Equation of Motion and Geodesics

The equation of motion in Newtonian dynamics is $\vec{F} = m\vec{a}$, so for a given mass and force the acceleration is $\vec{a} = \vec{F}/m$. If we generalize to spacetime, we would therefore expect that the equation of motion is

$$a^\alpha = F^\alpha/m \ , \tag{1}$$

and we would be right. However, in curved spacetime things are complicated because the coordinates themselves twist and turn. Therefore, a^α is not as simple an expression as one might have imagined. We're not going to go into details of this (feel free to consult the grad lectures if you want some more information), but we will concentrate on *geodesics*, which are paths in which particles fall freely, meaning that the force $F^\alpha = 0$. Note that an observer on such a path would "feel" no net force, i.e., would measure nothing on an accelerometer. However, the particle can execute quite complicated orbits that are geodesics. See Figure 1 for examples of geodesics on a sphere.

For a Schwarzschild spacetime, let's consider motion in the $r\phi$ plane (i.e., no θ motion, which one can always arrange in a spherically symmetric spacetime just by redefinition of coords). Then the radial equation of geodesic motion is (here we use the proper time τ as our affine parameter)

$$\frac{d^2r}{d\tau^2} + \frac{M}{r^2} - (1 - 3M/r)u_\phi^2/r^3 = 0 \ . \tag{2}$$

Let's think about what all this means. First, let's check this in the Newtonian limit. In that limit, $M/r \ll 1$ and can be ignored, and at low velocities $d\tau^2 \approx dt^2$ so we get the usual expression

$$\frac{d^2r}{dt^2} = -\frac{M}{r^2} + u_\phi^2/r^3 \ . \tag{3}$$

In particular, that means that for a circular orbit, $d^2r/dt^2 = 0$, the specific angular momentum is given by $u_\phi^2 = Mr$.

What about in strong gravity? First consider radial motion, $u_\phi = 0$. Then $d^2r/d\tau^2 = -M/r^2$. This has the same form as the Newtonian expression, but remember that the coordinates mean different things, so you have to be careful. Now consider circular orbits. Again we set $d^2r/d\tau^2 = 0$, to find $u_\phi^2 = Mr^2/(r-3M)$. But wait! Something's strange here. That $r-3M$ in the denominator means that the specific angular momentum goes to infinity at $r = 3M$, but the horizon is at $r = 2M$. Have we reached a contradiction of some sort?

No, but we have happened upon one of the most important features and predictions of general relativity. To explore this more closely, note that the minimum value of $Mr^2/(r-3M)$ occurs at $r = 6M$, for which $u_\phi = \sqrt{12}M$. This is completely different from Newtonian gravity, in which the specific angular momentum is happily monotonic down to arbitrarily small radius. Let's figure out what this means for circular particle motion near a very

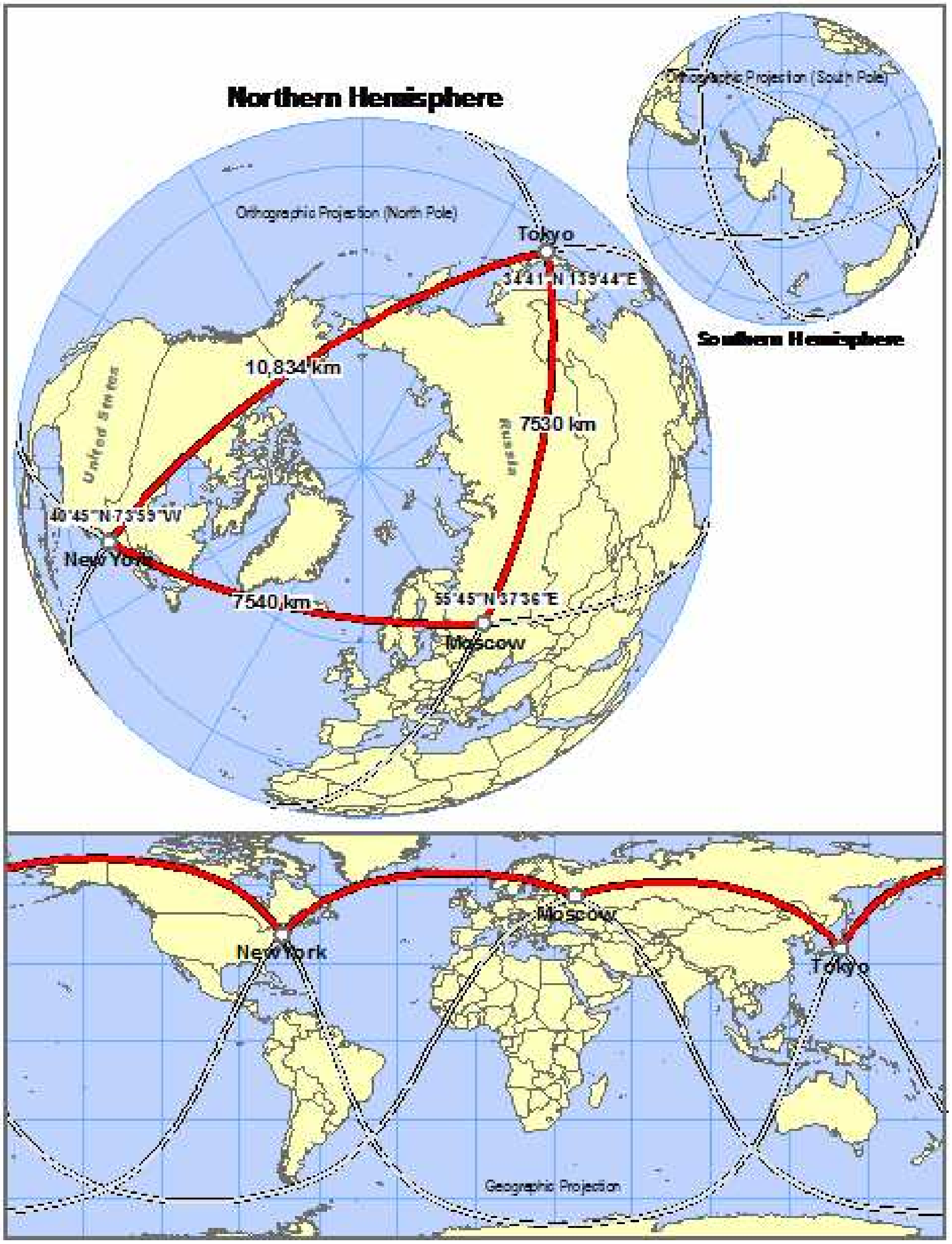

Fig. 1.— On a sphere, geodesics are simply great circles (minimum distance). From http://people.hofstra.edu/geotrans/eng/ch1en/conc1en/img/greatcircledistance.gif

compact object, like a black hole or a neutron star. First, consider the Newtonian part far from the object. Suppose we have a particle in a circular orbit. Imagining that it keeps its angular momentum, give it a kick inward. Now, it has more angular momentum than a particle in a circular orbit at the new radius. **Ask class:** what happens to the particle as a result? It moves outward. Now, **Ask class:** what if we give it a kick outward, so that it has less angular momentum than a particle in a circular orbit at the new radius? It moves inward. Fine, now suppose that we think of a particle orbiting in a circle at the minimum of the angular momentum. If we give it a kick outward, its angular momentum is smaller than that of a circular orbit at the same radius. **Ask class:** what happens to the particle? It moves inward. Now, what if we give it a kick inward, in which case (since it was at a minimum) it is still at a lower angular momentum than a particle in a circular orbit at the same radius. **Ask class:** what happens then? It moves in, where it still has less u_ϕ than it needs, so it moves in faster, and so on. In fact, it pretty much spirals right into the central object. Thus $r = 6M$ is the innermost stable circular orbit, shortened to ISCO by cool people.

Ask class: what does this mean for gas spiraling close to a black hole or neutron star? It means that even if the gas was moving in almost circular orbits at larger distances, then (neglecting other forces) when it reaches this critical radius it'll go right in without having to lose more angular momentum. This radius is called the innermost stable circular orbit, and it plays a fundamental role in the physics of accretion disks around very compact objects.

Qualitatively, one can think of it like this. A fundamental feature of the Schwarzschild geometry is the so-called "pit in the potential". That is, near a compact object gravity is "stronger" than you would have expected based on an extrapolation of the Newtonian law. To compensate for this, the angular velocity has to be higher than it would have been otherwise, so the angular momentum is higher than it would have been in the Newtonian limit, and eventually u_ϕ reaches a minimum and then increases as the radius is decreased further. This predicted behavior is an example of a phenomenon that only occurs in strong gravity, and so can only be tested by observing compact objects.

If we plug the u_ϕ for a circular orbit into the formula for specific energy we found earlier, we find

$$-u_t(\mathrm{circ}) = \frac{r-2M}{\sqrt{r(r-3M)}} \ . \tag{4}$$

At the ISCO, $-u_t = \sqrt{8/9}$, so 5.7% of the binding energy is released in the inspiral to this point.

Now, to help build up our calculational skills, we will do a number of derivations in the Schwarzschild spacetime.

We argued above that to compensate for the stronger gravity, particles have to move faster near a compact object. That would suggest that the angular velocity observed at infinity would be higher than in Newtonian gravity. However, there is also a redshift, which decreases frequencies. Let's calculate the frequency of a circular orbit observed at infinity, to see which effect wins.

We said a while back that the Schwarzschild time coordinate t is simply the time at infinity, and the azimuthal coordinate ϕ is also valid at infinity (in fact, unlike t, ϕ has constant meaning at all radii). Therefore, the angular velocity is $\Omega = d\phi/dt$. To calculate this, we relate it to components of the four-velocity: $d\phi/dt = (d\phi/d\tau)/(dt/d\tau) = u^\phi/u^t$. Now, we express this in terms of our conserved quantities u_ϕ and u_t:

$$\frac{u^\phi}{u^t} = \frac{g^{\alpha\phi}u_\alpha}{g^{\alpha t}u_\alpha} = \frac{g^{\phi\phi}u_\phi}{g^{tt}u_t} = \frac{u_\phi/r^2}{-u_t/(1-2M/r)} \ . \tag{5}$$

Then

$$\Omega = \frac{1-2M/r}{r^2}\frac{u_\phi}{-u_t} = \frac{1-2M/r}{r^2}\frac{\sqrt{Mr^2/(r-3M)}}{(r-2M)/\sqrt{r(r-3M)}} = \sqrt{M/r^3} \ . \tag{6}$$

This is exactly the Newtonian expression! By a lovely coincidence, in Schwarzschild coordinates the angular velocity observed at infinity is exactly the same as it is in Newtonian physics.

As our next calculation, let's use our expression for the specific angular momentum of a circular orbit, and for the specific energy, to derive the radius of the *marginally bound* orbit, which is where $-u_t = 1$ and hence a slight perturbation outward could send the particle to infinity.

Answer:

Since $-u_t = 1$ this means that $u_t^2 = 1$ as well, so we can make our lives easier by squaring.

$$\begin{aligned} u_t^2 &= (1-2M/r)[1+(Mr^2/(r-3M))/r^2] \\ &= (1-2M/r)(1+M/(r-3M)) \\ &= (r-2M)^2/[r(r-3M)] \ . \end{aligned} \tag{7}$$

Setting this to 1 and solving gives $r = 4M$. One consequence of this is that if a test particle plunges in from a very large distance, it could in principle go in to a pericenter distance of $4M$ and come back out again. Any closer, though, and it spirals right in.

A particle is in a circular geodesic at radius r around a star of mass M. Assuming the Schwarzschild spacetime, what is the linear azimuthal velocity of the particle as measured by a local static observer? Recall that the angular velocity as seen at infinity is $d\phi/dt = (M/r^3)^{1/2}$.

Answer:

The local linear azimuthal velocity is $d\hat{\phi}/d\hat{t}$, or $u^{\hat{\phi}}/u^{\hat{t}}$. Using the transformation matrices,

$$v^{\hat{\phi}} = \frac{u^{\hat{\phi}}}{u^{\hat{t}}} = \frac{e^{\hat{\phi}}{}_{\phi} u^{\phi}}{e^{\hat{t}}{}_{t} u^{t}} = \frac{r u^{\phi}}{(1-2M/r)^{1/2} u^{t}} = \frac{r}{(1-2M/r)^{1/2}} \left(\frac{M}{r^3}\right)^{1/2} = \left(\frac{M}{r-2M}\right)^{1/2} . \tag{8}$$

For example, at the innermost stable circular orbit ($r = 6M$), $v^{\hat{\phi}} = 1/2$.

A particle is moving in a Schwarzschild spacetime around a star of mass M. It moves in a circle at a Schwarzschild coordinate radius r at an angular frequency as seen at infinity of Ω. That is, $d\phi/dt = \Omega$, where t is the Schwarzschild coordinate time. What is the specific angular momentum at infinity u_ϕ of the particle? This is a useful calculation for matter on slowly rotating stars, for which the Schwarzschild spacetime is a good approximation.

Answer:

We have

$$\Omega = \frac{d\phi}{dt} = \frac{u^\phi}{u^t} = \frac{g^{\phi\phi}}{g^{tt}} \frac{u_\phi}{u_t} = -\frac{1-2M/r}{r^2} \frac{u_\phi}{u_t} \Rightarrow u_t = -\frac{1-2M/r}{r^2} \frac{u_\phi}{\Omega} . \tag{9}$$

We also know $u^2 = -1$, so

$$u^t u_t + u^\phi u_\phi = -1 , \qquad g^{tt} u_t^2 + g^{\phi\phi} u_\phi^2 = -1 . \tag{10}$$

Substituting in the expression for u_t we find

$$u_\phi^2 = \left(\frac{-g^{tt}}{g^{\phi\phi}}\right) \frac{\Omega^2}{g^{tt}\Omega^2 + g^{\phi\phi}} . \tag{11}$$

Finally, substituting $g^{tt} = -(1-2M/r)^{-1}$ and $g^{\phi\phi} = 1/r^2$ we get

$$u_\phi^2 = \frac{r^4\Omega^2}{(1-2M/r) - r^2\Omega^2} . \tag{12}$$

We've taken a long diversion here to discuss the radial component of the equation of geodesic motion and some of its implications. Let's briefly consider the azimuthal component, specifically $u_{\phi;\alpha} u^\alpha = 0$. This can be expressed as

$$\frac{du_\phi}{d\tau} + ??? = 0 . \tag{13}$$

We can certainly go through the same procedure of calculating the connection coefficients. But here is a place where we should apply our intuition to shortcut those calculations. Recalling that u_ϕ is the specific angular momentum, and that we are considering geodesic motion (no nongravitational forces), **Ask class:** what should the "???" be in this equation for the Schwarzschild spacetime? It should be zero! Angular momentum is conserved for Schwarzschild geodesics, so $du_\phi/d\tau$ had better vanish. You can confirm this explicitly if you want.

One last note about geodesics is that they represent extrema in the integrated path length ds^2 between two events. The reason for this is extremely deep and ultimately comes down to the same reason that optical paths are extrema in length. Basically, if you represent light as a wave, then two paths with different lengths will have different numbers of cycles and hence different phases along the way. With different phases, there is destructive interference and the amplitude is small. Only near an extremum, where nearby paths differ in path length by a small second-order quantity, are the phases close to each other, so only there is the interference constructive and the amplitude high. For massive particles the principle is the same, according to quantum mechanics. Again, a particle can be represented by a wave (or a wave function), and again if nearby paths have significantly different phases the interference will be destructive. Only near an extremum is the amplitude high. For this case, however, it isn't simply the length of the path, but instead the integral of the Lagrangian that matters (this integral is called the action). Extremization of the action is one of the unifying principles of physics, and provides (for example) a different way of looking at general relativity than the geometric approach we've taken.

Intuition Builder

Some of the results we derived are specific to the set of coordinates we defined. For example, the radius at the ISCO is $6M$ in Schwarzschild coordinates, but we could define other coordinates in which this isn't so. Which of our results is coordinate-independent?