

## EQS. OF MAGNETOHYDRODYNAMICS

Up To now: Eq. of hydrodynamics  $\rightarrow$  small perturb.  
 These eqs. apply when mean free path  $\ll$  physical size of system. Also only forces considered were gravitational, not EM.

Most of The universe is fully or partially ionized  $\Rightarrow \vec{E}$  and  $\vec{B}$  fields exist, hence EM forces.

$\Rightarrow$  Need To consider effects of fields on matter and matter on fields  $\Rightarrow$  Eqs. of MHD.

Consider Maxwell eqs.:

$$\nabla \cdot \vec{E} = 4\pi \rho_e \quad \text{Gauss' law}$$

$$\nabla \cdot \vec{B} = 0 \quad \text{No monopoles}$$

$$\nabla_x \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t} \quad \text{Faraday's law of induction}$$

$$\nabla_x \vec{B} = \frac{4\pi}{c} \vec{J} + \frac{1}{c} \frac{\partial \vec{E}}{\partial t} \quad \text{Ampere's law with Maxwell's displacement current.}$$

MHD approximations:

1) No free charges:  $\rho_e = 0 \Rightarrow \nabla \cdot \vec{E} = 0 \Leftrightarrow$  no electrostatic fields.

If There were, They would move around To short out The field.

It can be shown That The resulting  $|\vec{E}|$  is very small Such That  $\rho_e$  or  $\rho_m \ll 1$ .

Local charge neutrality  $\not\Rightarrow$  zero EM forces, because of The relative drift of + and - charges.

$e^-$  velocities  $\vec{v}_e$ , ion velocity  $\vec{v}_i$ ,  $m_i z_i e = m_e e$  (charge neutrality)

$\Rightarrow$  local current density (charge / time / area)  $\Rightarrow$

(2)

$$\vec{j}_c = -ne\vec{v}_e + n_i e \vec{v}_i = -ne e (\vec{v}_e - \vec{v}_i) = \vec{J}$$

2) Displacement current is small : i.e., Time variations are small  $\Rightarrow \frac{\partial \vec{E}}{c \partial t} \ll |\nabla \times \vec{B}|$

Although ~~discrete~~ electrostatic fields are small, Time dependent  $\vec{E}$  is non-negligible  $\Rightarrow$  we need to relate it to the other terms. To do so, Think in rest frame of the ions.

Lorentz Transformation :  $\vec{E}' = \vec{E} + \frac{\vec{v}_i}{c} \times \vec{B}$   
 $\vec{B}' = \vec{B}$

$\Rightarrow$  Eq. of motion of e in the ion frame

$$me \frac{d}{dt} (\underbrace{\vec{v}_e - \vec{v}_i}_{\vec{v}'_e}) = -e \left( \vec{E}' + \frac{(\vec{v}_e - \vec{v}_i)}{c} \times \vec{B}' \right) - \underbrace{me v_e \vec{v}_e'}_{\text{drag force on e due to collisions with ions}} + \underbrace{\text{grav. pressure terms}}_{\nu_c = \text{collision frequency}}$$

$$\Rightarrow me \frac{d \vec{v}'_e}{dt} = -e \left( \vec{E}' + \frac{\vec{v}_i}{c} \times \vec{B} + \frac{\vec{v}'_e \times \vec{B}}{c} \right) - me v_e \vec{v}'_e + \text{grav., pressure, etc}$$

Since  $\vec{v}'_e \ll \vec{v}_i \Rightarrow$  drop  $\vec{v}'_e \times \vec{B}$  term. Assume inertial and gravity terms are small compare to the rest  $\Rightarrow$

$$\vec{E}' = \vec{E} + \frac{\vec{v}_i}{c} \times \vec{B} = -\frac{me \nu_c}{e} \vec{v}'_e$$

$\Rightarrow$  in ion frame the e reach a terminal velocity  $\vec{v}'_e$  such that the  $\vec{E}'$  field force is balanced with the collisional drag.

(3)

The current in the ion frame,  $-\dot{m}e\vec{v}_i^*$ , is thus

$$\vec{J}' = \left(-\dot{m}e\right) \left(-\frac{e}{\dot{m}e\nu_c}\right) \vec{E}' = \frac{\dot{m}e^2}{\dot{m}e\nu_c} \vec{E}' \triangleq \sigma \vec{E}'$$

This is Ohm's law in the ion frame, current density equals conductivity  $\times \vec{E}$  field ( $V = iR$ ,  $V = E \cdot L \Rightarrow R = \frac{L}{\sigma A}$ )

$$\Rightarrow \vec{E}' = \vec{E} + \frac{\vec{v}_i}{c} \times \vec{B} = \frac{\vec{J}'}{\sigma} \Rightarrow \vec{E} = -\frac{\vec{v}_i \times \vec{B}}{c} + \frac{\vec{J}'}{\sigma}$$

$$\Rightarrow \nabla \times \left( -\frac{\vec{v}_i \times \vec{B}}{c} + \frac{\vec{J}'}{\sigma} \right) = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t} \quad \text{Faraday}$$

$$\vec{J}' = \frac{c}{4\pi} \nabla \times \vec{B} \quad \text{from Ampere's with } \frac{\partial \vec{E}}{\partial t} = 0 \quad (\text{recall } \vec{B}' = \vec{B})$$

$$\Rightarrow \nabla \times \left( \frac{\vec{v}_i \times \vec{B}}{c} - \frac{c}{4\pi\sigma} \nabla \times \vec{B} \right) = \frac{1}{c} \frac{\partial \vec{B}}{\partial t}$$

$\Rightarrow$  This yields an evolution equation for  $\vec{B}$

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{v}_i \times \vec{B}) - \nabla \times \left( \frac{c^2}{4\pi\sigma} \nabla \times \vec{B} \right)$$

$$\gamma \triangleq \frac{c^2}{4\pi\sigma} \quad \text{resistivity}, \quad \vec{v}_e \approx \vec{v}_i = \vec{v}$$

$$\Rightarrow \frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{v} \times \vec{B}) - \nabla \gamma \times (\nabla \times \vec{B}) - \gamma \underbrace{\nabla \times (\nabla \times \vec{B})}_{\frac{\nabla(\nabla \cdot \vec{B})}{\sigma} - \nabla^2 \vec{B}}$$

$$\Rightarrow \boxed{\frac{\partial \vec{B}}{\partial t} = \underbrace{\nabla \times (\vec{v} \times \vec{B})}_{\text{advection}} + \underbrace{\gamma \nabla^2 \vec{B}}_{\text{diffusion}} - \underbrace{\nabla \gamma \times (\nabla \times \vec{B})}_{\frac{\nabla(\nabla \cdot \vec{B})}{\sigma} - \nabla^2 \vec{B}}$$

Induction equation: evolution of  $\vec{B}$  in response to velocity field  $\vec{v}$  and diffusion from  $\gamma$  terms.

Eqs. of motion:

Mass conservation is still  $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$

Momentum eq. now has Lorentz force term:

$$\frac{\text{Lorentz force}}{\text{volume}} = \frac{\vec{J}}{c} \times \vec{B} = \frac{1}{4\pi} (\nabla \times \vec{B}) \times \vec{B} \quad (\vec{J} = \frac{c}{4\pi} \nabla \times \vec{B} \text{ Ampere})$$

$$\Rightarrow \frac{\text{force}}{\text{mass}} = \frac{\text{force}}{\text{volume}} \quad \left( \frac{\text{volume}}{\text{mass}} = \frac{1}{\rho} \right)$$

$$\Rightarrow \boxed{\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} = -\frac{\nabla P}{\rho} + \frac{(\nabla \times \vec{B}) \times \vec{B}}{4\pi\rho} - \nabla \Phi}$$

Note that although there is a term for collisions in the induction eq., here it doesn't figure because this is for eions, and the momentum sum is conserved (collisions cancel out).

These collisions can convert ordered motion into heat  $\Rightarrow$

The energy eq. has a term associated with "resistive heating" or.k.a "Joule heating" or "ohmic dissipation".

The energy/time/volume associated with a current flowing along  $\vec{E}$  field is  $\vec{J}' \cdot \vec{E}'$  (in frame) =  $\frac{\vec{J}'}{\sigma} \cdot \vec{J}'$

$$\Rightarrow \frac{|\vec{J}'|^2}{\sigma} = \frac{4\pi M}{c^2} |\vec{J}'|^2$$

Total energy equation

$$\Rightarrow \frac{\partial}{\partial t} \left( \rho \left( \epsilon + \frac{1}{2} v^2 \right) \right) + \nabla \cdot \left( \rho \vec{v} \left( \epsilon + \frac{1}{2} v^2 \right) + \vec{P} \vec{v} \right) = -\rho \vec{v} \cdot \nabla \Phi + \frac{4\pi M}{c^2} |\vec{J}'|^2$$

$$\text{with } \vec{J}' = \frac{c}{4\pi} \nabla \times \vec{B} \Rightarrow \frac{4\pi M}{c^2} |\vec{J}'|^2 = \frac{M}{4\pi} |\nabla \times \vec{B}|^2$$

(5)

The internal energy equation becomes

$$\frac{\partial}{\partial t} \left( \rho \epsilon \right) + \nabla \cdot \left( \rho \vec{v} \left( \epsilon + \frac{P}{\rho} \right) \right) = \vec{v} \cdot \nabla P + \frac{\eta}{4\pi} |\nabla \times \vec{B}|^2$$

$$\Rightarrow \boxed{\frac{\partial}{\partial t} \left( \rho \epsilon \right) + \nabla \cdot \left( \rho \vec{v} \epsilon \right) = -P \nabla \cdot \vec{v} + \frac{\eta}{4\pi} |\nabla \times \vec{B}|^2}$$

For a perfect gas with  $\epsilon = \frac{1}{\gamma-1} \frac{P}{\rho} \Rightarrow$

$$\frac{1}{\gamma-1} \left[ \frac{\partial P}{\partial t} + \nabla \cdot (\vec{v} \cdot \vec{P}) \right] = -P \nabla \cdot \vec{v} + \frac{\eta}{4\pi} |\nabla \times \vec{B}|^2$$

$$\frac{1}{\gamma-1} \left( \frac{\partial P}{\partial t} + \vec{v} \cdot \nabla P \right) + \frac{\gamma}{\gamma-1} P \nabla \cdot \vec{v} = \frac{\eta}{4\pi} |\nabla \times \vec{B}|^2$$

Using  $\nabla \cdot \vec{v} = -\frac{1}{\rho} \left( \frac{\partial \rho}{\partial t} + \vec{v} \cdot \nabla \rho \right)$  (mass conservation)

$$\Rightarrow \frac{P}{\gamma-1} \frac{D \ln P}{Dt} - \frac{\gamma}{\gamma-1} P \frac{D \ln \rho}{Dt} = \frac{\eta}{4\pi} |\nabla \times \vec{B}|^2$$

$$\Rightarrow \boxed{\frac{P}{\gamma-1} \frac{D}{Dt} \ln(P\rho^{-\gamma}) = \frac{\eta}{4\pi} |\nabla \times \vec{B}|^2}$$

Steadily  
entropy increases along  
streamlines from resistive  
heating

Ideal MHD :

Consists of evolution equations for  $\vec{v}, \rho, \vec{B}$ , and  $\rho \epsilon \propto P$  with  $\eta \rightarrow 0$ . This ideal limit represents infrequent inelastic electron collisions.

$$\text{Induction Eq.: } \frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{v} \times \vec{B}) + \eta \nabla^2 \vec{B} - \nabla \eta \times (\nabla \times \vec{B})$$

$$\text{with } \eta = \frac{c^2 m_e v_c}{4\pi m_e e^2}$$

$$\text{diffusion term } \sim \frac{\eta}{L^2} B, \text{ advection term } \sim \frac{v B}{L}$$

$$\Rightarrow \text{diffusion/advection} \sim \frac{\eta}{L v} \sim \frac{c^2 m_e v_c}{4\pi m_e e^2 L v}$$

(6)

What is the collision frequency?

Thermal speed of electrons is  $v_T \sim \left(\frac{kT}{me}\right)^{1/2}$

To exchange momentum with the ions, They need to approach closely enough (To a distance  $r_{\min}$ )

To estimate  $r_{\min}$ :

$$me v_T^2 \sim \frac{e^2 Z}{r_{\min}} \Rightarrow r_{\min} \sim \frac{e^2 Z}{v_T^2 me}$$

$\Rightarrow$  Cross section for encounters is  $\sigma \sim \pi r_{\min}^2 \sim \frac{\pi e^4 Z^2}{(kT)^2}$

$\Rightarrow$  The collision rate is  $\nu_c \sim n_i v_T \sigma$

where  $n_i = \frac{me}{Z} \Rightarrow \nu_c \sim \frac{\pi e^2 Z^2}{(kT)^2} \frac{me}{Z} \left(\frac{kT}{me}\right)^{1/2}$

$$\nu_c \sim \frac{e^4 Z me}{me^{1/2} (kT)^{3/2}}$$

Exact calculations find

$$\nu_c = \frac{me e^4 me^{-1/2} Z \pi^{3/2} \ln \Lambda}{Z (2kT)^{3/2}} \cdot \frac{1}{\gamma}$$

where  $\gamma = 0.582$  for  $Z=1$

,  $\gamma \rightarrow 1$  for  $Z \rightarrow \infty$

(Spitzer 5-34, 5-36)

$\ln \Lambda$  = "Coulomb logarithm"  $\sim 10$

$$\Rightarrow \eta = \frac{c^2}{4\pi} \frac{m_e}{n_e e^2} v_c^2 = \frac{1}{4\pi} \frac{c^2 e^2 m_e^{1/2} Z \pi^{3/2} \ln \Lambda}{2(2kT)^{3/2}} \frac{1}{\gamma}$$

Note That  $\eta_{\text{sho}} = \frac{1}{4\pi} \eta_{\text{Spitzer}}$

$$= \frac{1}{4\pi} \cdot 3.8 \times 10^{12} \frac{Z \ln \Lambda}{T^{3/2} \gamma} \quad \text{from Phys. of Fully Ionized Gas, eqs 5-34, 5-37}$$

$$= \frac{1}{4\pi} \cdot 6.53 \times 10^{12} \frac{\ln \Lambda}{T^{3/2}}$$

$$\Rightarrow \frac{\eta}{L^r} = \frac{1.7 \times 10^{-12}}{(\gamma_{pc})(r/km s^{-1})} \frac{\ln \Lambda}{(T/K)^{3/2}} \quad \text{Very small}$$

$\Rightarrow$  down to very small scales  $iT \approx 1$  is a good approx.  
To neglect resistivity