THERMAL EFFECTS IN THE ULTRARELATIVISTIC TWO-STREAM INSTABILITY

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Abstract. The dispersion relation for longitudinal waves in a one-dimensional ultrarelativistic plasma is calculated. Analytical and numerical results for the growth rate and frequency of the two-stream instability are presented as a function of the energy spread in the denser stream when the dilute stream is cold. The case of energy spreads in both beams is investigated numerically: it is found that relatively small energy spreads in both streams can lead to suppression of the instability.

1. Introduction

The ultrarelativistic two-stream instability has been proposed as a mechanism for bunching radio-emitting electrons in the magnetosphere of a pulsar (e.g., Ruderman and Sutherland, 1975). The idea is that the magnetospheric plasma above the polar cap consists of several components streaming away from the pole: it may include ‘primary’ positrons created in the ‘spark gap’ (with Lorentz factors $\gamma \sim 10^7$; Ruderman and Sutherland, 1975), ‘secondary’ electron-positron plasma from pair creation ($\gamma \sim 10^5$) and iron nuclei ripped from the crust of the neutron star ($\gamma \sim 500$; Cheng and Ruderman, 1980). These components are streaming relative to each other, and the relative streaming provides a source of free energy which may drive an instability. The instability is assumed to result in the growth of electrostatic waves, and particles may be bunched in the spatially-oscillating wave fields. The bunched particles radiate by coherent curvature emission, and can explain the observed radio-emission from pulsars.

In the strong magnetic field of a pulsar the plasma is forced to follow the magnetic field lines, and it may be treated as one-dimensional. This paper is concerned with the dispersion relation and growth rate of electrostatic waves in a one-dimensional ultrarelativistic two-stream plasma. Due to the nature of the origins of the streams they are unlikely to be monoenergetic: indeed, with the exception of nuclei torn from the polar cap which are all accelerated through the spark-gap potential drop (Cheng and Ruderman, 1980), one expects that the spread of energies ($\Delta \gamma mc^2$) of particles in a stream will be comparable to the mean streaming energy ($\gamma mc^2$). This spread of energies was included in the analyses of the ultrarelativistic two-stream instability by Hinata (1976) and Cheng and Ruderman (1977). It is necessary, however, to analyze this instability further because the above authors used a form for the dispersion relation which does not correctly include the effects of the energy spread. Here we calculate the appropriate form of the dispersion relation, and investigate the growth rate of the instability.

In Section 2 the longitudinal part of the dielectric tensor is calculated for a single
ultrarelativistic streaming plasma component. Section 3 considers the dispersion relation for the two-stream instability in the case when the denser stream is slower and includes a spread of energies, while the faster dilute stream is a cold beam. Numerical results in a specific case are presented; the dispersion relation is investigated analytically in the Appendix using a method due to Hinata (1976). The effect of an energy spread in the faster beam is investigated numerically in Section 4.

2. The Dielectric Tensor of an Ultrarelativistic Stream

For a one-dimensional plasma with a momentum distribution \( f(p) \) the longitudinal part of the dielectric tensor (which describes electrostatic waves) takes the form (e.g., Melrose, 1980; p. 40)

\[
\varepsilon' = 1 + \frac{4\pi e^2}{\omega^2} \int dp \frac{\omega v}{\omega - kv} \frac{\partial}{\partial p} f(p),
\]

where \( v \) is the velocity corresponding to the momentum \( p \). Equation (1) omits any possible effects due to field-line curvature, and for pulsars one assumes that the particle motion is one-dimensional and, hence, the magnetic field does not affect the instability. The validity of these assumptions for application to pulsars has been criticized by Asseo et al. (1980, 1983): their calculations include curvature effects, but not the effect of a spread of energy on the instability, which is the main point of interest here.

For ultrarelativistic particles the following approximations are assumed to be valid:

\[
\gamma \approx \frac{p}{mc}; \quad \frac{v}{c} \approx 1 - \frac{m^2c^2}{2p^2}; \quad \frac{\partial v}{\partial p} \approx \frac{1}{m\gamma^3}.
\]

Using (2), integrating (1) by parts and rearranging, one finds that

\[
\varepsilon' = 1 + \frac{4\pi e^2}{m} \frac{\partial}{\partial \omega} \int_{-\infty}^{\infty} \frac{f(p)}{\gamma^3} \frac{dp}{\omega - kv}.
\]

Evaluation of (3) must take into account the poles of the factor \((\omega - kv)^{-1}\), which occur on the real axis provided that \( kc \geq \omega \). The latter condition is appropriate here since if \( kc < \omega \) the wave may not have a resonant interaction with particles (Ginzburg and Zheleznyakov, 1975). To integrate around poles on the real axis one must use the causal prescription of Landau: the frequency \( \omega \) is given a small positive imaginary part, \( \omega \rightarrow \omega + i0 \). Using (2) one writes

\[
\omega - kv = \frac{\omega - kc}{p^2} (p^2 - D^2 - i0),
\]
with

\[ D^2 = \frac{kcm^2c^2}{2(kc - \omega)} \geq 0; \]

and \( D \) is assumed to be positive.

Suppose that the plasma consists of a single component with a Lorentzian distribution of momenta,

\[ f(p) = \frac{n}{\pi} \frac{\Delta p}{(p - p_0)^2 + \Delta p^2}, \quad \text{(4)} \]

where \( p_0 \) is the streaming momentum, \( \Delta p \) the spread of momenta in the stream and \( n \) is the number density of particles. Then the integral in (3) takes the form

\[ I = \int_{-\infty}^{+\infty} dp \frac{p^2}{(1 + (p^2/m^2c^2))^{3/2}} \frac{1}{(p - p_0)^2 + \Delta p^2} \frac{1}{p^2 - D^2 - i0}. \quad \text{(5)} \]

The integral in (5) is not amenable to the usual technique of reduction to a standard form. The presence of the square-root leads to difficulties in an exact evaluation of the integral by use of Cauchy’s residue theorem. Here we use an accurate approximation to the integral in (5), relying on the following observation. The function \( x^2(1 + x^2)^{-3/2} \) behaves \( 1/|x| \), except in a region of width of order unity around \( x = 0 \). We replace it by the function \( |x|/(1 + x^2) \), which has the same behaviour except in a narrow region close to \( x = 0 \). Since by assumption \( p_0/mc, \Delta p/mc, \text{ and } D/mc \) are all much larger than unity the contribution from this narrow region is small. It should then be valid to approximate \( I \) by \( I' \), where

\[ I' = \int_{-\infty}^{\infty} dp \frac{|p|}{1 + (p/mc)^2} \frac{1}{(p - p_0)^2 + \Delta p^2} \frac{1}{p^2 - D^2 - i0}. \]

\( I' \) may be evaluated exactly by separating the denominator into partial fractions and integrating each term separately. After considerable algebra, and making use of the assumptions that \( p_0, \Delta p, D \gg mc \), one finds that

\[ \epsilon' = 1 + \omega_p^2 \frac{m}{k} \frac{\delta}{\delta\omega} \left\{ \frac{2x_+}{p_0 + i\Delta p} \frac{D^2}{D^2 - (p_0 - i\Delta p)^2} + \frac{D^2}{p_0 + i\Delta p} \frac{D^2}{D^2 - (p_0 - i\Delta p)^2} \right\} \times \left[ \frac{1}{p_0 - i\Delta p} - \frac{1}{p_0 + i\Delta p} \right] \text{,} \quad \text{(6)} \]
where

\[
\alpha_{\pm} = \frac{1}{\pi} \tan^{-1} \left[ \frac{p_0}{\Delta p} \right] \pm \left[ -\frac{1}{2} + \frac{i}{2\pi} \log \left( \frac{p_0^2 + \Delta p^2}{m^2c^2} \right) \frac{kc - \omega}{kc} \right].
\]

Note that the poles \((p_0 + i\Delta p)\) and \((p_0 - i\Delta p)\) do not appear symmetrically in (6), due to the causal prescription. Carrying out the differentiation in (6) and replacing \(D\) by physical variables, we find (with the notations \(\Delta \gamma = \Delta p/mc\) and \(\gamma = p_0/mc\))

\[
e' = 1 - \omega_p^2 \left\{ \frac{\alpha_-}{(\gamma - i\Delta \gamma)^3} \left[ \omega - ku_- \right]^{-2} + \frac{\alpha_+}{(\gamma + i\Delta \gamma)^3} \left[ \omega - ku_+ \right]^{-2} - \frac{i}{\pi(\gamma - i\Delta \gamma)} \frac{1}{kc[\omega - ku_-]} + \frac{i}{\pi(\gamma + i\Delta \gamma)} \frac{1}{kc[\omega - ku_+]} \right\},
\]

(7)

with

\[
u_{\pm} = c \left[ 1 - \frac{1}{2(\gamma \pm i\Delta \gamma)^2} \right].
\]

Consider the modes of a single ultrarelativistic stream. Equation (7) may be written as a fourth-order polynomial in the wave-vector \(k\) and, hence, has four roots. In the limit of very small thermal spread two of the roots must lie close to the pole at \(k = \omega/u_+\) (this is apparent by comparing (7) with (8), see below). Since \(\text{Im}(k) < 0\), these modes appear to be growing. Their growth rate increases with frequency, and with thermal spread. All the roots of the dispersion relation satisfy Briggs’s (1964) criterion for amplified waves (this is easily seen in the form of (7): the main dependence on \(k\) is in terms of the form \((\omega - ku)\) with \(u \approx c\); such terms are effectively invariant under the transformation \(\omega \rightarrow \omega + i\delta, kc \rightarrow kc + i\delta\), which is the required test). To check whether these growing modes arise from the approximation used in replacing \(I\) with \(I'\) (which may be interpreted as altering the particle distribution in the vicinity of \(p = 0\) so as to produce a spurious stream there) the dispersion relation was recalculated using the approximation \(|x|^3/(1 + x^2)^2\) to the factor \(x^2/(1 + x^2)^{3/2}\) which appears in (5). This new approximation is identical to the first approximation except near \(x = 0\). After considerable algebra it may be shown that in the limit that \(p_0 \gg mc\) and \(D \gg mc\) the result (7) is exactly reproduced. It follows that, as assumed earlier, the region around \(p = 0\) is not an important determinant of the form of the dispersion relation (7). Since a single stream should be stable the explanation of these apparently growing modes is not clear. Now in the case of a single nonrelativistic stream with a thermal spread there are only two modes, and both are damped (Montgomery and Tidman, 1964; p. 178). We, therefore, assume that the additional apparently growing modes in the ultrarelativistic case are not real, and they are omitted from further discussion.

We now derive the form of the dielectric tensor used by Hinata (1976) and Cheng and Ruderman (1977). Consider the limit \(\Delta \gamma/\gamma \ll 1\). The appropriate approximation may be
derived from (7), or alternatively the following physical argument holds. When \( \Delta \gamma \ll \gamma \) there should be no contribution from the region of \( p = 0 \) (i.e., slow particles) in (5), and we may treat the poles at \( p = p_0 \pm i\Delta p \) as poles on the real axis (this assumption omits some first-order terms present in (7)). One then finds that, in this limit, the dielectric tensor is given by

\[
\varepsilon' = 1 - \frac{\omega_p^2}{(\gamma - i\Delta \gamma)^2} \left[ \omega - kc + \frac{kc}{2(\gamma - i\Delta \gamma)^2} \right]^{-2}.
\]  

(8)

Note that only the pole \((p_0 - i\Delta p)\) contributes in (8).

Now if one assumes that \( \Delta \gamma \ll \gamma \) and that the momentum spread is only important in the resonant denominator of (8), one finds that

\[
\varepsilon' = 1 - \frac{\omega_p^2}{\gamma^3} \left[ \omega - ku + i(\Delta p/\gamma mc)kc \right]^2
\]

(9)

with \( \gamma = p_0/mc \) and \( u = c(1 - m^2c^2/2p_0^2) \). Equation (9) exactly reproduces the dielectric tensor used by Hinata (1976, Equation (3)) and Cheng and Ruderman (1977, Equation (14)). However, these authors discuss electrostatic waves in the limit \( \Delta \gamma = \gamma \), when (9) is not a valid approximation to (7). Even in the limit \( \Delta \gamma \ll \gamma \), Equation (9) gives incorrect results because of the neglect of first-order terms present in (7). The next section considers the growth rates of waves in a two-stream plasma using (7): there it is shown that the approximation (9) underestimates the growth rate of the two-stream instability.

3. The Two-Stream Instability

Following Hinata (1976) and Cheng and Ruderman (1977) we analyze the modes of a plasma consisting of two components: a dilute cold beam (subscript \( b \)) together with a slower hot stream which acts as a background plasma (subscript \( p \)). From the previous section it follows that the modes of this system satisfy the dispersion relation

\[
\varepsilon' = 1 - \frac{\omega_p^2}{(\gamma_p - i\Delta \gamma)^3} \left[ \omega - ku_+ \right]^2 - \frac{\omega_p^2}{(\gamma_p + i\Delta \gamma)^3} \left[ \omega - ku_- \right]^2 - \\
- \frac{i\omega_p^2}{\pi(\gamma_p - i\Delta \gamma)} \frac{1}{kc[\omega - ku_-]} + \frac{i\omega_p^2}{\pi(\gamma_p + i\Delta \gamma)} \frac{1}{kc[\omega - ku_+]} - \\
- \frac{\omega_b^2}{\gamma_b^3} \left[ \omega - kc + \frac{kc}{2\gamma_b^2} \right]^2 = 0,
\]

(10)

where \( \Delta \gamma mc^2 \) is the energy spread in the slower plasma stream. The addition of the dilute beam term to (7) converts the dispersion relation to a sixth-order polynomial. Assuming that

\[
\frac{n_b \gamma_p^3}{n_p \gamma_b^3} \ll 1,
\]

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the four modes corresponding to the background plasma stream are essentially unchanged by the addition of the beam. One of the two new roots of the dispersion relation becomes the mode driven unstable by the two-stream instability. An analytic approximation to the solution of (10) is discussed in the Appendix, following closely a method used by Hinata (1976). Due to the greater complexity of (10) as compared with the dispersion relation solved by Hinata (see Appendix, Equation (11)) the analytic approximation found there is not as accurate as the corresponding result of Hinata (1976). In this section we present numerical calculations of the instability.

To illustrate the behaviour of the instability we present the results of numerical calculations for a system with parameters \( \gamma_p = 100, \gamma_b = 500, n_p/n_p = 0.1 \), for a range of values of \( \Delta \gamma \). These are plausible parameters for the pulsar application. The roots of the sixth-order polynomial corresponding to the dispersion relation (10) were obtained using the complex root-finding program CROOTS (Botten et al., 1983; note that CROOTS requires analytic functions and, hence, the search area had to be chosen so as to exclude the branch cut in the logarithmic term in \( u_x \)). For any given real frequency there are six complex roots \( k \). Roots with \( \text{Im}(k) < 0 \) correspond to amplified waves.

Figure 1 shows the dimensionless spatial growth rate, \(- \text{Im}(k c \gamma_p^{3/2}/\omega_p)\), as a function of the dimensionless frequency variable, \( \omega / \gamma_p^{1/2} \omega_p \) (see Appendix), for various values of

\[ -\text{Im}(k c \gamma_p^{3/2}/\omega_p) \]

![Fig. 1. Numerical solutions of the dispersion relation (10), showing the dimensionless spatial growth rate \(- \text{Im}(k c \gamma_p^{3/2}/\omega_p)\) as a function of the dimensionless frequency variable \( \omega / \gamma_p^{1/2} \omega_p \) for four values of the relative energy spread in the denser stream \( \Delta \gamma / \gamma_p \) = 0.01, 0.10, 0.50, 1.00. Other parameters are \( \gamma_p = 100, \gamma_b = 500, \) and \( n_p/n_p = 0.1 \).]

the energy spread parameter \( \Delta \gamma \). The growth rate is a sharp function of frequency at small values of \( \Delta \gamma / \gamma_p \), becoming broader as \( \Delta \gamma / \gamma_p \) increases; growth occurs at all (positive) frequencies, but the growth rate goes to zero at large frequencies. All three
curves show a maximum growth rate occurring at about $\omega \approx 2\gamma_p^{1/2} \omega_p$. This result is further investigated in Figure 2, which shows the frequency corresponding to maximum growth as a function of $\Delta\gamma/\gamma_p$. For comparison, Figure 2 also shows the curve resulting

![Graph](image)

**Fig. 2.** Graphs of the frequency corresponding to the maximum instability growth rate as a function of the relative energy spread in the denser stream. The three curves represent the numerical solution of the full dispersion relation (10); the numerical solution of Hinata’s dispersion relation, Equation (11) in the Appendix; and the analytical approximation to the solution of (11) discussed in part (a) of the Appendix. Relevant parameters are $\gamma_p = 100$ and $\gamma_b = 500$.

from Hinata’s analytical formula (see Appendix) and the curve found by numerical solution of Hinata’s dispersion relation (11). It is apparent that whereas Hinata’s dispersion relation gives an instability frequency which falls off as $\Delta\gamma$ increases, in fact the instability frequency remains close to, but slightly above, the value of $2\gamma_p^{1/2} \omega_p$.

The maximum growth rate as a function of $\Delta\gamma/\gamma_p$ is presented in Figure 3, again using dimensionless variables. Three curves are shown: numerical solutions of the full dispersion relation (10); numerical solutions of Hinata’s dispersion relation (11); and the corresponding analytical approximation of Section (a). Figure 3 indicates that Hinata’s dispersion relation consistently underestimates the growth rate of the instability, by a factor of about four for moderate values of $\Delta\gamma/\gamma_p$. As in Hinata’s (1976) results, however, the growth rate only decreases slowly over a range of large thermal spreads in the denser stream.

4. The Effect of a Thermal Spread in the Beam

Cheng and Ruderman (1977) comment that the thermal spread in the dilute stream is unlikely to be important in determining the growth rate of the instability: they argue that the effect of the thermal spread is of order $(\gamma_p/\gamma_b)^3$ smaller than the thermal spread of
Fig. 3. The maximum growth rate of the two-stream instability as a function of the relative energy spread of the denser stream $\Delta\gamma_0/\gamma_0$ when the dilute fast beam is cold. The three curves represent the numerical solution of the full dispersion relation (10); the numerical solution of Hinata's dispersion relation (11); and the analytic approximation to the solution of (11) given in part (a) of the Appendix. Parameters are $\gamma_0 = 100$ and $\gamma_b = 500$.

the denser stream. To investigate this, the dispersion relation for the two-stream instability has been solved numerically in the case when there are thermal spreads of energy in both streams.

The allowance for a thermal spread in the distribution of beam energies as well as in the background stream leads to the presence of two more roots of the dispersion relation. In the limit $n_b \to 0$, each of the poles $\omega = k_c[1 - \frac{1}{2}(\gamma_b \pm i\Delta\gamma)^{-2}]$ is a double root of the dispersion relation. Thus two roots apparently correspond to growing modes in the limit that no beam is present. As in the case of a single stream, it is argued here that these modes are spurious: in the limit of no beam there should not be any free energy present to drive an instability. The two-stream instability corresponds to the mode which is damped in the limit of no beam (due to Landau damping by the background stream) and then becomes amplified for a sufficiently dense beam. The other modes are not irrelevant, however: they can lead to confusion in the identification of the two-stream instability; and in a related problem, there appears to be a form of mode-coupling between one of these modes and the two-stream instability mode in the limit of large $\Delta\gamma_b$. Identification of the appropriate mode was only found to be difficult at a couple of points, and there the following criteria were applied: the instability should be damped in the limit of no beam; driven unstable for sufficiently large beam density; its growth
rate should always increase as the beam density increases; it should not be amplified at arbitrarily large frequencies; and its growth rate should decrease as the thermal spread of the beam is increased. The results of the numerical calculations are shown in Figure 4.

Fig. 4. A graph of the growth rate of the two-stream instability as a function of the relative energy spread in the dilute stream $\Delta \gamma_b/\gamma_b$ for four values of the relative energy spread in the dense plasma component $\Delta \gamma_p/\gamma_p = 0.10, 0.30, 0.50, 1.00$, with $\gamma_p = 100$ and $\gamma_b = 500$.

There the instability growth rate is plotted as a function of $\Delta \gamma_b/\gamma_b$ for four values of $\Delta \gamma_p$, where $\Delta \gamma_b mc^2$ is the spread of energies in the beam particles, with $\gamma_b = 500$ and $\gamma_p = 100$. It is found that, unlike the modes shown in Figure 1 with $\Delta \gamma_b = 0$, the mode is only growing over a finite range of frequencies, and is damped outside this range. Like the case $\Delta \gamma_b = 0$, however, the instability frequency does not change greatly as $\Delta \gamma_b$ is changed. Figure 4 shows that the growth rate of the instability drops significantly as $\Delta \gamma_b$ increases. The instability ceases to act altogether beyond a certain threshold value of $\Delta \gamma_b$. Physically we expect the growth rate to decrease with an increase in $\Delta \gamma_b$ because the positive slope in the momentum distribution is decreased. Note that mode identification problems occurred only in a narrow range of $\Delta \gamma_b$ in the vicinity of the ‘knee’ in the curves for $\Delta \gamma_p/\gamma_p = 0.1$ and 0.3. This is where the mode-coupling referred to earlier takes place.

The threshold value of $\Delta \gamma_b$ at which the instability ceases can be relatively small, as shown in Figure 4. In particular, in the case $\Delta \gamma_p = \gamma_p = 100$ with $\gamma_b = 500$, a spread of $\Delta \gamma_b/\gamma_b = 0.21$ in the beam is sufficient to suppress the instability. We conclude that the neglect of the energy spread of the dilute stream, even when it is relatively small, may not be justified if the energy spread of the denser stream is large.

5. Summary

In this paper we have investigated the effects of a spread of energies on the ultrarelativistic two-stream instability. The appropriate dispersion relation, correctly including the effects of a spread of energies in the particles of the streams, has been calculated and
solved numerically for parameters likely to be plausible in a pulsar magnetosphere. An analytic approximation to the growth rate of the instability has been derived in the case when the dilute stream has no energy spread. This work corrects previous calculations of the same instability by Hinata (1976) and Cheng and Ruderman (1977): it is found that the dispersion relation used by these authors underestimates both the growth rate and the frequency of the instability. The growth rate found here is about four times larger than that predicted by (11). However, for the application to pulsars where the relevant parameters are but poorly known, the difference between the growth rates found from (11), which have hitherto been used, and the corrected growth rates found from (10) is unlikely to be large enough to change current models. More important is the result that the instability may be completely suppressed by a relatively small spread of energies in the dilute fast stream. This is in contrast to a previous suggestion that the spread of energies in the fast stream should be unimportant (Cheng and Ruderman, 1977). Since it is likely that electron-positron components of a pulsar magnetosphere will have a large spread of energies this result favours models such as that of Cheng and Ruderman (1980), in which the two-stream instability is driven by a beam of nearly monoenergetic iron nuclei, rather than earlier models (Ruderman and Sutherland, 1975; Cheng and Ruderman, 1977) in which the instability is driven by a beam of electrons.

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Appendix

Here we present an analytic approximation to the solution of (10). The calculation follows that used by Hinata (1976), and it is convenient to first outline his calculation.

(a) Hinata's Analysis

Hinata (1976) used the following dispersion relation:

\[
\epsilon' = 1 - \frac{\omega_p^2}{\gamma_p^3} \left[ \omega - kc + \frac{k c}{2 \gamma_p} + \frac{\Delta \gamma k c}{\gamma_p} \right]^{-2} - \frac{\omega_b^2}{\gamma_b^3} \left[ \omega - kc + \frac{k c}{2 \gamma_b} \right]^{-2} = 0. \tag{11}
\]

We seek solutions of (11) for complex \( k \) with given real frequency. Solutions with \( \text{Im}(k) < 0 \) should correspond to amplified waves, and \( \text{Re}(k) \) then determines the scalelength of likely bunching. Cheng and Ruderman (1977) solve (11) for complex \( \omega \) with real \( k \); numerically we find that

\[
[\text{Im}(\omega)/\text{Re}(\omega)]_{\text{real } k} \approx -[\text{Im}(k)/\text{Re}(k)]_{\text{real } \omega},
\]

as expected from the relativistic condition \( \omega \approx kc \).
It is known that, when both beams are cold, the instability occurs at the frequency $\omega \approx \text{Re}(k c) \approx 2\gamma_p^{1/2} \omega_p$ with infinite growth rate. It is convenient to work with the following dimensionless quantities:

$$x = \omega \gamma_p^{3/2} / \omega_p, \quad \kappa = \kappa c \gamma_p^{3/2} \left( 1 - 1 / 2 \gamma_p^2 \right) / \omega_p, \quad \Delta_p = \gamma_p \left( 1 - 1 / 2 \gamma_p^2 \right)^{-1},$$

$$\Delta = \left( \frac{1}{2 \gamma_p^2} - \frac{1}{2 \gamma_p^2} \right) \left( 1 - 1 / 2 \gamma_p^2 \right)^{-1}, \quad \alpha = n_p / n_p \gamma_p^3.$$

Equation (12) may be written as

$$\lambda^{1/2} \hat{z} = x - \kappa(1 + \Delta), \quad \zeta = \Delta_p / \Delta, \quad y = x \Delta,$$  

(13)

Equation (11) may be written as

$$1 = \left\{ \frac{1 + i \zeta}{1 + \Delta} \right\}^{-2} (1 + \zeta)^{-2} + \lambda^{-2},$$  

(14)

with

$$\zeta = \alpha^{1/2} \frac{1 - \text{i} \Delta_p \hat{\lambda}}{1 + \text{i} \zeta} \frac{\lambda}{y}.$$  

(15)

When $\Delta_p = 0$ the most unstable mode ($\omega = 2\omega_p \gamma_p^{1/2}$) satisfies $y \approx 1$. We expect that the two terms on the right-hand side of (11) will both be of order unity and, hence, $| \hat{\lambda} |$ will be of order unity; $\Delta_p$ and $\zeta$ will be of this order or smaller. Therefore, we postulate that $| \zeta | \ll 1$, and neglect $\zeta$ in (14). This assumption is found to be valid later in the numerical calculations. Noting that the imaginary part of $\hat{\lambda}$ is, to within a numerical factor, the same as the imaginary part of $k$, we calculate $\hat{\lambda}_r = \text{Im}(\hat{\lambda})$. Equation (14) can then be written as

$$\lambda^2 = \frac{1}{1 - l e^{-2i\omega}},$$  

(16)

with

$$l = \frac{(1 + \Delta)^2}{(1 + \zeta^2) y^2}$$  

(17)

and

$$\varphi = \tan^{-1} \left[ \zeta \right].$$  

(18)
Equation (16) may be solved to give

$$\lambda_I^2 = \frac{1}{2\sqrt{F}} - \frac{1 - l \cos 2\varphi}{2F}$$

(19)

with the real part of $\lambda$ found from

$$\lambda_R = -\frac{l \sin 2\varphi}{2\lambda_I F}$$

where $F = l^2 - 2l \cos 2\varphi + 1$.

The most unstable mode should give the maximum value for $|\text{Im}(k)|^2$ and, hence, also for $\lambda_I^2$; it is found by considering the roots of the equation $\delta\lambda_I^2/\delta l = 0$. This gives four roots, whereby the value of $l$ is determined from $\varphi$; the two which lead to growing modes are, in Hinata’s notation,

$$l_3 = \cos \frac{2}{3} \varphi - \sqrt{3} \sin \frac{2}{3} \varphi$$

(20a)

and

$$l_4 = \cos \frac{2}{3} \varphi + \sqrt{3} \sin \frac{2}{3} \varphi$$

(20b)

In Hinata’s analysis $0 \leq \varphi \leq \frac{1}{2} \pi$, and $l_4$ always describes the fastest growing mode. To calculate the growth rate one proceeds as follows: given $\omega_p$, $\omega_b$, $\gamma_p$, and $\gamma_b$, the spread $\Delta\gamma$ determines the angle $\varphi$. Then (20b) fixes $l$; it is straightforward to use (19) to find $\text{Im}(k)$. The value of $l$ also fixes the frequency corresponding to maximum growth (called here the ‘instability’ frequency) via (17), since $y$ is proportional to frequency as in (13). Thus one finds that the spatial growth rate of the instability is given by

$$\text{Im}(k) \approx -\frac{\omega_p}{c\gamma_p^{3/2}} \chi^{1/2} \lambda_I$$

(21)

at the frequency

$$\omega = \text{Re}(k) \approx 2 \omega_p \gamma_p^{1/2} \frac{\cos \varphi}{l_4^{1/2}}$$

(22)

Note that the instability frequency shifts as $\Delta\gamma/\gamma_p$ (and, hence, the angle $\varphi$) changes.

We now give the analytic results for several limits. When the ratio $\Delta\gamma/\gamma_p$ is small, one finds that the maximum growth rate of the instability is

$$\text{Im}(k) \approx -\frac{\omega_p}{c\gamma_p^{3/2}} \chi^{1/2} \left[ \frac{3 \sqrt{3}}{16\varphi} \right]^{1/2}$$

(23)

with $\varphi \approx 2\Delta\gamma/\gamma_p$ when $\gamma_b \gg \gamma_p$; thus the growth rate varies as $(\Delta\gamma)^{-1/2}$.

In the important limit $\Delta\gamma = \gamma_p$, thought to be relevant to pulsar magnetospheres,
Hinata finds that the maximum spatial growth rate is

\[ \text{Im}(k) = -2^{-3/2} \frac{\omega_p}{c} \gamma_p^{3/2} x^{1/2}, \]

occurring at a frequency \( \omega = \text{Re}(kc) \approx (2/\sqrt{3})^{1/2} \omega_p \gamma_p^{1/2} \).

(b) **Analysis of (10)**

We follow Hinata’s method closely: due to the greater complexity of (10) some approximations must be made in order to obtain a useful formula. Since the denominators in (10) differ from those of (11) it is necessary to use dimensionless parameters of generally different forms. The parameters \( x \) and \( a \) are unchanged, but \( \kappa, \Delta_p, \) and \( \Delta \) are replaced by

\[
\begin{align*}
\kappa &= \frac{kc}{\omega_p} \gamma_p^{3/2} \left( 1 - \frac{1 - (\Delta \gamma / \gamma_p)^2}{2 \gamma_p^2 (1 + \Delta \gamma^2 / \gamma_p^2)^2} \right), \\
\kappa \Delta_p &= \frac{kc}{\omega_p} \gamma_p^{3/2} \frac{\Delta \gamma / \gamma_p}{\gamma_p^2 (1 + \Delta \gamma^2 / \gamma_p^2)^2}, \\
\kappa \Delta &= \frac{kc}{\omega_p} \gamma_p^{3/2} \left( 1 - \frac{1 - \Delta \gamma^2 / \gamma_p^2}{2 \gamma_p^2 (1 + \Delta \gamma^2 / \gamma_p^2)^2} - \frac{1}{2 \gamma_p^2} \right).
\end{align*}
\]

Note that in (24), in contrast to (12), the parameter \( \Delta \) depends on \( \Delta \gamma \); in particular, \( \Delta \) is negative when \( \Delta \gamma = \gamma_p \). The quantities \( \lambda, \xi, \) and \( y \) are now defined by (13) with (24).

As in part (a) we write the pole \([\omega - ku_-]\) in the form

\[ x - \kappa + i \kappa \Delta_p = \frac{1 - i \xi}{1 + \Delta} y(1 + \xi); \quad (25) \]

now the pole \([\omega - ku_+]\) is similarly replaced as

\[ x - \kappa - i \kappa \Delta_p = \frac{1 + i \xi}{1 + \Delta} y(1 + \xi), \]

with

\[ \xi' = \alpha^{1/2} \frac{\lambda}{y} \frac{1 + i \Delta_p}{1 - i \xi}; \quad |\xi'| = |\xi|. \quad (26) \]

We proceed now by assuming that \( |\xi| \ll 1 \). Again the dispersion relation can be manipulated into the form

\[ \lambda^2 = \frac{1}{1 - L e^{-2i\theta}}, \quad (27) \]
cf. Equation (16). To utilize this equation simply we need to be able, as in (a), to
determine the value of the angle $\vartheta$ from the initial conditions, independently of $L$.
Difficulties arise in handling the $[\omega - ku_+]^{-1}$ terms in $x_+$. These terms are unimportant
in the limit of small thermal spread, and so we assume that they are smaller than the
remaining terms for moderate thermal spreads. Then a crude approximation may be
appropriate to evaluate these terms. We proceed by assuming that the roots of the
dispersion relation are close to the pole $k = \omega/u_p$, which is known to be approximately
true from the numerical work. This allows the following approximations:

$$
\left(1 + i \frac{\Delta \gamma}{\gamma_p}\right)^{-1} [\omega - ku_+]^{-1} - \left(1 - i \frac{\Delta \gamma}{\gamma_p}\right)^{-1} [\omega - ku_-]^{-1} =
$$

$$
\frac{i \gamma_p \Delta \gamma}{(\gamma_p^2 + \Delta \gamma^2)^2} \frac{1}{[\omega - ku_+][\omega - ku_-]};
$$

$$
\log \left[\frac{\gamma_p^2 + \Delta \gamma^2}{m^2 c^2 \gamma_p^2 + \Delta \gamma^2} \frac{kc - \omega}{kc}\right] \approx \log \left[\frac{\gamma_p^2 + \Delta \gamma^2}{\gamma_p^2}\right].
$$

Thus we write

$$
L e^{-2i\vartheta} = \left[1 + \frac{\Delta \gamma^2}{\gamma^2}\right]^{-3/2} \left[x_+ e^{i(2\varphi - 3\vartheta)} + x_- e^{-i(2\varphi - 3\vartheta)} -
\frac{1}{\pi} \frac{\Delta \gamma}{\gamma_p} \left(1 + \frac{\Delta \gamma^2}{\gamma_p^2}\right)^{-1/2}\right],
$$

(28)

where $l$ and $\varphi$ are defined by (17) and (18) with $\Delta$ and $\Delta_p$ given by (24), while

$$
\psi = \tan^{-1} \left[\frac{\Delta \gamma}{\gamma_p}\right].
$$

Since (27) is identical in form to (16) the analysis now proceeds exactly as in part (a),
with results given by (19) and (20), but with $l$ and $\varphi$ now replaced by $L$ and $\vartheta$; $\vartheta$ may
be determined unambiguously from (28).

We write down the results of the analysis in the case when $\Delta \gamma/\gamma_p$ is small. We find
that $\varphi \approx 2\Delta \gamma/\gamma_p$, while $\psi \approx \Delta \gamma/\gamma_p$. Then

$$
\vartheta \approx \frac{\Delta \gamma}{2\gamma_p} \approx \frac{\varphi}{4},
$$

and the maximum growth rate is found to be approximately given by

$$
\Im(kc) = -\frac{\omega_p}{\gamma_p^{3/2}} \varphi^{1/2} \left\{3 \sqrt{3} \frac{\gamma_p}{8 \Delta \gamma}\right\}^{1/2};
$$

(29)
this is a factor of two larger than Hinata's result. This factor of two may be attributed to Hinata's neglect of a term of order $\Delta \gamma/\gamma_p$ arising from the factor $(\gamma_p - i\Delta \gamma)^{-3}$ in (10).

The analytic approximation derived here is found to give growth rates correct to about $\pm 20\%$ for moderate values of $\Delta \gamma$ in the specific case discussed in Section 3 (too large at small $\Delta \gamma$ and too small at larger $\Delta \gamma$). The assumption $|\xi| \ll 1$ breaks down at small values $\Delta \gamma < 0.1$ (this is true in part (a) also). The remaining discrepancies are attributed to the approximations required above to achieve analytic results.

References