ASTR430 Homework # 2 – Orbital Motion
Due Wednesday, October 8, 2003

Read the rest of the “Building Physical Intuition in Mechanics” handout and try some examples from each section. In the following problems - and in all future homeworks - make sure to check units, limits, symmetry, and your common sense! You will catch many errors this way. Show me that you have thought about your answers rather than just deriving them.

1. Show that \( C = -\frac{GM_\odot}{2a} \) in the energy equation \( \frac{v^2}{2} - \frac{GM_\odot}{r} = C \) by using equations for the specific angular momentum \( h \). Hint: Since the left side of the energy equation is constant all along the orbit, evaluate it at a convenient point. Use your result to obtain an expression for the speed of a planet at an arbitrary point along its orbit. What are the maximum and minimum speeds for each of the planets? Use the Planetary Calculator to find out (follow Astronomy Workshop/Solar System Calculators links from the class webpage).

2. At apocenter, a 10-ton satellite on an \( a = 4R_E, e = 0.5, i = 0 \) orbit uses a large coiled spring to launch a 1-ton communication satellite onto an \( a = 6R_E, e = 0.0, i = 0 \) orbit (\( R_E \) is the Earth’s radius). Find the new orbit of the (now 9-ton) parent satellite. Are satellite orbital energy and angular momentum lost, gained, or conserved? Explain your answer.

3. As the interstellar cloud collapsed to form our Solar System, the gas was heated significantly. In this problem, you will estimate how hot the disk was at 1AU assuming that no energy was lost via radiation during collapse. In reality, lots of energy is radiated away so this will be a serious overestimate! Imagine an \( H_2 \) molecule which is at rest at infinity and falls inward toward the Sun.
   a) Calculate the circular velocity at 1AU around a 1 Solar Mass star, and determine the total (kinetic + potential) energy of an \( H_2 \) molecule on a circular orbit at 1AU.
   b) Noting that the total energy of the molecule at rest at infinity is zero, calculate the temperature increase of the hydrogen gas assuming that it has not suffered radiative losses.

4. EXTRA CREDIT CHALLENGE PROBLEMS (Experts Only!):
   Do both, one or none.

   a) Show that the vector equation
      \[
      \frac{d^2r}{dt^2} = F(r)\hat{r}
      \]
      reduces to two scalar equations
      \[
      \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = F(r) \quad \text{and} \quad 2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2} = 0
      \]
      and show that the second secular equation expresses the conservation of angular momentum.

   b) Convert the scalar equation
that we derived in class into one for $d^2u/d\theta^2$ using the transformation $u = 1/r$ and the expression for angular momentum $h = r^2d\theta/dt$. Solve for $u$ and show that the solution can be written as

$$r = \frac{h^2/GM_\odot}{1 + e \cos(\theta - \varpi)}$$

where $e$ and $\varpi$ are integration constants. These solutions are ellipses with the Sun at one focus.