Class 23. PDEs, Part 2

Solving Hyperbolic PDEs, Continued

Upwind differencing

- In addition to amplitude errors (instability or damping), scheme may also have phase errors (dispersion) or transport errors (spurious transport of information).

- Upwind differencing helps reduce transport errors:

\[
\frac{u_j^{n+1} - u_j^n}{\Delta t} = -v_j^n \left\{ \begin{array}{ll}
\frac{u_j^n - u_{j-1}^n}{\Delta x}, & v_j^n > 0, \\
\frac{u_{j+1}^n - u_j^n}{\Delta x}, & v_j^n < 0,
\end{array} \right.
\]

where here we’ve supposed that \( v \) is not constant, for illustration.

- Schematically, only use information upwind of grid point \( j \) to construct differences:

  - Upwind difference is only first order in space. Still, it has lower transport error than second-order centered difference. Better? Can construct higher-order upwind difference schemes...

Second-order accuracy in time

- We have been dealing with two derivatives, \( \partial/\partial x \) and \( \partial/\partial t \). We have constructed higher-order schemes in space. What about \( t \)?

- Staggered leapfrog is 2nd-order in time:

\[
\frac{\partial}{\partial t} \frac{u_j^{n+1} - u_j^{n-1}}{\Delta t} = - \left( \frac{F_{j+1}^n - F_{j-1}^n}{\Delta x} \right).
\]

But, subject to a mesh-drift instability. Think of space-time discretization:

- Odd-integer \( n \) coupled to even-integer \( j \),
- Even-integer \( n \) coupled to odd-integer \( j \)

(“red-black” ordering; odd and even mesh points decoupled). Schematically,
Can be fixed by adding diffusion to couple grid points (add $\epsilon(F^n_{j-1} - 2F^n_j + F^n_{j+1})$, $\epsilon \ll 1$ to RHS).

- **Two-step Lax-Wendroff**: another 2nd-order scheme.

  1. Use Lax step to estimate fluxes at $n + \frac{1}{2}$ and $j \pm \frac{1}{2}$:

     $$u^{n+1/2}_{j-1/2} = \frac{u^n_{j-1} + u^n_j}{2} - \frac{\Delta t}{2\Delta x} (F^n_j - F^n_{j-1})$$

     $$u^{n+1/2}_{j+1/2} = \frac{u^n_j + u^n_{j+1}}{2} - \frac{\Delta t}{2\Delta x} (F^n_{j+1} - F^n_j).$$

  2. Using these half-step values of $u$, calculate $F(u^{n+1/2}_{j \pm 1/2}) \equiv F^{n+1/2}_{j \pm 1/2}$.

  3. Then use leapfrog to get updated values:

     $$u^{n+1}_j = u^n_j - \frac{\Delta t}{\Delta x} \left( F^{n+1/2}_{j+1/2} - F^{n+1/2}_{j-1/2} \right).$$

Schematically,

Fixes dissipation and mesh drifting but introduces phase error (dispersion). Often first-order upwind scheme is as good as/better than 2nd-order L-W.
Summary: Hyperbolic methods

- Many IVPs can be cast in flux-conservative form.

- Solving methods:
  1. FTCS — unconditionally unstable. Never use.
  2. Lax — equivalent to adding diffusion, damps small scales.
  3. Upwind differencing — reduces transport errors, but only 1st-order in space.
  4. Staggered leapfrog — 2nd-order in time, but subject to mesh-drift instability. Fix with diffusion.
  5. Two-step Lax-Wendroff — 2nd-order in time, but suffers from phase error.

- NRiC recommends staggered leapfrog (presumably with diffusion), particularly for problems related to the wave equation.

- For problems sensitive to transport errors, NRiC recommends upwind differencing schemes.

Solving Parabolic PDEs (Diffusive IVPs)

- NRiC §19.2.

- Prototypical parabolic PDE is diffusion equation:

\[
\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2},
\]

where we have taken \(D > 0\) to be constant (\(D = 0\) is trivial and \(D < 0\) leads to physically unstable solutions).

- Consider FTCS differencing:

\[
\frac{u_j^{n+1} - u_j^n}{\Delta t} = D \left[ \frac{u_{j-1}^{n} - 2u_j^n + u_{j+1}^n}{(\Delta x)^2} \right],
\]

- von Neumann analysis gives (E.F.T.S.)

\[
\xi(k) = 1 - \frac{4D\Delta t}{(\Delta x)^2} \sin^2 \left( \frac{k\Delta x}{2} \right).
\]

This is stable provided (E.F.T.S.)

\[
\frac{2D\Delta t}{(\Delta x)^2} \leq 1.
\]

The 2nd derivative makes all the difference (we saw adding diffusion via the Lax method stabilizes FTCS for the hyperbolic equation).
• Diffusion time over scale $L$ is $\tau_D \sim L^2/D$. So stability criterion says $\Delta t \lesssim \tau_D/2$ across one cell.

• Often interested in evolution of time scales $\gg \tau_D$ of one cell. How can we build stable scheme for larger $\Delta t$?

Implicit differencing

• Evaluate RHS of difference equation at $n+1$:

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = D \left[ \frac{u_{j-1}^{n+1} - 2u_j^{n+1} + u_{j+1}^{n+1}}{(\Delta x)^2} \right].$$

• To solve this, rewrite as:

$$-\alpha u_{j-1}^{n+1} + (1 + 2\alpha)u_j^{n+1} - \alpha u_{j+1}^{n+1} = u_j^n,$$

where $\alpha \equiv D\Delta t/(\Delta x)^2$.

– In 1-D, this is a tri-di matrix.
– In 3-D, get large, sparse, banded matrix.
– Solve the usual way.

• What is limit of (1) as $\Delta t \to \infty$ ($\alpha \to \infty$)? Divide through by $\alpha$ to find FD form of $\partial^2 u/\partial x^2 = 0$, i.e., static solution.

• Fully implicit scheme is unconditionally stable (E.F.T.S.) and gives correct equilibrium structure, but cannot be used to follow small-timescale phenomena.

Crank-Nicholson differencing

• Form average of explicit and implicit schemes (in space):

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = D \left[ \frac{(u_{j-1}^{n+1} - 2u_j^{n+1} + u_{j+1}^{n+1}) + (u_{j-1}^{n} - 2u_j^{n} + u_{j+1}^{n})}{2(\Delta x)^2} \right].$$

• Unconditionally stable (E.F.T.S.), 2nd-order accurate in time (both sides centered at $n + 1/2$).

• Schematically,
• “Freezes” small-scale phenomena. Can use fully implicit scheme at end to drive fluctuations to equilibrium.

**Nonlinear diffusion problems**

• For nonlinear diffusion problems, e.g., where \( D = D(x) \), then implicit differencing more complex.

• Must linearize system and use iterative methods.