This exercise is intended to give you a quantitative impression of how important stellar parameters such as the central pressure and the gravitational potential energy depend upon the density distribution within the star. We may also use these results later to make some simple analytic models of stars.

Consider a star with a total mass \( M \) and radius \( R \). The mean density of the star, \( \bar{\rho} \), is given by

\[
\bar{\rho} = \frac{M}{\frac{4}{3} \pi R^3}.
\]

We will assume a linear variation of the density from the central value \( \rho_c \) to a surface value \( \rho_s \):

\[
\rho(r) = \rho_c (1 - \beta x), \text{ where } x = (r/R), \text{ and } \beta = (\rho_c - \rho_s)/\rho_c.
\]

If \( \beta = 0 \), we have a constant density model, while if \( \beta = 1 \), the surface density \( \rho_s \) is zero – this is Stein’s linear density model. (We only consider \( \beta \) in the range \( 0 \leq \beta \leq 1 \)).

1. Find \( M_r \) for this density distribution. Write the results in the form \( M_r = M f(x) \), where \( f(x) \) will be a polynomial.

2. What is the ratio of central density to mean density, \( \rho_c / \bar{\rho} \), in this model?

3. Find the central pressure, \( P_c \), for the model from the integral over the star of the equation of hydrostatic equilibrium (H,K&T, eqn. 1.6):

\[
\int_{P(0)}^{P(R)} -dP = P_c = \int_0^R \frac{G M_r}{r^2} \rho(r) \, dr.
\]

Your result for \( P_c \) will have the form \( P_c = (C/8\pi)GM^2/R^4 \) where \( C \) is some constant.

4. Find the gravitational potential energy, \( \Omega \) from

\[
\Omega = -\int_0^M \frac{G M_r}{r} dM_r = -\frac{GM^2}{R} \int_0^1 \frac{(M_r/M)}{(r/R)} d(M_r/M).
\]

Your answers should have the form \( \Omega = -\text{constant} \times GM^2/R \). Comment on the variation of potential energy with the degree of central condensation.

5. Look again at case \( \beta = 1 \). Find the pressure, \( P(x) \), not just at the center, but also for all radial points \( x \). If the equation of state were that of an ideal gas, with a constant mean molecular weight, find the corresponding temperature variation, \( T(x) \), that would follow from \( P(x) \) and \( \rho(x) \). Plot \( P(r) \) and \( T(r) \) for this case. Do you think such a temperature gradient would be likely (or possible)?