

STEADY SOLUTIONS FOR ADIABATIC, ① ISOTHERMAL & BAROTROPIC FLOW

We derived the eqs. of motion in the previous lectures.
In addition to those, we need an "eq. of state".

Eq. of motion \rightarrow changes in mass, density, volume,
velocity, and energy density

Eq. of state \rightarrow relates energy density and pressure

Perfect gas: $\frac{\text{internal energy}}{\text{mass}}$ is related to pressure and
density, $E = \frac{1}{\gamma-1} \frac{P}{\rho}$ where $\frac{2}{\gamma-1} = \#$ of DOF (internal) per
particle

Ex: monatomic gas \rightarrow 3 translations, no internal DOF,
 $\Rightarrow \frac{2}{\gamma-1} = 3 \Rightarrow \gamma = 5/3$

diatomic gas \rightarrow 3 translations plus 2 (internal) rotations
 $\Rightarrow \frac{2}{\gamma-1} = 5 \Rightarrow \gamma = 7/5$

$$P = nkT, \rho = \mu n \Rightarrow E = \frac{1}{\gamma-1} \frac{P}{\rho} = \frac{1}{\gamma-1} \frac{nkT}{\mu n} = \frac{1}{\gamma-1} \frac{kT}{\mu}$$

\Rightarrow in the absence of heating and cooling

$$\frac{DE}{Dt} = -P \frac{D\rho^{-1}}{Dt} \quad \text{where} \quad \frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{v} \cdot \nabla$$

$$\Rightarrow \frac{1}{\gamma-1} \frac{P}{Dt} \left(\frac{P}{\rho} \right) = -P \frac{D\rho^{-1}}{Dt} \quad \text{for a perfect gas.}$$

$$\Rightarrow \frac{1}{\gamma-1} \left[\rho^{-1} \frac{D\rho}{Dt} + \rho \frac{D\rho^{-1}}{Dt} \right] = -\rho \frac{D\rho^{-1}}{Dt}$$

(2)

$$\Rightarrow \rho^{-1} \frac{D\rho}{Dt} = -\gamma \rho \frac{D\rho^{-1}}{Dt} \Rightarrow \frac{D \ln \rho}{Dt} = -\gamma \frac{D \ln \rho^{-1}}{Dt}$$

$$\Rightarrow \frac{D}{Dt} \ln(\rho \rho^{-\gamma}) = 0 \Rightarrow \boxed{\rho \rho^{-\gamma} = \text{constant}}$$

following the flow of Lagrangian fluid elements

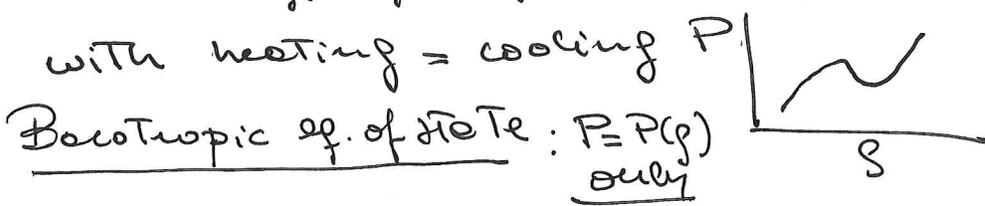
Since the specific entropy of a perfect gas is $S \propto \ln(\rho \rho^{-\gamma})$
 $(S = \frac{3k}{2m} \ln(\rho \rho^{-\gamma})) \Rightarrow$ entropy is conserved along streamlines

This is known as adiabatic flow (no heat is added or removed).

If all streamlines originate from a region of uniform conditions (e.g. inflow or outflow) $\Rightarrow P = K \cdot \rho^\gamma$ everywhere.
 \Rightarrow Need not integrate the energy eqn. to update ϵ , instead update P from $P = (\gamma-1)\epsilon \rho$ and use $P = K \rho^\gamma$

Another common astrophysical situation: instead of very slow heating/cooling compared w/ dynamical processes, to have very fast heating/cooling equilibrium $\Rightarrow T$ is nearly constant \Rightarrow no need to solve the energy eqn., just use $P = n k T = \frac{\rho k T}{\mu}$ with $T = \text{constant}$

A related situation happens when $T \neq \text{constant}$ but follows an equilibrium temperature curve that is a function of ρ
 $\Rightarrow P = P(\rho) = \rho k T(\rho)$. Example is the ISM P - ρ relation with heating = cooling



Barotropic flow solutions

(3)

$$\frac{\partial \rho}{\partial t} + \vec{v} \cdot \nabla \rho + \rho \nabla \cdot \vec{v} = 0 \quad (1) \quad \text{mass continuity}$$

$$\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} = -\frac{\nabla P}{\rho} - \nabla \Phi \quad (2) \quad \text{momentum eq.}$$

$$\vec{v} \cdot \nabla \vec{v} = \nabla \left(\frac{1}{2} v^2 \right) + (\nabla \times \vec{v}) \times \vec{v}$$

$$\Rightarrow (2) \cdot \vec{v} \Rightarrow \frac{\partial}{\partial t} \left(\frac{1}{2} v^2 \right) + \vec{v} \cdot \nabla \left(\frac{1}{2} v^2 \right) = -\vec{v} \cdot \frac{\nabla P}{\rho} - \vec{v} \cdot \nabla \Phi$$

$$\text{Barotropic} \Rightarrow P = P(\rho) \Rightarrow \frac{\nabla P}{\rho} = \frac{1}{\rho} \frac{dP}{d\rho} \cdot \nabla \rho$$

$$= \nabla \left(\int \frac{dP}{\rho} \right)$$

$$\triangleq \nabla h$$

pressure head
(internal energy of
fluid due to pressure)

$$\text{Ex: For } P = K \rho^\gamma \Rightarrow \frac{\nabla P}{\rho} = K \gamma \rho^{\gamma-2} \nabla \rho = \nabla \left(\frac{K \gamma}{\gamma-1} \rho^{\gamma-1} \right)$$

$$\Rightarrow h = \frac{\gamma}{\gamma-1} \frac{P}{\rho}$$

$$\Rightarrow \frac{\partial}{\partial t} \left(\frac{1}{2} v^2 \right) + \vec{v} \cdot \nabla \left(\frac{1}{2} v^2 + h + \Phi \right) = 0$$

$$\text{If the flow is steady} \Rightarrow \frac{\partial}{\partial t} = 0$$

$$\Rightarrow B = \frac{1}{2} v^2 + h + \Phi \quad \text{is conserved along streamlines}$$

This is Bernoulli's Theorem

Bernoulli's function B is clearly a generalized function of the energy that accounts for fluid flow processes.

Ex., the thermal energy/mass is increased by compression by PdV work.

$$B = \underbrace{\frac{1}{2} v^2}_{\text{kinetic energy per mass}} + \underbrace{\frac{P}{\rho}}_{\text{Thermal energy per mass}} + \underbrace{\frac{P}{\rho}}_{\text{additional term}} + \underbrace{\Phi \rho}_{\text{grav. energy / mass}}$$

Now, let's go back and take the $\nabla \times (2)$

$$\frac{\partial}{\partial t} (\nabla \times \vec{v}) + \underbrace{\nabla \times \left(\nabla \frac{1}{2} v^2 \right)}_0 + \nabla \times ((\nabla \times \vec{v}) \times \vec{v}) = \underbrace{\nabla \times (-\nabla h - \nabla \Phi)}_0$$

Barotropic assumption

$$\Rightarrow \frac{\partial}{\partial t} (\nabla \times \vec{v}) + \nabla \times ((\nabla \times \vec{v}) \times \vec{v}) = 0$$

Define $\nabla \times \vec{v} \triangleq \vec{\omega}$, $\vec{\omega}$ is the vorticity

$$\Rightarrow \frac{\partial \vec{\omega}}{\partial t} + \nabla \times (\vec{\omega} \times \vec{v}) = 0$$

Define $\Gamma \triangleq \oint \vec{\omega} \cdot \hat{n} dA$ flow of vorticity through a surface

$$\Gamma = \int \nabla \times \vec{v} \cdot d\vec{A} = \oint \vec{v} \cdot d\vec{\ell}$$

↑
is the total circulation around a curve

↑
Stoke's Theorem



let's follow this surface in the flow. what happens to Γ ?

$$\frac{d\Gamma}{dt} = \frac{d}{dt} \int (\nabla \times \vec{v}) \cdot d\vec{A} \stackrel{\text{pois}}{=} \int \left[\frac{\partial \vec{\omega}}{\partial t} \cdot d\vec{A} + \vec{\omega} \cdot (\vec{v} \times d\vec{\ell}) \right]$$

area element added/subtracted

$$= \int \frac{\partial \vec{\omega}}{\partial t} \cdot d\vec{A} + \int (\vec{\omega} \times \vec{v}) \cdot d\vec{\ell}$$

$$= \int \left[\frac{\partial \vec{\omega}}{\partial t} + \nabla \times (\vec{\omega} \times \vec{v}) \right] \cdot d\vec{A} \stackrel{\text{Stokes}}{=} \int 0 \cdot d\vec{A} = 0$$



Core Claim: The total circulation doesn't change with time. This is Kelvin's circulation theorem.

We think of this as meaning that fluid elements that have no initial local rotation / shear cannot ~~acquire~~ acquire any, if $P = P(\rho)$. (inviscid, barotropic flow)

Let's now consider some examples using Bernoulli's theorem.

① Flow from faucet: incompressible $\Rightarrow \rho = \text{constant}$



$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 \Rightarrow \nabla \cdot (\rho \vec{v}) = 0 \xRightarrow{\text{Gauss}} \oint \rho \vec{v} \cdot d\vec{A} = 0$$

$$\vec{v} = v_z \cdot \hat{z} \Rightarrow \rho \cdot v_z \cdot A = \text{constant}$$



Since $\rho = \rho_0 \Rightarrow A \propto \frac{1}{v_z}$

$$B = \frac{1}{2} v^2 + h + \Phi = \text{constant} \quad \Phi(z) = \rho g z$$

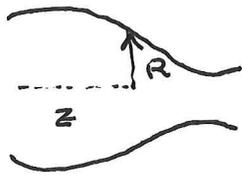
\downarrow
neglect changes

$$\Rightarrow v = -\sqrt{2(\Phi_0 - \Phi(z))}$$
$$v(z) = -\sqrt{2(z_0 - z)g}$$
$$\Rightarrow A(z) \propto \frac{1}{v(z)} \propto \frac{1}{(z_0 - z)^{1/2}}$$

\Rightarrow as $z \uparrow$, $A(z) \downarrow \Rightarrow$ stream narrowing

Try this at home!

② Flow of compressible gas; De Laval nozzle Shw 6 ⑥



Now consider flow through shaped chamber. Let's ignore gravity (e.g. motions are rapid w.r.t. g, jet exhaust)

Steady flow, $\nabla \cdot (\rho \vec{v}) = 0$ $\left(\frac{\partial \rho}{\partial t} = 0\right)$

Integrate over x-sectional volume element $\oint \rho \vec{v} \cdot d\vec{A} = 0$
 $\Rightarrow \rho v_{\perp} \cdot A(z) = \text{constant}$ (\vec{v} into walls is zero by definition)

$$v_{\perp} = \vec{v} \cdot \hat{z}$$



Define $v_{\perp} = u$

$$\rho u A = \text{const} = F \text{ (flux of matter)}$$

$$\frac{1}{2} u^2 + h = \text{const} = B$$

Now $d\rho(uA) + du(\rho A) + \rho u dA = 0$

$$\Rightarrow \frac{d\rho}{\rho} + \frac{du}{u} + \frac{dA}{A} = 0 \quad (1)$$

And $u du + dh = 0$

$$\Rightarrow u du + \frac{dP}{\rho} = u du + \frac{dP}{d\rho} \cdot \frac{d\rho}{\rho} = 0$$

($h = \int \frac{dP}{\rho}$)

Define $\frac{dP}{d\rho} \equiv a^2$, the sound speed ($P = k\rho^\gamma \Rightarrow \frac{dP}{d\rho} = \frac{\gamma P}{\rho}$)

$$\Rightarrow \frac{d\rho}{\rho} = -\frac{u du}{a^2} \quad (2)$$

~~adiabatic EOS~~
adiabatic EOS

\Rightarrow Going back to (1) and substituting (2)

$$-\frac{u du}{a^2} + \frac{du}{u} = -\frac{dA}{A} \Rightarrow \boxed{\frac{du}{u} \left(1 - \frac{u^2}{a^2}\right) = -\frac{dA}{A}}$$

Subsonic flow: $\frac{u}{a} < 1 \Rightarrow \text{sgn}(du) = -\text{sgn}(dA)$ (7)

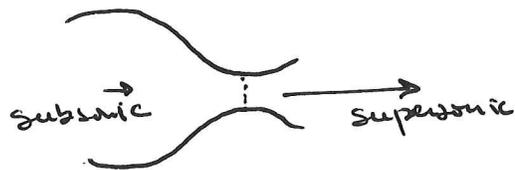
increase in one produces decrease in flow velocity and vice-versa (as in incompressible flow)

Supersonic flow: $\frac{u}{a} > 1 \Rightarrow \text{sgn}(du) = \text{sgn}(dA)$

increase in one corresponds to increase in speed! Flow has to move faster to fill increased volume. This corresponds to intuition for non-interacting particle flow

A transition from subsonic to supersonic flow (i.e., as in a jet engine) requires $1 - \frac{u^2}{a^2} = 0 \Rightarrow dA = 0 \Rightarrow a$

Throat in the nozzle:



Presence of the nozzle does not guarantee subsonic \rightarrow supersonic transition, since another valid solution would be $du = 0$ (i.e., local max or min in speed) \Rightarrow need engineering!

(what we need is $dA = 0$ @ the point where $\frac{u}{a} = 1$ (sonic point))

③ Bondi ⁱⁿ flow / Parker out flow

⑧

Spherically symmetric radial flow in the grav. field of a point mass. Simplest treatment of inflow (accretion) or outflow (wind).

Steady adiabatic/isothermal flow, $\vec{v} = v \cdot \hat{r}$
 $\rho = \rho(r)$

$$\nabla \cdot (\rho \vec{v}) = 0 \Rightarrow \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \rho v) = 0 \Rightarrow r^2 \rho v = \text{constant}$$

$$\Rightarrow \boxed{4\pi r^2 \rho v \triangleq \dot{M}}$$

$$B = \frac{1}{2} v^2 + h + \Phi$$

$$\Phi = -\frac{GM_*}{r} \quad h = \int \frac{dP}{\rho} = \begin{cases} \frac{\gamma}{\gamma-1} \frac{P}{\rho} + \text{const, adiabatic} \\ \frac{kT}{\mu} \ln \rho + \text{const, isothermal} \end{cases}$$

④ Inflow problem:

Inflow from medium in which

$$\begin{cases} \rho \rightarrow \rho_1 & \text{ambient } \rho \\ P \rightarrow P_1 & \text{ambient } P \\ v \rightarrow 0 \\ \Phi \rightarrow 0 \end{cases}$$

adiabatic

Define $c_1^2 = \gamma \frac{P_1}{\rho_1} = \gamma K \rho_1^{\gamma-1}$ for $P = K \rho^\gamma$

Sound speed ($c_s^2 = \frac{\partial P}{\partial \rho}$)

$$\Rightarrow \frac{1}{2} v^2 + \frac{\gamma}{\gamma-1} K \rho^{\gamma-1} - \frac{GM}{r} = \frac{\gamma}{\gamma-1} K \rho_1^{\gamma-1} \quad \text{constancy of Bernoulli}$$

$$\Rightarrow \gamma K = \frac{c_1^2}{\rho_1^{\gamma-1}} \Rightarrow \boxed{\frac{1}{2} v^2 + \frac{c_1^2}{\gamma-1} \left[\left(\frac{\rho}{\rho_1} \right)^{\gamma-1} - 1 \right] - \frac{GM}{r} = 0}$$

isothermal

$$c_1^2 = \frac{P_1}{\rho_1} \Rightarrow \boxed{\frac{1}{2} v^2 + c_1^2 \ln \left(\frac{\rho}{\rho_1} \right) - \frac{GM}{r} = 0}$$

We can introduce the Mach number

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$$M_0 = \frac{v(r)}{c(r)} = \frac{v}{c_1 \left(\frac{\rho}{\rho_1}\right)^{\frac{\gamma-1}{2}}}$$

⇒ Bernoulli becomes

$$\frac{1}{2} c_1^2 M_0^2 \left(\frac{\rho}{\rho_1}\right)^{\gamma-1} + \frac{1}{\gamma-1} c_1^2 \left[\left(\frac{\rho}{\rho_1}\right)^{\gamma-1} - 1\right] - \frac{GM}{r} = 0 \quad \text{adiabatic case}$$

$$\frac{1}{2} c_1^2 M_0^2 + c_1^2 \ln\left(\frac{\rho}{\rho_1}\right) - \frac{GM}{r} = 0 \quad \text{isothermal case}$$

Dividing by $c_1^2 \left(\frac{\rho}{\rho_1}\right)^{\gamma-1}$:

$$\frac{1}{2} M_0^2 + \frac{1}{\gamma-1} = \left(\frac{\rho}{\rho_1}\right)^{-(\gamma-1)} \frac{1}{\gamma-1} + \frac{GM}{rc_1^2} \left(\frac{\rho}{\rho_1}\right)^{-(\gamma-1)} \quad \text{adiabatic}$$

$$\frac{1}{2} M_0^2 + \ln\left(\frac{\rho}{\rho_1}\right) = \frac{GM}{rc_1^2} \quad \text{isothermal}$$

Define $r_B \triangleq \frac{GM}{c_1^2}$ radius where $E_{Th} \sim E_G$

and define $\xi \triangleq r/r_B \Rightarrow$

$$\frac{1}{2} M_0^2 + \frac{1}{\gamma-1} = \left(\frac{\rho}{\rho_1}\right)^{-(\gamma-1)} \left[\frac{1}{\gamma-1} + \frac{1}{\xi}\right] \quad \text{adiabatic}$$

$$\frac{1}{2} M_0^2 + \ln\left(\frac{\rho}{\rho_1}\right) = \frac{1}{\xi} \quad \text{isothermal}$$

The mass continuity equation now is

$$4\pi r_B^2 \xi^2 \rho_1 \left(\frac{\rho}{\rho_1}\right) M_0 c_1 \left(\frac{\rho}{\rho_1}\right)^{\frac{\gamma-1}{2}} = \dot{M}$$

$$\Rightarrow \xi^2 M_0 \left(\frac{\rho}{\rho_1}\right)^{\frac{\gamma+1}{2}} = \frac{\dot{M}}{4\pi r_B^2 \rho_1 c_1} = \frac{\dot{M}}{4\pi (GM)^2 \rho_1 c_1} = \frac{\dot{M}}{\left[\frac{4\pi (GM)^2 \rho_1}{c_1^3}\right]}$$

⇒ Define dimensionless mass accretion rate \dot{M}
 (mass accretion relative to flow at $r \sim r_B$, at distance $r \sim r_B$, at density $\rho \sim \rho_{\infty}$)

$$\xi^2 M_0 (\rho/\rho_1)^{\frac{\gamma+1}{2}} = \dot{M}$$

$$\Rightarrow \left[\rho/\rho_1 = \dot{M}^{\frac{2}{\gamma+1}} M_0^{-2/\gamma+1} \xi^{-4/\gamma+1} \right]$$

$$\Rightarrow \frac{1}{2} M_0^2 + \frac{1}{\gamma-1} = \dot{M}^{-2(\frac{\gamma-1}{\gamma+1})} M_0^{2(\frac{\gamma+1}{\gamma+1})} \xi^{4(\frac{\gamma-1}{\gamma+1})} \left[\frac{1}{\gamma-1} + \frac{1}{\xi} \right]$$

adiabatic

$$\frac{1}{2} M_0^2 + \ln(\dot{M} M_0^{-1} \xi^{-2}) = \frac{1}{\xi} \quad \text{isothermal}$$

$$\Rightarrow \frac{1}{2} M_0^{\frac{4}{\gamma+1}} + \frac{1}{\gamma-1} M_0^{-2(\frac{\gamma-1}{\gamma+1})} = \dot{M}^{-2(\frac{\gamma-1}{\gamma+1})} \left[\xi^{-(-\frac{3\gamma+5}{\gamma+1})} + \frac{1}{\gamma-1} \xi^{4(\frac{\gamma-1}{\gamma+1})} \right]$$

adiabatic

$$\frac{1}{2} M_0^2 - \ln M_0 = -\ln \dot{M} + 2 \ln \xi + \frac{1}{\xi} \quad \text{isothermal}$$

$$\Rightarrow \exp\left(\frac{1}{2} M_0^2 - \ln M_0\right) = \dot{M}^{-1} \exp(2 \ln \xi + \xi^{-1})$$

Define $\mathcal{O}_f^e(M_0) \triangleq \begin{cases} \frac{1}{2} M_0^{\frac{4}{\gamma+1}} + \frac{1}{\gamma-1} M_0^{-2(\frac{\gamma-1}{\gamma+1})} & \text{adiabatic} \\ \exp\left(\frac{1}{2} M_0^2 - \ln M_0\right) & \text{iso.} \end{cases}$

$$\mathcal{G}(\xi) \triangleq \begin{cases} \xi^{-(-\frac{3\gamma+5}{\gamma+1})} + \frac{1}{\gamma-1} \xi^{4(\frac{\gamma-1}{\gamma+1})} & \text{adiabatic} \\ \exp(2 \ln \xi + \xi^{-1}) & \text{iso.} \end{cases}$$

⇒ Solution ~~is~~ $M_0(\xi)$ on \pm trajectories for a given value of \dot{M} (ie, \dot{M}), as determined by the implicit solution of

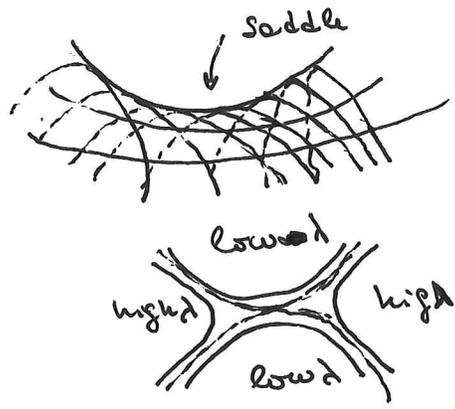
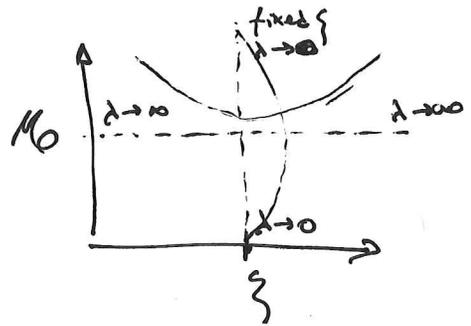
$$\frac{\mathcal{O}_f^e(M_0)}{\mathcal{G}(\xi)} = \begin{cases} \dot{M}^{-2(\frac{\gamma-1}{\gamma+1})} & \text{adiabatic} \\ \dot{M}^{-1} & \text{isothermal} \end{cases}$$

⇒ Coe find $M_0(\xi)$ by drawing contours of $\frac{\mathcal{F}(M_0)}{\mathcal{G}(\xi)} = \text{const.}$ in the $M_0-\xi$ plane (see homework).

Let's look @ the structure of possible solutions.

RHS is $R(\lambda)$, a function of M_0 and ξ treated as general variables ranging from $0 \rightarrow \infty$. For a fixed ξ , $\Rightarrow \mathcal{F}(M_0)$ varies from ∞ through some minimum back to ∞ as M_0 goes from 0 to ∞ , provided $\gamma \geq 1$. \Rightarrow @ fixed ξ , λ goes from 0 to a max back to 0 again as M_0 varies.

Similarly, @ fixed M_0 and $\frac{5}{3}\gamma \geq 1$, \mathcal{G} varies from ∞ to a minimum to ∞ again $\Rightarrow \lambda$ goes from $\infty \rightarrow \text{min} \rightarrow \infty$



⇒ The min of $\mathcal{F}(M_0)$ for fixed ξ corresp. to the max of λ for fixed $\xi \Rightarrow$ occurs @ $\frac{d\mathcal{F}}{dM_0} = 0$

$$\Rightarrow \begin{cases} \frac{2}{\gamma+1} M_0^{\frac{4}{\gamma+1}-1} - \frac{2}{\gamma+1} M_0^{-2} \left(\frac{\gamma-1}{\gamma+1}\right) = 0 \Rightarrow M_0 = 1 \text{ adiabatic} \\ M_0 - \frac{1}{M_0} = 0 \Rightarrow M_0 = 1 \text{ isothermal} \end{cases}$$

Similarly, min of $\gamma(\xi)$ for $M_0 = \text{const.}$ corresp. To min of d for fixed $M_0 \Rightarrow$ occurs @ $\frac{d\gamma}{d\xi} = 0$

$$\left\{ \begin{aligned} -\frac{5-3\gamma}{\gamma+1} \xi^{-\frac{5-3\gamma}{\gamma+1}-1} + \frac{4}{\gamma+1} \xi^{4\left(\frac{\gamma-1}{\gamma+1}\right)-1} &= 0 \Rightarrow \xi = \frac{5-3\gamma}{4} \text{ adiabatic} \\ \frac{2}{\xi} - \frac{1}{\xi^2} = 0 &\Rightarrow \xi = \frac{1}{2} \text{ isothermal} \end{aligned} \right.$$

The separatrix of $d(M_0, \xi)$ ~~occurs at~~ passes through the point where $\left. \frac{d\sqrt{\gamma}}{dM_0} \right|_{\xi} = 0$ and $\left. \frac{d\gamma}{d\xi} \right|_{M_0} = 0$

$$\Rightarrow M_0 = 1 \text{ and } \xi = \xi_s \begin{cases} \frac{5-3\gamma}{4} \text{ adiabatic} \\ \frac{1}{2} \text{ iso.} \end{cases}$$

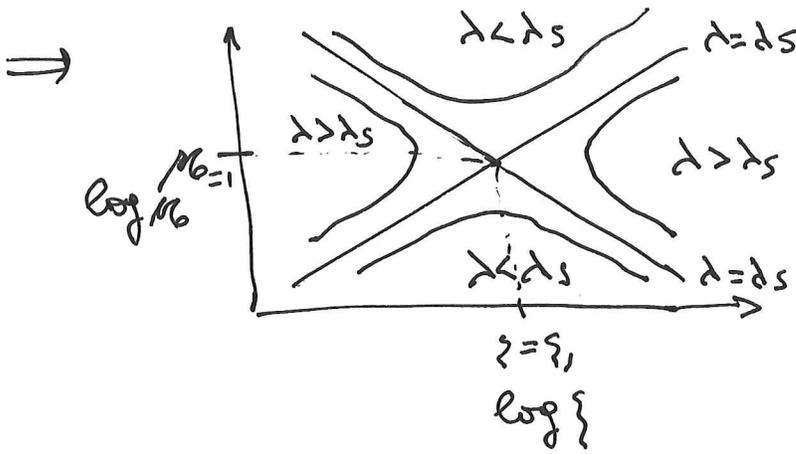
Sonic point

$$\Rightarrow f(M_0)|_s = \begin{cases} \frac{1}{2} + \frac{1}{\gamma-1} & \text{ad.} \\ \exp(1/2) & \text{iso.} \end{cases}$$

$$\gamma(M_0)|_s = \begin{cases} \xi_s^{-\frac{5-3\gamma}{\gamma+1}} + \frac{1}{\gamma+1} \xi_s^{4\left(\frac{\gamma-1}{\gamma+1}\right)} & \text{ad.} \\ \exp(2e^{1/2} + 2) = \frac{e^2}{4} & \text{iso.} \end{cases}$$

$$\Rightarrow \left[\frac{\frac{1}{2} + \frac{1}{\gamma-1}}{\xi_s^{-\frac{5-3\gamma}{\gamma+1}} + \frac{1}{\gamma+1} \xi_s^{4\left(\frac{\gamma-1}{\gamma+1}\right)}} \right]^{-\frac{(\gamma+1)}{2(\gamma-1)}} = ds \text{ ad.}$$

$$\frac{e^2/4}{e^{1/2}} = \frac{e^{3/2}}{4} = ds \text{ iso. } (ds = 1.120)$$



The inflow must follow $\lambda = \text{const.}$ and start from $N_6 \rightarrow 0$ at $\zeta \rightarrow \infty$ (lower right)

$\lambda > \lambda_s \rightarrow$ not in flow ζ decreases to a min then increases again (multivalued)

$\lambda < \lambda_s \rightarrow$ goes to $\zeta \rightarrow 0$ but have $N_6 \rightarrow 0 \Rightarrow$

$$\frac{1}{r-1} N_6^{-2 \left(\frac{\gamma-1}{\gamma+1} \right)} \rightarrow \lambda^{-2 \left(\frac{\gamma-1}{\gamma+1} \right) - \left(\frac{5-3\gamma}{\gamma+1} \right)}$$

dominant terms in $r \rightarrow 0$

$$\Rightarrow \left(\frac{v}{\rho r^{\frac{\gamma-1}{2}}} \right)^{-2(\gamma-1)} \propto r^{-(5-3\gamma)}$$

$$\Rightarrow \rho^{(\gamma-1)^2} \propto v^{2(\gamma-1)} r^{-(5-3\gamma)}$$

$$\Rightarrow \rho \propto v^{\frac{2}{\gamma-1}} r^{-\frac{5-3\gamma}{(\gamma-1)^2}} \rightarrow \infty \text{ for any finite } v \text{ when } r \rightarrow 0$$

$N_6 \rightarrow 0, \zeta \rightarrow \infty$, unphysical

~~The~~ $\lambda = \lambda_s$ has no unphysical problems associated with it.

IT starts subsonic, makes sonic transition @ $\zeta = \zeta_s$, and remains supersonic until it hits the star. For $r \rightarrow 0$

$$\frac{1}{2} N_6^{\frac{4}{\gamma+1}} = \lambda_s^{-2 \left(\frac{\gamma-1}{\gamma+1} \right) - \left(\frac{5-3\gamma}{\gamma+1} \right)}$$

$$\Rightarrow \left(\frac{v}{\rho r^{\frac{\gamma-1}{2}}} \right)^{\frac{4}{\gamma+1}} \propto r^{-\frac{5-3\gamma}{\gamma+1}} \Rightarrow \rho \propto \left[r^{\frac{5-3\gamma}{4}} v \right]^{\frac{2}{\gamma-1}} \rightarrow 0 \text{ for finite } v \text{ as } r \rightarrow 0$$

Nowhere have we demanded inflow \Rightarrow same reasoning applies for steady, spherically symmetric outflow

Must start at small M_0 and ρ (lower left)

$d > d_s$ don't work, $d < d_s$ have too much mass. Only $d = d_s$ solution is physical. See Sec 6.

$$\gamma = 4/3$$

