1 Fermi-Dirac and Bose-Einstein distributions (15pts)

In class we have calculated the mean particle number $\langle N \rangle$ and the pressure for a gas of photons (Bosons with chemical potential $\mu = 0$).

(a) Derive expressions for $\langle N \rangle$ and the pressure, $P$, for a gas of fermions in the relativistic and non-relativistic limits. Start from the expressions derived in class for the grand potential and grand canonical partition function for Fermions. The general expression for $\langle N \rangle$ and the pressure $P$ can be written in terms of the Fermi integrals:

$$ f_n(z) = \frac{1}{\Gamma(n)} \int_0^\infty \frac{x^{n-1} dx}{z^{-1}\exp(x) + 1}, $$

where $z = \exp(\mu/kT)$ and $\Gamma$ is the Gamma function.

(b) Write down the general expressions (for $\mu \neq 0$) for $\langle N \rangle$ and $P$ for a Boson gas in the relativistic and non-relativistic limits. You can write the results in terms of the Bose integrals:

$$ g_n(z) = \frac{1}{\Gamma(n)} \int_0^\infty \frac{x^{n-1} dx}{z^{-1}\exp(x) - 1}. $$

(c) Derive the equation of state, $P(n, T)$, for Fermions and Bosons in the relativistic and non-relativistic limits. Explore the differences between degenerate ($\mu > kT$) and non-degenerate ($\mu \ll kT$) non-relativistic gas and degenerate ($\mu > mc^2$) relativistic gas.

Note: this problem is long but quite easy. Please, be concise.

2 Three-level quantum System (15pts)

Consider a system of two particles which can be in any of three quantum states of energy 0, $\Delta$ or $3\Delta$. The system is in contact with a heat reservoir at temperature $T$. Write down the partition function $Z(T)$ and the grand partition function $Q(\mu, T)$ for three cases:

(a) Maxwell-Boltzmann (distinguishable) particles.

(b) Fermi-Dirac (indistinguishable) particles (Fermions).

(c) Bose-Einstein (indistinguishable) particles (Bosons).

Be sure to account for the multiplicity of ways ($g_i$) of obtaining a given total energy $E_i = \sum n_i \epsilon_i$, where $n_i$ is the occupation number and $\epsilon_i$ is the energy level. Hint: For Maxwell-Boltzmann (distinguishable) particles the partition function for $N$ particles is $Z_N = (Z_1)^N$, where $Z_1$ is the partition function for one particle (you can verify that). The grand partition function is related to $Z_N$ by: $Q = \sum_{N=0}^{\infty} \exp[N\mu/kT]Z_N$. 
