

# EQUATIONS OF HYDRODYNAMICS

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Continuous medium approximation  $\leftarrow$   
mean free path is smaller than smallest length scale of interest.

Neutral particles; Typ. elastic collision X section is  
 $\sigma \sim 10^{-15} \text{ cm}^2$  ( $\pi a^2$ , with  $a \sim 1 \text{ \AA} = 10^{-8} \text{ cm}$ )

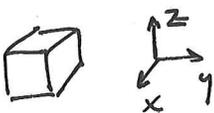
$$\Rightarrow \ell = \frac{1}{n \sigma} \quad \text{mean free path}$$

For  $n \sim 1 \text{ cm}^{-3}$ ,  $\ell \sim 10^{15} \text{ cm} \sim 100 \text{ AU} \ll$  than sizes of  
Typical ISM structures.

In high density regions (e.g. protostellar cores  $n \sim 10^4 \text{ cm}^{-3}$ )  
scales are smaller, but sizes are also smaller.

Basic eqs. are conservation laws: mass, momentum, energy

## Mass conservation

Consider cell 

Change in mass is just due to fluid entering or leaving its  
walls  $\Rightarrow$

$$\delta m = \left[ \delta z \delta y (p v_x|_x - p v_x|_{x+\delta x}) + \delta x \delta y (p v_z|_z - p v_z|_{z+\delta z}) + \delta x \delta z (p v_y|_y - p v_y|_{y+\delta y}) \right] \cdot \delta t$$

$$\Rightarrow \frac{\delta m}{\delta x \delta y \delta z} = - \delta t \left[ \frac{\partial}{\partial x} (p v_x) + \frac{\partial}{\partial y} (p v_y) + \frac{\partial}{\partial z} (p v_z) \right]$$

$$\delta m = \rho \delta x \delta y \delta z \Rightarrow \left. \frac{\delta \rho}{\delta t} \right|_{x,y,z} = - \nabla \cdot (\rho \vec{v})$$

$$\Rightarrow \boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0}$$

"continuity" equation  
(mass conservation)

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Define  $\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{v} \cdot \nabla$  operator: Lagrangian derivative<sup>Time</sup>

this is the total derivative w.r.t. time following a fluid element  
(aka "convective derivative")

$$\Rightarrow \boxed{\frac{D\rho}{Dt} + \rho \nabla \cdot \vec{v} = 0} \quad (\nabla \cdot (\rho \vec{v}) = \vec{v} \cdot \nabla \rho + \rho \nabla \cdot \vec{v})$$

Alternatively  $\frac{1}{\rho} \frac{D\rho}{Dt} = -\nabla \cdot \vec{v}$

$$\Rightarrow \boxed{\frac{D \ln \rho}{Dt} = -\nabla \cdot \vec{v}}$$

Rate of change of  $\ln \frac{\text{volume}}{\text{mass}}$  = divergence of velocity

Since for a Lagrangian element mass doesn't change,  
this simply means that the rate of change of  $\ln$  volume  
is the divergence of  $\vec{v}$  field.

Since  $\rho \vec{v}$  is the flux of mass, conservation laws look

like  $\frac{\partial \rho Q}{\partial t} + \nabla \cdot (\vec{T}_Q) = 0$

$\swarrow$  density of Q  
 $\searrow$  flux of Q

# Momentum conservation

$$\delta P_i = \delta (\rho \delta x \delta y \delta z v_i)$$



$$= \left[ \frac{-\partial}{\partial x} (\rho v_x v_i) \delta x \delta y \delta z - \frac{\partial}{\partial y} (\rho v_y v_i) \delta x \delta y \delta z - \frac{\partial}{\partial z} (\rho v_z v_i) \delta x \delta y \delta z \right] \delta t$$

flux terms

$$+ \vec{F}_i \delta t \quad \leftarrow \text{force terms}$$

pressure looks like:  $\vec{F}_i = (P(x_i) - P(x_i + \delta x_i)) \delta x_i \delta x_i \frac{\delta x_i}{\delta x_i}$

gravity:  $\vec{F}_i = - \left[ \frac{\partial \Phi}{\partial x_i} \right] \rho \delta x \delta y \delta z$

All Together,

$$\frac{\partial}{\partial t} (\rho \vec{v}) = - \nabla \cdot (\rho \vec{v} \vec{v}) - \nabla P - \rho \nabla \Phi$$

Eulerian form of momentum equation

$$\Rightarrow \rho \frac{\partial \vec{v}}{\partial t} + \underbrace{\vec{v} \cdot \frac{\partial \rho}{\partial t}}_{=0 \text{ from mass continuity}} = - \vec{v} \cdot \nabla (\rho \vec{v}) - (\rho \vec{v}) \cdot \nabla \vec{v} - \nabla P - \rho \nabla \Phi$$

$$\Rightarrow \rho \frac{\partial \vec{v}}{\partial t} + \rho \vec{v} \cdot \nabla \vec{v} = - \nabla P - \rho \nabla \Phi$$

$$\frac{D \vec{v}}{Dt} = \frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} = - \frac{\nabla P}{\rho} - \nabla \Phi$$

$$\Rightarrow \frac{D \vec{v}}{Dt} = - \frac{\nabla P}{\rho} - \nabla \Phi$$

Lagrangian form of momentum equation

Lagrangian derivative of  $\vec{v}$  force / mass

The Eulerian form can be re-arranged to become

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$$\frac{\partial}{\partial t} (\underbrace{\rho \vec{v}}_{\text{mom. density}}) + \nabla \cdot (\underbrace{\rho \vec{v}^2}_{\text{mom. flux}}) + \nabla P = -\rho \nabla \Phi$$

( $\rho \vec{v}^2$  is a pressure, "ram pressure")

$$\frac{\partial}{\partial t} (\rho \vec{v}) + \nabla \cdot (\rho \vec{v}^2 + P) = -\rho \nabla \Phi$$

### Energy Conservation

$$\delta E = \delta \left[ \rho \left( \epsilon + \frac{1}{2} v^2 \right) \right]$$

$\epsilon = \frac{\text{Thermal energy}}{\text{mass}}$

$E = \frac{\text{Total energy}}{\text{volume}}$

$$= -\nabla \cdot \left[ \rho \vec{v} \left( \epsilon + \frac{1}{2} v^2 \right) \right] \delta t \leftarrow \text{flux terms}$$

$$+ W \delta t \leftarrow \text{work done}$$

Pressure work  $\Rightarrow W_P = -\nabla \cdot (P \vec{r})$

$$W = F dx = \frac{F dx}{dt} dt$$

Gravitational work  $\Rightarrow W_G = \vec{g} \cdot \rho \cdot \vec{r}$

$$\Rightarrow \left[ \frac{\partial}{\partial t} \left( \rho \left( \epsilon + \frac{1}{2} v^2 \right) \right) + \nabla \cdot \left( \rho \vec{v} \left( \epsilon + \frac{1}{2} v^2 \right) + P \vec{r} \right) = -\rho \vec{v} \cdot \nabla \Phi \right]$$

Conservation law form of energy eqn.

Further simplification:  $\nabla \cdot (\gamma \cdot \vec{A}) = \vec{A} \cdot \nabla \gamma + \gamma \cdot \nabla \cdot \vec{A}$   
 (assuming  $P=0$ , no PdV work)

$$-\rho \vec{v} \cdot \nabla \Phi = -\nabla \cdot (\rho \vec{v} \Phi) + \cancel{\rho \Phi \cdot \nabla \cdot (\rho \vec{v})}$$

from continuity eqn.  $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$

$$\Rightarrow = -\frac{\partial \rho}{\partial t} \Phi$$

$$\Rightarrow \rho \vec{v} \cdot \nabla \Phi = -\nabla \cdot (\rho \vec{v} \Phi) - \frac{\partial \rho}{\partial t} \Phi$$

$$= -\nabla \cdot (\rho \vec{v} \Phi) - \frac{\partial}{\partial t} (\rho \Phi) + \rho \frac{\partial \Phi}{\partial t} \quad (\text{pois})$$

Mass and gravitational potential are related through Poisson's 5  
~~equation~~ equation for the grav. field

$$4\pi G \rho = \nabla^2 \Phi \Rightarrow \Phi \frac{\partial \rho}{\partial t} = \Phi \frac{\partial}{\partial t} \left( \frac{\nabla^2 \Phi}{4\pi G} \right) = \Phi \nabla \cdot \left( \frac{\partial}{\partial t} \frac{\nabla \Phi}{4\pi G} \right)$$

$$= \nabla \cdot \left( \Phi \frac{\partial}{\partial t} \frac{\nabla \Phi}{4\pi G} \right) - \frac{\partial}{\partial t} \frac{|\nabla \Phi|^2}{4\pi G} \cdot \frac{1}{2}$$

$$\left[ \frac{\partial}{\partial t} (\nabla \Phi)^2 = 2 \nabla \Phi \frac{\partial}{\partial t} (\nabla \Phi) \right]$$

Similarly:  $\int \frac{\partial \Phi}{\partial t} = \frac{\nabla^2 \Phi}{4\pi G} \cdot \frac{\partial \Phi}{\partial t} = \nabla \cdot \left( \frac{\nabla \Phi}{4\pi G} \cdot \frac{\partial \Phi}{\partial t} \right) - \frac{\partial}{\partial t} \frac{|\nabla \Phi|^2}{8\pi G}$

Although these two terms aren't equal, their integrals over all space are. This is because the integrals ~~are~~ of the divergence  $\iiint \nabla \cdot \vec{F} dV = \oint_S \vec{F} \cdot \vec{n} dS$  (Gauss' theorem) are the same surface integral

$$\Rightarrow \iiint \rho \frac{\partial \Phi}{\partial t} d^3x = \iiint \Phi \frac{\partial \rho}{\partial t} d^3x \Rightarrow \iiint \frac{\partial}{\partial t} (\rho \Phi) d^3x = 2 \iiint \rho \frac{\partial \Phi}{\partial t} d^3x = 2 \iiint \Phi \frac{\partial \rho}{\partial t} d^3x$$

$$\iiint -\frac{\partial}{\partial t} (\rho \Phi) + \rho \frac{\partial \Phi}{\partial t} d^3x = -\frac{1}{2} \frac{d}{dt} \iiint \rho \Phi d^3x$$

So, interpreting over all space (dropping  $\iiint$  signs)

$$\frac{d}{dt} \int \rho \left( \epsilon + \frac{1}{2} v^2 \right) d^3x = -\frac{1}{2} \frac{d}{dt} \int \rho \Phi d^3x$$

$$\Rightarrow \boxed{\frac{d}{dt} \int \left[ \rho \left( \epsilon + \frac{1}{2} v^2 \right) + \frac{1}{2} \rho \Phi \right] d^3x = 0}$$

Conservation of  
Total energy, no PdV work

There is no purely local energy conservation when gravity is included, because gravity is a long-range force.

The energy eqn. may also be written in a different form

$$\underbrace{\left(\epsilon + \frac{1}{2}v^2\right) \frac{\partial \rho}{\partial t} + \rho \frac{\partial}{\partial t} \left(\frac{1}{2}v^2 + \epsilon\right) + \left(\epsilon + \frac{1}{2}v^2\right) \nabla \cdot (\rho \vec{v}) + \rho \vec{v} \cdot \nabla \left(\epsilon + \frac{1}{2}v^2\right)}_{=0}$$

$$+ \rho \vec{v} \cdot \nabla \left(\frac{P}{\rho}\right) + \frac{P}{\rho} \nabla \cdot (\rho \vec{v}) = -\rho \vec{v} \cdot \nabla \Phi$$

$$\left(\epsilon + \frac{1}{2}v^2\right) \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v})\right] = 0 \text{ because of mass continuity}$$

$$\Rightarrow \frac{\partial}{\partial t} \left(\frac{1}{2}v^2 + \epsilon\right) + \vec{v} \cdot \nabla \left(\epsilon + \frac{1}{2}v^2 + \frac{P}{\rho}\right) - \frac{P}{\rho^2} \frac{\partial \rho}{\partial t} = -\vec{v} \cdot \nabla \Phi$$

$$\Rightarrow \frac{\partial}{\partial t} \left(\epsilon + \frac{1}{2}v^2\right) + \vec{v} \cdot \nabla \left(\epsilon + \frac{1}{2}v^2 + \frac{P}{\rho}\right) = -P \frac{\partial \rho^{-1}}{\partial t} - \vec{v} \cdot \nabla \Phi \quad \left. \vphantom{\frac{\partial}{\partial t}} \right\} \begin{array}{l} \text{we'll come back} \\ \text{to this in a} \\ \text{moment} \end{array}$$

Recall that Lagrangian form of momentum eqn. is.

$$\left(\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} = -\frac{\nabla P}{\rho} - \nabla \Phi\right) \cdot \vec{v}$$

Taking dot product with  $\vec{v}$ , and using  $\vec{v} \cdot \nabla \vec{v} = \nabla \left(\frac{1}{2}v^2\right) + (\nabla \times \vec{v}) \times \vec{v}$   
 $\left(\frac{1}{2} \nabla (\vec{A} \cdot \vec{A}) = \vec{A} \times (\nabla \times \vec{A}) + \vec{A} \cdot \nabla \vec{A}\right)$

when projected  
 onto  $\vec{v}$   
 since  
 $\vec{v} \cdot (\nabla \times \vec{v}) \times \vec{v} = 0$

$$\frac{\partial}{\partial t} \left(\frac{1}{2}v^2\right) + \vec{v} \cdot \nabla \left(\frac{1}{2}v^2\right) = -\vec{v} \cdot \frac{\nabla P}{\rho} - \vec{v} \cdot \nabla \Phi$$

$$= -\vec{v} \cdot \nabla \left(\frac{P}{\rho}\right) + P \vec{v} \cdot \nabla \left(\frac{1}{\rho}\right) - \vec{v} \cdot \nabla \Phi$$

$$\frac{\partial}{\partial t} \left(\frac{1}{2}v^2\right) + \vec{v} \cdot \nabla \left(\frac{1}{2}v^2 + \frac{P}{\rho}\right) = P \vec{v} \cdot \nabla \rho^{-1} - \vec{v} \cdot \nabla \Phi$$

We had arrived to this form of the energy eqn. after

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Some manipulations:

$$\frac{\partial}{\partial t} \left( \epsilon + \frac{1}{2} v^2 \right) + \vec{v} \cdot \nabla \left( \epsilon + \frac{1}{2} v^2 + \frac{P}{\rho} \right) = -P \frac{\partial \bar{\rho}^{-1}}{\partial t} - \vec{v} \cdot \nabla \Phi$$

Now we can subtract the momentum eqn.

$$\frac{\partial}{\partial t} \left( \frac{1}{2} v^2 \right) + \vec{v} \cdot \nabla \left( \frac{1}{2} v^2 + \frac{P}{\rho} \right) = P \vec{v} \cdot \nabla \bar{\rho}^{-1} - \vec{v} \cdot \nabla \Phi$$

$$\Rightarrow \frac{\partial \epsilon}{\partial t} + \vec{v} \cdot \nabla \epsilon = -P \frac{\partial \bar{\rho}^{-1}}{\partial t} - P \vec{v} \cdot \nabla \bar{\rho}^{-1}$$

$$= -P \left( \frac{\partial \bar{\rho}^{-1}}{\partial t} + \vec{v} \cdot \nabla \bar{\rho}^{-1} \right)$$

$$\Rightarrow \boxed{\frac{D\epsilon}{Dt} = -P \frac{D\bar{\rho}^{-1}}{Dt}}$$

$\epsilon = \text{energy/mass}$ ,  $\bar{\rho}^{-1} = \text{volume/mass}$

$$\Rightarrow \boxed{\frac{D\epsilon}{Dt} = -P \frac{DV}{Dt}}$$

The familiar law of Thermodynamics with no entropy change

Taking the dot-product of the Lagrangian momentum eqn. with  $\rho \vec{v} \Rightarrow$

$$\rho \vec{v} \cdot \frac{\partial}{\partial t} (\vec{v}) + \rho \vec{v} \cdot (\nabla \frac{1}{2} v^2) = -\vec{v} \cdot \nabla P - \rho \vec{v} \cdot \nabla \Phi$$

$$\Rightarrow \rho \frac{\partial}{\partial t} \left( \frac{1}{2} v^2 \right) + \rho \vec{v} \cdot \nabla \left( \frac{1}{2} v^2 \right) = -\vec{v} \cdot \nabla P - \rho \vec{v} \cdot \nabla \Phi$$

$$\Rightarrow \boxed{\frac{\partial}{\partial t} \left( \frac{1}{2} v^2 \rho \right) + \nabla \cdot \left( \frac{1}{2} v^2 \rho \vec{v} \right) = -\vec{v} \cdot \nabla P - \rho \vec{v} \cdot \nabla \Phi} \quad (\text{after using continuity eqn.})$$

$\underbrace{\frac{\partial}{\partial t} \left( \frac{1}{2} v^2 \rho \right) + \nabla \cdot \left( \frac{1}{2} v^2 \rho \vec{v} \right)}_0$

This is the "work equation"

Subtracting work eqn. from Total energy eq.:

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$$\frac{\partial}{\partial t} (\rho \epsilon) + \nabla \cdot (\rho \vec{v} \epsilon) + \nabla \cdot (P \vec{v}) = \vec{v} \cdot \nabla P$$

$$\Rightarrow \frac{\partial}{\partial t} (\rho \epsilon) + \nabla \cdot (\rho \vec{v} \epsilon) + P \nabla \cdot \vec{v} = 0$$

$$\Rightarrow \boxed{\frac{\partial}{\partial t} (\rho \epsilon) + \nabla \cdot (\rho \vec{v} \epsilon) = -P \nabla \cdot \vec{v}} \quad \text{"Internal energy eqn."}$$