Lecture 15: Cosmological Models I

- Standard cosmological models - 3 cases
- Hubble time and other terminology
- The Friedmann equation
- The Critical Density and $\Omega$

Reading for lectures 15 & 16: Chapter 11

O: Recap

- Use framework of GR to examine Universe as a whole... need to make some simplifying assumptions (Cosmological Principles)
  - Assume homogeneous and isotropic
  - Observations show this is true on largest scales

- Cosmological Principles imply that there are only three possible geometries for space:
  - Spherical (“closed” universe: finite volume)
  - Flat (“open” : infinite volume, no curvature)
  - Hyperbolic (“open” : infinite volume)
(Recap continued)

- Expansion of the Universe described by the Scale Factor (R)
  - New way of thinking about Hubble’s law... galaxies are “at rest” but the space itself is expanding (i.e. the scale factor is increasing)

- Exactly how the scale factor changes with time is determined by putting the spatial geometries into Einstein’s equation
  - Einstein found it was impossible to make the Universe “stand still”... led him to include the Cosmological Constant
  - When Hubble found Universe to be expanding, Einstein considered this to be a huge mistake

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II : Standard cosmological models

- Let’s return to question of how scale factor changes over time
  - Equations of GR relates geometry to dynamics
  - That means curvature must relate to evolution
  - It turns out that there are three possibilities...
Important features of standard models...

✦ All models begin with $R \rightarrow 0$ at a finite time in the past
  ✦ This time is known as the BIG BANG
  ✦ Space and time come into existence at this moment... there is no time before the big bang!
  ✦ The big bang happens everywhere in space... not at a point!
There is a connection between the geometry and the dynamics

- Closed solutions for universe expand to maximum size then re-collapse
- Open solutions for universe expand forever
- Flat solution for universe expands forever (but only just barely... almost grinds to a halt).

III : The Friedman equation (or “let’s get a bit technical!”)

When we go through the GR stuff, we get the Friedman Equation... this is what determines the dynamics of the Universe

\[
\left(\frac{dR}{dt}\right)^2 = \frac{8\pi G}{3} \rho R^2 - kc^2
\]

Here, “k” is the curvature constant...

- k=+1 for spherical case
- k=0 for flat case
- k=-1 for hyperbolic case
Divide this equation by $R^2$ and we get...

$$H^2 = \frac{8\pi G}{3} \rho - \frac{kc^2}{R^2}$$

Let’s examine this equation

- $H^2$ must be positive (square of a real number), so the RHS of this eq. must also be positive
- Then, we must have negative $k$ if $\rho=0$ (i.e., $k=-1$)
- So, empty universes are open and expand forever
- Flat and spherical Universes can only occur in presence of matter.

Now, suppose the Universe is flat ($k=0$)

- Friedmann equation then gives
  $$H^2 = \frac{8\pi G}{3} \rho$$
  
  - So, this case occurs if the density is exactly equal to the critical density...
  $$\rho = \rho_{\text{crit}} = \frac{3H^2}{8\pi G}$$
Value of critical density

- For present best-observed value of the Hubble constant, $H_0 = 72 \text{ km/s/Mpc}$, the critical density, $\rho_{\text{critical}} = \frac{3H_0^2}{8\pi G}$, is equal to $\rho_{\text{critical}} = 10^{-26} \text{ kg/m}^3$; i.e. 6 H atoms/m$^3$.

- Compare to:
  - $\rho_{\text{water}} = 1000 \text{ kg/m}^3$
  - $\rho_{\text{air}} = 1.25 \text{ kg/m}^3$ (at sea level)
  - $\rho_{\text{interstellar gas}} = 2 \times 10^{-21} \text{ kg/m}^3$

- In general case, we can define the density parameter...

$$\Omega = \frac{\rho}{\rho_c}$$

- Can now rewrite Friedmann’s equation yet again using this... we get

$$\Omega = 1 + \frac{kc^2}{H^2R^2}$$
(ok... my brain hurts... what do I need to remember from the past 4 slides???)

- Important take-home message... within context of the “standard model”:
  - $\Omega<1$ means universe is hyperbolic and will expand forever
  - $\Omega=1$ means universe is flat and will (just manage to) expand forever
  - $\Omega>1$ means universe is spherical and will recollapse
- Physical interpretation... if there is more than a certain amount of matter in the universe, the attractive nature of gravity will ensure that the Universe recollapses.

III: SOME USEFUL DEFINITIONS

- Have already come across...
  - **Standard model** (Homogeneous & Isotropic GR-based models, ignoring Dark Energy!!)
  - **Critical density** $\rho_c$ (average density needed to just make the Universe flat)
  - **Density parameter** $\Omega=\rho/\rho_c$
- We will also define...
  - **Cosmic time** (time as measured by a clock which is stationary in co-moving coordinates, i.e., stationary with respect to the expanding Universe)
- **Hubble time**, $t_H = 1/H$ (cosmic time since the big bang if the universe were expanding at a constant rate.)

- **Hubble distance**, $D = ct_H$ (distance that light travels in a Hubble time). This gives an approximate idea of the size of the observable Universe.

- **Age of the Universe**, $t_{age}$ (the amount of cosmic time since the big bang). In standard models, this is always less than the Hubble time.

- **Look-back time**, $t_{lb}$ (amount of cosmic time that passes between the emission of light by a certain galaxy and the observation of that light by us)

- **Particle horizon** (a sphere centered on the Earth with radius $ct_{age}$; i.e., the sphere defined by the distance that light can travel since the big bang). This gives the edge of the actual observable Universe.
IV: WHERE HAS RELATIVITY GONE?

- **Question:** Cosmology is based upon General Relativity, a theory that treats space and time as relative quantities. So, how can we talk about concepts such as...
  - Galaxies being stationary with respect to the expanding Universe? Relativity tells us that there is no such thing as being stationary!
  - The age of the Universe? Surely doesn’t time depending upon the observer?
  - What do you think??