In this exercise you will integrate numerically the Lane-Emden equation for the structure of a polytrope (Hansen, Kawaler & Trimble, p334, eqn 7.26):

\[
\frac{1}{\xi^2} \frac{d}{d\xi} \left[ \xi^2 \frac{d\theta_n}{d\xi} \right] = - \theta_n^n ,
\]

and the equation of Chandrasekhar (Chandrasekhar,”Stellar Structure”, p417, eqn 28):

\[
\frac{1}{\eta^2} \frac{d}{d\eta} \left[ \eta^2 \frac{d\phi}{d\eta} \right] = - \left[ \frac{\phi^2}{y_0^2} - \frac{1}{y_0} \right]^{3/2}.
\]

You are to write a FORTRAN or C program to do this, but to reduce the necessary programming to a minimum, you may make use of a general-purpose subroutine for the integration of ordinary differential equations, such as \texttt{odeint} from ”Numerical Recipes” by Press, Flannery, Teukolsky, and Vetterling. (The FORTRAN and C routines in this book can be found at

/local/src/lib/NumRecF/recipes/ and /local/src/lib/NumRecC/recipes/)

Numerical integration of second order differential equations is almost always done by first rewriting the equation as two simultaneous first-order equations. Thus, for the Lane-Emden equation, we could define a variable \( w \) such that \( \theta_n = d\theta_n/d\xi \). Then we have a pair of first order equations:

\[
\frac{d\theta_n}{d\xi} = w
\]

\[
\frac{dw}{d\xi} = - \theta_n^n - \frac{2w}{\xi}
\]

The initial conditions are then \( \theta_n = 1 \) and \( w = 0 \) at \( \xi = 0 \). But you can see that equation (4) is not well behaved at this point. Thus we must use the series expansion to start the integration for some value of \( \xi \) close to zero.

First, test the integration subroutine by integration of the equation

\[
\frac{d^2y}{dx^2} = - y ,
\]

with \( y = 1 \) and \( dy/dx = 0 \) at \( x = 0 \). The solution is of course just \( y = \cos(x) \). Thus you can compare the numerical result to the exact solution and insure that you are getting an accuracy of say one part in 10^{-5}.

Next, integrate the Lane-Emden equation for three values of the index: \( n = 1.5 \), \( n = 3.0 \), and \( n = 4.8 \). Find \( \xi_1 \) (the first zero) and also \( d\theta_n/d\xi \) at \( \xi = \xi_1 \) for these cases (For the first two, check with the results in the book). For the \( n = 1.5 \) and \( n = 3.0 \) polytropes, tabulate \( \theta_n(\xi) \) at intervals of 0.2 in \( \xi \), i.e., 0.0, 0.2, 0.4, ... up to \( \xi_1 \). For the \( n = 4.8 \) case, tabulate at intervals of 5 in \( \xi \).
Finally, construct white dwarf models for two masses: $M = 0.8 \, M_\odot$ and $M = 1.2 \, M_\odot$. Assume a composition of pure carbon, which will fix the value of $\mu_e = 2$. You will have to integrate the equation many times to find the proper values of $y_0^2$ which will correspond to the required masses. For a given $y_0^2$, integration of equation (2) gives the quantity

$$\frac{\Omega(y_0)}{\omega_3} = \left[ \eta^2 \frac{d^2 \bar{\phi}}{d \eta^2} \right]_{\eta = \eta_1}, \quad (5)$$

and the mass of the corresponding white dwarf is

$$M(y_0) = \frac{\Omega(y_0)}{\omega_3} \, M_3,$$

where $\omega_3 = 2.01824$ and $M_3 = 5.739 \mu_e^{-2} \, M_\odot$.

Calculate the density $\rho(r)$ as a function of the radius $r$ in g cm$^{-3}$, and plot this against $r$ in cm. Plot both models on the same figure. Recall that in terms of $\phi(\eta)$,

$$\frac{\rho}{\rho_c} = \left[ \frac{(y_0 \phi)^2 - 1}{y_0^2 - 1} \right]^{3/2} \quad \text{and} \quad \rho_c = B \, (y_0^2 - 1)^{3/2},$$

Also, the physical radius $r = \alpha \eta$, where the constants $B$ and $\alpha$ are given by

$$B = \frac{8 \pi m_e^3 c^3}{3 h^3} \mu_e m_H = 9.8155 \times 10^5 \mu_e$$

and

$$\alpha = \frac{1}{4 \pi m_e \mu_e m_H y_0} \left( \frac{3 h^3}{2 c G} \right)^{1/2} = \frac{7.7094 \times 10^8}{\mu_e y_0^2}. \quad (8)$$

Finally, recall that the parameter $x$ is a measure of relativistic ($x >> 1$) vs non-relativistic ($x << 1$) conditions. Using $x = \sqrt{y_0^2 \phi^2 - 1}$, plot $x$ vs $r$ for your two models.

For your information:

The Lane-Emden function has the following expansion for small $\xi$:

$$\theta_n(\xi) = 1 - \frac{1}{6} \xi^2 + \frac{n}{120} \xi^4 - \frac{n(-5 + 8n)}{15120} \xi^6 + \frac{n(70 - 183n + 122n^2)}{3265920} \xi^8 - \ldots$$

Chandrasekhar gives the following expansion for $\phi$ (Stellar Structure, p 426):

$$\phi(\eta) = 1 - \frac{q^3}{6} \eta^2 + \frac{3 q^4}{120} \eta^4 - \frac{q^5(14 + 5q^2)}{5040} \eta^6 + \frac{q^6(280 + 339q^2)}{1088640} \eta^8 - \ldots,$$

where the quantity $q$ is defined as $q^2 = (y_0^2 - 1)/y_0^2$. \quad (9)