(k)}{d[log(k)]} = \frac{\epsilon}{V(\phi)}\frac{d}{d[log(k)]}\left[\frac{V(\phi)}{\epsilon}\right] & \\ &= \frac{1}{V(\phi)}\frac{dV(\phi)}{d[log(k)]} - \frac{1}{\epsilon}\frac{d\epsilon}{d[log(k)]} = -6\epsilon + 2\eta = n-1. & (25) \end{aligned}$$

One could also find the derivatives of η and of $(n-1)$ using this method, from which it can be concluded that spectral index $n(k)$ has a slight dependence on k.

In most models of inflation and their values of the slow-roll parameters [3], the spectral index will have a conservative constraint $|n-1| < 0.30$, so $n > 0.70$

in "natural" inflation. In extended inflation theories, which are based on Brans-Dicke theory, one would find $n > 0.84$. However, in order for GW from bubble wall collisions to not be observed [3] in LIGO (as no GW background has been observed yet), $n < 0.75$ is required. This fact may be used to rule out extended inflation theories "experimentally".

5 Gravitational wave perturbations produced during slow-roll inflation

In order to relate gravitational waves to the inflaton field, we consider the Lagrangian density in clasical general relativity which occurs in the Einstein-Hilbert action. In terms of reduced Planck mass M_P, it is proportional to the Ricci scalar and requires an additional matter Lagrangian density. Minimizing the Einstein-Hilbert action for arbitrary metric variations recovers the Einstein equation as its field equation:

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2}M_P^2 R(g_{\mu\nu}) + \mathcal{L}_{matter} \qquad (26)\\
\delta S &= \delta \int d^4x\sqrt{-g}\mathcal{L} = \frac{M_P^2}{2}\int d^4x \left[\frac{\delta(\sqrt{-g}R)}{\delta g_{\mu\nu}} + \frac{2}{M_P^2}\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g_{\mu\nu}}\right]\delta g_{\mu\nu} = 0\\
&\Rightarrow\ R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{1}{M_P^2}\left(g_{\mu\nu}\mathcal{L}_m - 2\frac{\delta\mathcal{L}_m)}{\delta g_{\mu\nu}}\right) \equiv \frac{8\pi G}{c^4}T_{\mu\nu}.
\end{aligned}$$

If one compares this field equation with the one for a massless scalar field discussed earlier, they are identical when the identification $\delta\psi_{+,\times} = (M_P/sqrt(2))h_{+,\times}$ is made. So, the time-evolution of the GW amplitudes is the same as that of the inflaton field perturbation ϕ, but now since $\kappa = sqrt(2)/M_P$ is the factor, the power density of gravitational radiation is

$$P_{GW}(k) = \frac{2}{M_P^2}\left(\frac{H}{2\pi}\right)|_{k=aH} = \frac{1}{2\pi^2}\frac{1}{M_P^2}H^2 = \frac{1}{6\pi^2}\frac{1}{M_P^4}V. \qquad (27)$$

As before, suppose that $P_{GW}(k)$ has a power law dependence on k, with a spectral index n_{grav} defined by $P_\delta(k) \simeq P_{GW0}k^{n_{grav}}$ by convention. We can find this spectral index for gravitational waves by the same algebraic manipulation using the slow-roll conditions as earlier. At horizon exit, we have

$$\begin{aligned}
n_{grav} &\equiv \frac{d[log(P_{GW}(k))]}{d[log(k)]} = \frac{1}{P_{GW}(k)}\frac{dP_{GW}(k)}{d[log(k)]} = \frac{1}{V(\phi)}\frac{d[V(\phi)]}{d[log(k)]} \qquad (28)\\
&= \frac{1}{V(\phi)}\left(-M_P^2\frac{V'(\phi)}{V(\phi)}\frac{dV(\phi)}{d\phi}\right) = -2\epsilon = n_{grav}, \qquad (29)
\end{aligned}$$

so $|n_{grav}| \ll 1$ in practice, which means that the power spectrum of the GW background due to inflation is white-noise like and independent of frequency, as can be seen in the plot in Allen[1]. Also the ratio of the power spectrum of GW is simply related to that of density perturbations: $P_{GW}(k)/P_\delta(k) = 25\epsilon$, which is still less than one in any inflation model.

Finally, if one uses an approximate theoretical result (Starobinsky, 1985) for the anisotropy coefficients C_ℓ(tensor) of the multipole expansion in the fractional background of CMB contributed by tensor perturbations (gravitational radiation) at the large-scale (for $\ell < 100$), one can compare the contributions to the CMB anisotropy of scalar (energy density perturbations) and tensor (gravitational waves) components. The large-scale anisotropy of scalar perturbations is governed by the Sachs-Wolfe effect, according to which they are nearly constant at the large scale. From eq. (5.40,5.41) and (6.42) in [1]:

$$\ell(\ell+1)C_\ell(\text{tensor}) = \frac{\pi}{9}\left(1+\frac{48\pi^2}{385}\right)c_\ell P_{GW}(k) \text{ , while} \tag{30}$$

$$C_\ell(\text{scalar}) = \frac{\pi}{2}\left\{\frac{\sqrt{\pi}}{2}\frac{\Gamma(1-\frac{n-1}{2})}{\Gamma(3/2-\frac{n-1}{2})}\frac{\Gamma(\ell+\frac{n-1}{2})}{\Gamma(\ell+2-\frac{n-1}{2})}\right\}P_\delta(k). \tag{31}$$

Here, the dimensionless factor c_ℓ is close to 1 and $c_\ell \to 1$ for $\ell > 10$. Also, the term in the curly brackets equals $(\ell(\ell+1))^{-1}$ when $n-1=0$, so it can be ignored since the spectral index $n-1=-6\epsilon+2\eta$ is small. This gives a ratio for the anisotropy contributions to the CMB:

$$\frac{C_\ell(\text{tensor})}{C_\ell(\text{scalar})} = 12.4\epsilon. \tag{32}$$

In the multipole anisotropy plot of the cosmic microwave background, the tensor contribution is about a fourth of the total (for $\epsilon \sim 0.02$) up to about $\ell = 100$ and drops off at lower scales.

6 Bibliography

(Derivations are mostly from the first textbook reference)

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