FLUID INSTABILITIES

Frequently we can find a steady solution $\left(\frac{\partial}{\partial t}=0\right)$ of the hydrodynamic eqs., representing a long-term config.. Such solutions are typically "smooth" — e.g. circular flow with differential rotation for gas around a star representing a proto-stellar disk, or spherically-stratified solutions for stellar interiors.

In nature, however, small perturbations abound, so it is key to check for stability: will the perturbations remain small and oscillatory or will they grow, ultimately disrupting the basic solution? Or will they saturate, thus producing small scale structure affecting the long-term flow? (e.g., turbulence)

$\Rightarrow$ Perturbation theory: flow consists of a "simple" solution + variations with <u>small</u> amplitude. Then we linearize eqs. to obtain eqs. of motion for perturbations. In many (but not all) situations we can consider the evolution of separate Fourier modes separately: i.e., assume space-time variation is $\propto e^{i(k\vec{x}-\omega t)}$ and solve eigenvalue problem for relation among amplitudes (eigenvectors) and dispersion relation (ω-k relation). In some other cases, it's necessary to integrate t-dependent eqs. directly.

Sound Waves

Perhaps the simplest kind of disturbance. Good example to develop intuition about perturbation theory.

Suppose we have a uniform, adiabatic medium with EOS $P = \mathbb{K}\rho^\gamma$ everywhere.

$$\Rightarrow \quad P_0 + P_1 = \mathbb{K}(\rho_0+\rho_1)^\gamma = \mathbb{K}\rho_0^\gamma\left(1+\frac{\rho_1}{\rho_0}\right)^\gamma \approx \mathbb{K}\rho_0^\gamma\left(1+\gamma\,\rho_1/\rho_0\right)$$

$$P_0 = \mathbb{K}\rho_0^\gamma$$

$$P_1 = \mathbb{K}\rho_0^\gamma\,\gamma\frac{\rho_1}{\rho_0} = \gamma\frac{P_0}{\rho_0}\rho_1 = \left.\frac{dP}{d\rho}\right|_0 \cdot \rho_1$$

We can write the linearized mass and momentum eqs.:

$$\left[\ \frac{\partial\rho_0}{\partial t} = 0, \quad \vec{v}_0 = 0 \ \text{main solution steady}, \quad \rho = \rho_1 + \rho_0 \approx \rho_0\ \right]$$

$$\frac{\partial(\rho_0+\rho_1)}{\partial t} + \nabla\cdot\left((v_0+v_1)(\rho_0+\rho_1)\right) = 0$$

$$\Rightarrow \begin{cases} \dfrac{\partial\rho_1}{\partial t} + \nabla\cdot(\vec{v}_1\rho_0) = 0 \ \Rightarrow \ \dfrac{\partial}{\partial t}(\rho_1/\rho_0) + \nabla\cdot\vec{v}_1 = 0 \\[2ex] \dfrac{\partial\vec{v}_1}{\partial t} = -\dfrac{\nabla P_1}{\rho_0} = -\dfrac{\gamma P_0}{\rho_0}\nabla\left(\dfrac{\rho_1}{\rho_0}\right) \end{cases}$$

$+\vec{v}_1\nabla\vec{v}_1 \sim$ small $O(2)$

$$\frac{D\vec{v}}{Dt} = -\frac{\nabla P}{\rho}$$

$$\frac{\partial(v_0+v_1)}{\partial t} + (v_0+v_1)\nabla(v_0+v_1) = -\frac{\nabla(P_0+P_1)}{\rho_0+\rho_1}$$

$$\frac{\partial v_1}{\partial t} + \underbrace{v_1\nabla v_1}_{\sim O(2)} + \underbrace{v_1\nabla v_0}_{0} = -\frac{\nabla P_1}{\rho_0}$$

Assume $\dfrac{\rho_1}{\rho_0}, \vec{v}_1, \propto e^{i(\vec{k}\vec{x}-\omega t)}$

$$\Rightarrow -i\omega(\rho_1/\rho_0) + i\vec{k}\vec{v}_1 = 0 \ \Rightarrow \ \rho_1/\rho_0 = \frac{\vec{k}\cdot\vec{v}_1}{\omega} \quad (1)$$

$$-i\omega\vec{v}_1 = -i\vec{k}\left(\frac{\gamma P_0}{\rho_0}\right)\frac{\rho_1}{\rho_0} \ \Rightarrow \ \vec{v}_1 = \frac{\vec{k}}{\omega}\left(\frac{\gamma P_0}{\rho_0}\right)\frac{\rho_1}{\rho_0} \quad (2)$$

$$\Rightarrow \vec{v}_1 = \frac{\vec{k}}{\omega^2}\left(\frac{\gamma P_0}{\rho_0}\right)\vec{k}\cdot\vec{v}_1 \ \Rightarrow \ \text{dot-product with } \vec{k}, \text{ factor } (\vec{k}\cdot\vec{v}_1)$$

$$\Rightarrow \omega^2 = k^2 \left(\frac{\gamma P_0}{\rho_0}\right) \Rightarrow \omega = \pm\left(\frac{\gamma P_0}{\rho_0}\right)^{1/2} |\vec{k}|$$

This is a <u>real</u> value $\Rightarrow$ wave propagation is stable

Note that: 1) $\vec{v}_1$ is parallel to $\vec{k}$

2) phase velocity $\omega/|k| = \left(\frac{\gamma P_0}{\rho_0}\right)^{1/2}$

3) group velocity $d\omega/d|k| = \left(\frac{\gamma P_0}{\rho_0}\right)^{1/2} = C_0$

This is the sound speed

For $\hat{x}$ along $\hat{k}$ direction, $\vec{v} = v_x \hat{x} \Rightarrow$

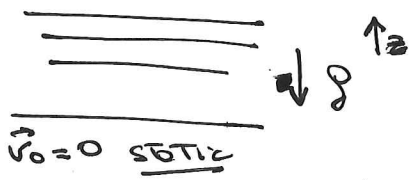
- Forward prop. wave has $\omega > 0$ and ρ_1/ρ_0 max. at the max. of v_{1x}
- Backward prop. wave has $\omega < 0$ and ρ_1/ρ_0 min at the max of v_{1x}

$$\frac{\rho_1}{\rho_0} = \frac{\vec{k}\cdot\vec{v}_1}{\omega} = \frac{k v_{1x}}{\omega} e^{i(kx-\omega t)}$$

Rayleigh-Taylor instability

Static stratified atmosphere in external grav. field $\vec{g}$ (e.g. stellar interior). We will take the plane-parallel limit.

$$\frac{D\vec{v}}{Dt} = -\frac{\nabla P}{\rho} - \nabla\Phi \Rightarrow$$



$$0 = \frac{1}{\rho}\frac{\partial P}{\partial z} - g(z) \quad \text{static solution}$$

$$\begin{cases} \dfrac{\partial \vec{v}_1}{\partial t} + \underbrace{\vec{v}\cdot\nabla\vec{v}}_{\substack{2^{\text{nd}}\text{ order} \\ \Rightarrow \text{drop}}} = \dfrac{\nabla P_0}{\rho_0^2}\rho_1 - \dfrac{\nabla P_1}{\rho_0} & \text{momentum} \quad (1) \\ \dfrac{\partial \rho_1}{\partial t} + \nabla\cdot(\vec{v}_1\rho_0) = 0 & \text{mass} \quad (2) \end{cases}$$

Internal energy eq: $\frac{\partial}{\partial t}(\rho\varepsilon) + \nabla\cdot(\rho\vec{v}\varepsilon) = -P\nabla\cdot\vec{v}$

$\varepsilon = \frac{1}{\gamma-1}\frac{P}{\rho}$ perfect gas $\Rightarrow \rho\varepsilon = \frac{1}{\gamma-1}P$

$\Rightarrow$ linearizing yields

$$\frac{\partial}{\partial t}\left(\frac{1}{\gamma-1}P_1\right) + \nabla\cdot\left(\frac{1}{\gamma-1}P_0\vec{v}_1\right) = -P_0\nabla\cdot\vec{v}_1 \quad (3)$$

Assume sinusoidal perturbation $\propto e^{i(\vec{k}\cdot\vec{x} - \omega t)}$

$\nabla \to i\vec{k}$, $\frac{\partial}{\partial t} \to -i\omega$

$$\Rightarrow \begin{cases} -i\omega\vec{v}_1 = \dfrac{\nabla P_0}{\rho_0^2}\rho_1 - \dfrac{i\vec{k}P_1}{\rho_0} & (1) \\ -i\omega\rho_1 + i\vec{k}\cdot\vec{v}_1\rho_0 + \vec{v}_1\cdot\nabla\rho_0 = 0 & (2) \\ -i\omega P_1 + i\vec{k}\cdot\vec{v}_1 P_0 + \vec{v}_1\nabla P_0 = -(\gamma-1)P_0(i\vec{k}\cdot\vec{v}_1) & (3) \end{cases}$$

Physical cause: <u>buoyancy</u>

Consider a "blob" in an atmosphere and displace it upward adiabatically.

$\delta M = \rho(z_0)(\delta x)^3$, $P = P(z_0)$ @ original location

Assume pressure equilibrium $\Rightarrow$ @ new location

$$P(z+\delta z) = P(z_0) + \delta z \left.\frac{\partial P}{\partial z}\right|_{z_0}$$

Adiabaticity implies:

$$P(z_0)\,\rho(z_0)^{-\gamma} = \left(P(z_0) + \left.\frac{\partial P}{\partial z}\right|_{z_0}\delta z\right)\left(\rho(z_0) + (\delta\rho)_{blob}\right)^{-\gamma}$$

[$(1+x)^{-\gamma} \sim 1-\gamma x$ and throw 2nd order terms]

$$\Rightarrow \frac{1}{P_0}\frac{\partial P}{\partial z}\delta z = \gamma\frac{\delta\rho_{blob}}{\rho_0} \Rightarrow (\delta\rho)_{blob} = \frac{1}{\gamma}\frac{\rho_0}{P_0}\left.\frac{\partial P}{\partial z}\right|_{z_0}\delta z$$

The density of the new surroundings is

$$\rho_0(z_0) + \left.\frac{\partial\rho}{\partial z}\right|_{z_0}\delta z \quad \Rightarrow (\delta\rho)_{background} = \left.\frac{\partial\rho}{\partial z}\right|_{z_0}\delta z$$

$\Rightarrow$ density of blob relative to background is

$$(\delta\rho)_{blob} - (\delta\rho)_{background} = \left(\frac{1}{\gamma}\frac{\rho}{P}\frac{\partial P}{\partial z} - \frac{\partial\rho}{\partial z}\right)\delta z$$

$$= \frac{\rho}{\gamma}\left(\frac{\partial \ln P\rho^{-\gamma}}{\partial z}\right)\delta z$$

If the blob is underdense, then it will be buoyant and keep rising $\Rightarrow$ the condition for instability is

$$(\delta\rho)_{blob} - (\delta\rho)_{backg.} < 0 \quad \Rightarrow \frac{\partial \ln P\rho^{-\gamma}}{\partial z} < 0$$

$\Rightarrow$ <u>Specific entropy decreases upward</u>, just as we found in the formal analysis.

$$\Rightarrow \begin{cases} P_1 = \frac{1}{i\omega}\left(P_0(\gamma) \vec{k}\,\vec{v}_1 \cdot i + \vec{v}_1 \nabla P_0 \right) & (3) \\ \vec{v}_1 = \frac{1}{i\omega}\left(\frac{i\vec{k}}{\rho_0} P_1 - \frac{\nabla P_0}{\rho_0}\frac{\rho_1}{\rho_0} \right) & (1) \\ \rho_1 = \frac{1}{i\omega}\left(i\vec{k}\cdot\vec{v}\,\rho_0 + \vec{v}_1 \nabla \rho_0 \right) & (2) \end{cases}$$

$\Rightarrow$ After substituting for P_1 in $\vec{v}_1$ and for ρ_1 in $\vec{v}_1$, and multiplying by $i\omega$, we find:

$$\omega^2 \vec{v}_1 = -\gamma \frac{P_0}{\rho_0} \vec{k}\cdot\vec{v}_1 \vec{k} + i\vec{v}_1 \cdot \frac{\nabla P_0}{\rho_0}\vec{k} - \frac{\nabla P_0}{\rho_0^2}\left(i\vec{k}\vec{v}_1 \rho_0 + \vec{v}_1 \nabla \rho_0 \right)$$

Simplest case: one component of $\vec{k} \parallel$ to $\nabla P_0, \nabla \rho_0$ (call it $\hat{z}$)

one component $\perp$ (call it $\hat{x}$)

$$\Rightarrow -\omega^2 \left(v_x \hat{x} + v_z \hat{z} \right) = -\gamma \frac{P_0}{\rho_0}\left(k_x v_x + k_z v_z \right)\left(k_x \hat{x} + k_z \hat{z} \right)$$
$$+ i \frac{v_z}{\rho_0}\frac{\partial P_0}{\partial z}\left(k_x \hat{x} + k_z \hat{z} \right)$$
$$- \frac{1}{\rho_0^2}\frac{\partial P_0}{\partial z}\hat{z}\left(i(k_z v_z + k_x v_x)\rho_0 + v_z \frac{\partial \rho_0}{\partial z} \right)$$

Equating like terms:

$$-\omega^2 v_x + \gamma \frac{P_0}{\rho_0}\left(k_x v_x + k_z v_z \right) k_x - i \frac{v_z}{\rho_0}\frac{\partial P_0}{\partial z} k_x = 0$$

$$-\omega^2 v_z + \gamma \frac{P_0}{\rho_0}\left(k_x v_x + k_z v_z \right) k_z - i \frac{v_z}{\rho_0}\frac{\partial P_0}{\partial z} k_z + \frac{i}{\rho_0}\frac{\partial P_0}{\partial z}\left(k_z v_z + k_x v_x \right)$$
$$+ \frac{1}{\rho_0^2}\frac{\partial \rho_0}{\partial z} v_z \frac{\partial \rho_0}{\partial z} = 0$$

After some manipulation, we find that:

$$\frac{v_x}{v_z} = \frac{\frac{\gamma P_0}{\rho_0} k_z k_x - \frac{i}{\rho_0}\frac{\partial P_0}{\partial z} k_x}{-\omega^2 + \gamma \frac{P_0}{\rho_0} k_x^2} = \frac{-\omega^2 + \frac{\gamma P_0}{\rho_0} k_z^2 + \frac{1}{\rho_0^2}\frac{\partial P_0}{\partial z}\frac{\partial \rho_0}{\partial z}}{\frac{\gamma P_0}{\rho_0} k_x k_z + \frac{i}{\rho_0}\frac{\partial P_0}{\partial z} k_x}$$

Some more manipulation yields a quadratic equation in ω

$$(\omega^2)^2 - \omega^2 c_0^2 \left(k^2 + \frac{1}{\gamma}\frac{\partial \ln P_0}{\partial z}\frac{\partial \ln \rho_0}{\partial z}\right) + (c_0^2)^2 \left(\frac{1}{\gamma}\frac{\partial \ln P_0}{\partial z}\frac{\partial \ln \rho_0}{\partial z} - \frac{1}{\gamma^2}\left(\frac{\partial \ln P_0}{\partial z}\right)^2\right)$$

$k_x = 0$

That has the solutions

$$\omega^2 = \frac{1}{2}\left(c_0^2\left(k^2 + \frac{1}{\gamma}\frac{\partial \ln P_0}{\partial z}\frac{\partial \ln \rho_0}{\partial z}\right) \pm \left[c_0^4\left(k^2 + \frac{1}{\gamma}\frac{\partial \ln P_0}{\partial z}\frac{\partial \ln \rho_0}{\partial z}\right)^2 - 4 c_0^4 \cdot k_x\left(\frac{1}{\gamma}\frac{\partial \ln P_0}{\partial z}\frac{\partial \ln \rho_0}{\partial z} - \frac{1}{\gamma^2}\left(\frac{\partial \ln P_0}{\partial z}\right)^2\right)\right]^{1/2}\right)$$

Instability requires $\omega^2 < 0$, so

ω^2 has the form $A \pm \sqrt{A^2 + B}$
$\Rightarrow B > 0$

$$\Rightarrow k_x^2\left[\frac{1}{\gamma^2}\left(\frac{\partial \ln P_0}{\partial z}\right)^2 - \frac{1}{\gamma}\frac{\partial \ln P_0}{\partial z}\frac{\partial \ln \rho_0}{\partial z}\right] > 0$$

$$\Rightarrow k_x^2 \neq 0 \quad \text{and} \quad \frac{1}{\gamma^2}\frac{\partial \ln P_0}{\partial z}\left(\frac{\partial \ln P_0}{\partial z} - \gamma\frac{\partial \ln \rho_0}{\partial z}\right) > 0$$

$$\Rightarrow \frac{1}{\gamma^2}\frac{\partial \ln P_0}{\partial z}\left(\frac{\partial \ln P_0 \rho_0^{-\gamma}}{\partial z}\right) > 0$$

Since $\frac{\partial \ln P_0}{\partial z} < 0$ for atm. in grav. field $\left(\frac{\partial P_0}{\partial z} = -\rho g(z)\right)$

$$\Rightarrow \text{need } \boxed{\frac{\partial \ln(P_0 \rho_0^{-\gamma})}{\partial z} < 0}$$

$\Rightarrow$ Entropy must decrease with increasing height. $(S \propto \ln(P\rho^{-\gamma}))$

This is <u>Schwarzschild's criterion for convective instability</u>.

Key physical concept: entropy is buoyant.

If the entropy gradient is opposite to the gradient of the grav. field then the medium is <u>stable</u>; if the entropy gradient is the same as gravity, then is <u>unstable</u>.

Particularly important example: interface where density increases upward, with $P \sim$ constant

$$\Rightarrow \frac{\partial \ln(P_0 \rho_0^{-\gamma})}{\partial z} = -\gamma \underbrace{\frac{\partial \ln \rho_0}{\partial z}}_{>0} < 0 \quad \Rightarrow \underline{\text{unstable}}$$

Rayleigh-Taylor "fingers" of heavy material penetrating light material

$\Rightarrow$ Phenomenon is at the heart of convection in stars: high opacity traps radiation, creating an adverse entropy gradient leading to convective instability.

The same general effect is responsible for the fact that expanding SN shell breaks up in pieces (with acceleration playing the role of gravity).

(later we'll see an analog of this with $\vec{B}$ playing the role of entropy)

Note that requiring $k_x \neq 0$ is crucial $\Rightarrow$ needs some flow $\perp$ to gravity in order for overturning / mixing to occur

If we assume $k_x = 0 \Rightarrow$

$$\omega^2 = c_0^2\left(k_z^2 + \frac{1}{\gamma}\frac{\partial \ln P_0}{\partial z}\frac{\partial \ln \rho_0}{\partial z}\right) \times \begin{cases} 0 \\ 1 \end{cases}$$

$\Rightarrow$ sound waves modified by pressure and density gradients ("1")
or zero-buoyancy horizontal displacement ("0")
(with $v_{1z} = 0$, $v_{1x} = \text{const}$, and $\frac{1}{P}\frac{\partial P_0}{\partial z} = i k_z \frac{P_1}{\rho_1}\frac{\rho_0}{P_0}$)

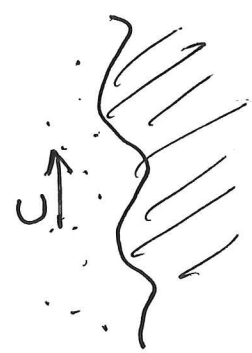
KELVIN-HELMHOLTZ instability

Associated with the presence of a discontinuity in a velocity field. E.g., interface between a rapid stream and slowly moving material.

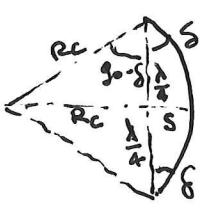
$\lambda_{ripple} = \frac{2\pi}{k} \Rightarrow$ growth rate of perturbation is $\gamma = \frac{k U}{1+r} r^{1/2}$

where $r = \rho_A/\rho_B$

$\Rightarrow$ maximally fast growth if $\rho_A = \rho_B$



Reason for growth: centrifugal effect. If initially in pressure balance $\Rightarrow$



bending of interface causes force $\sim \frac{U^2}{R_c}$

$\frac{\lambda/4}{R_c} = \cos(90-\delta) = \sin\delta \sim \delta$

with $R_c \sim \frac{\lambda/4}{\delta}$ radius of curvature

$s = s_0 e^{(i\omega t - kx)}$

$\Rightarrow \frac{\text{force}}{\text{mass}} \sim \frac{U^2\delta}{\lambda/4}$, displacement $s \sim \delta\lambda/4 \Rightarrow \ddot{s} \sim U^2\delta/(\lambda/4) \Rightarrow$
$\Rightarrow -\omega^2\delta\lambda \sim U^2\delta/\lambda \Rightarrow \omega^2 \sim -(U/\lambda)^2 \Rightarrow \gamma \sim U/\lambda \sim U k$
$(\lambda = \frac{2\pi}{k})$

Tends to increase δ

4 Rotational Infall

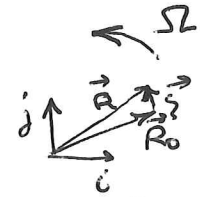
What about rotational motion? Is it stable?

Ignore p gradients and buoyancy effects.

Consider only eqs. of motion of fluid elements:

$\vec{\xi} \triangleq$ displacement of fluid element from initial position

@ $\vec{R_0}$. $\vec{\Omega}$ ang. velocity of rot. frame.



$$\vec{v} = \frac{d\vec{\xi}}{dt} + \vec{\Omega} \times (\vec{\xi} + \vec{R_0}) = \frac{D\vec{\xi}}{Dt}$$

$$\vec{a} = \frac{d^2\vec{\xi}}{dt^2} + \vec{\Omega} \times \frac{d\vec{\xi}}{dt} + \vec{\Omega} \times \left(\frac{d\vec{\xi}}{dt} + \Omega \times (\vec{\xi} + \vec{R_0}) \right) = \frac{D^2\vec{\xi}}{Dt}$$

$$\Rightarrow \vec{a} = \ddot{\vec{\xi}} + 2\vec{\Omega} \times \dot{\vec{\xi}} + \vec{\Omega} \times (\vec{\Omega} \times (\vec{\xi} + \vec{R_0}))$$

Assume $\vec{\Omega} \cdot \vec{\xi} = 0$, $\vec{\Omega} \cdot \vec{R_0} = 0$ Lagrangian approach

$$\Rightarrow \vec{a} = \ddot{\vec{\xi}} + 2\vec{\Omega} \times \dot{\vec{\xi}} - \Omega^2 \vec{\xi} - \Omega^2 \vec{R_0}$$

Force is $-\nabla\Phi \Big|_{\vec{R} = \vec{R_0} + \vec{\xi}} = -\nabla\Phi(R)$

$$|\vec{R}|^2 \equiv |\vec{R_0} + \vec{\xi}|^2 \simeq R_0^2 + 2\vec{R_0} \cdot \vec{\xi} \Rightarrow |R| \simeq (R_0^2 + 2\vec{R_0} \cdot \vec{\xi})^{1/2} \approx R_0 \left(1 + \frac{\vec{R_0} \cdot \vec{\xi}}{R_0^2}\right) = R_0 + \frac{\vec{R_0} \cdot \vec{\xi}}{R_0}$$

$$\vec{F} = \left(-\frac{\partial\Phi}{\partial R}\Big|_{R_0} - \frac{\partial^2\Phi}{\partial R^2} \frac{\vec{R_0} \cdot \vec{\xi}}{R_0} \right) \hat{R} \quad \text{(To first order)}$$

$$\vec{R} = (R_0 + \xi_r)\hat{r} + \xi_\varphi \hat{\varphi} \Rightarrow |\vec{R}| = (R_0^2 + 2R_0\xi)^{1/2} = R_0(1 + \xi_r/R_0) = R_0 + \xi_r$$

$$\hat{R} = \frac{\vec{R}}{|\vec{R}|} = \frac{\vec{R}}{R_0 + \xi_r}$$

$$\hat{R} \simeq \frac{R_0+\xi_r}{R_0+\xi_r}\cdot\hat{r} + \frac{\xi_\varphi}{R_0}\hat{\varphi} = \hat{r} + \frac{\xi_\varphi}{R_0}\hat{\varphi}$$

Using sinusoidal solutions: $\frac{d}{dt} \to i\omega$

$$\Rightarrow \vec{a} = -\nabla\Phi \quad \Rightarrow \quad -\omega^2\vec{\xi} + 2i\omega\,\vec{\Omega}\times\vec{\xi} - \Omega^2\vec{\xi} - \Omega^2 R_0\hat{r} =$$

$$-\left(\frac{\partial\Phi}{\partial R}\Big|_{R_0} + \frac{\partial^2\Phi}{\partial R^2}\Big|_{R_0}\xi_r\right)\left(\hat{r} + \frac{\xi_\varphi}{R_0}\hat{\varphi}\right)$$

$$\Rightarrow \begin{cases} -\omega^2\xi_r - 2i\omega\Omega\xi_\varphi - \Omega^2\xi_r - \Omega^2 R_0 = -\left(\frac{\partial\Phi}{\partial R}\Big|_{R_0} + \frac{\partial^2\Phi}{\partial R^2}\Big|_{R_0}\xi_r\right) \\ -\omega^2\xi_\varphi + 2i\omega\Omega\xi_r - \Omega^2\xi_\varphi = -\frac{\partial\Phi}{\partial R}\Big|_{R_0}\cdot\frac{\xi_\varphi}{R_0} \end{cases}$$

$$\Rightarrow \left.\begin{aligned} &\xi_\varphi\left(-\omega^2 - \Omega^2 + \frac{1}{R_0}\frac{\partial\Phi}{\partial R}\Big|_{R_0}\right) = -2i\omega\xi_r\Omega \\ &-\Omega^2 R_0 = -\frac{\partial\Phi}{\partial R}\Big|_{R_0} \Leftarrow \left\{\begin{array}{l}\text{centrifugal =}\\ \text{attraction}\\ \text{for steady solution}\end{array}\right. \\ &-\omega^2\xi_r - 2i\omega\Omega\xi_\varphi - \Omega^2\xi_r = -\frac{\partial^2\Phi}{\partial R^2}\Big|_{R_0}\xi_r \end{aligned}\right\}$$

$$\Rightarrow \xi_\varphi = +\frac{2i\omega\Omega}{\omega^2}\xi_r = \frac{2i\Omega}{\omega}\xi_r$$

$$\xi_r\left(-\omega^2 - \Omega^2 + \frac{\partial^2\Phi}{\partial R^2}\Big|_{R_0}\right) = 2i\omega\Omega\xi_\varphi = 2i\omega\Omega\left(\frac{2i\Omega}{\omega}\right)\xi_r$$

$$\Rightarrow -\omega^2 - \Omega^2 + \frac{\partial^2\Phi}{\partial R^2}\Big|_{R_0} = -(2\Omega)^2$$

$$\Rightarrow \omega^2 = 3\Omega^2 + \frac{\partial^2\Phi}{\partial R^2}\Big|_{R_0} = 3\Omega^2 + \frac{\partial}{\partial R}(\Omega^2 R) = 4\Omega^2 + R\frac{d\Omega^2}{dR}$$

$$\Rightarrow \omega^2 = \frac{1}{R^3}\frac{d}{dR}(\Omega^2 R^4) = \kappa^2,$$ The epicyclic frequency squared

$\kappa^2 = \frac{1}{R^3}\frac{d}{dR}(J^2)$ $\quad J = \Omega R^2$ = specific angular mom.

Instability occurs for $\omega^2 < 0 \Leftrightarrow$ if specific angular momentum <u>decreases outward</u>

Typical rotating systems:

Kepler has $\Omega \propto R^{-3/2} \Rightarrow J \propto R^{1/2}$ <u>increasing</u> outward

Galaxy with constant vel. rot. curve has $\Omega \propto R^{-1} \Rightarrow J \propto R$ <u>increasing</u>

So, for generic cases rotational motion is <u>stable</u>

The condition $\frac{d}{dR}(\Omega^2 R^4) > 0$ for stability is known as "Rayleigh's criterion"

<u>3D Grav. instability</u>

Standard Treatment introduced by Sir James Jeans.

Unperturbed state: uniform density and Temp. medium with no grav. forces

This is not self consistent: no equilibrium is possible in a finite medium against self gravity w/o pressure gradients.

Nevertheless, it does produce the correct answer for whether instability would occur.

Uniform, adiabatic medium: $P = K\rho^\gamma \Rightarrow P_0 + P_1 = K\rho_0^\gamma\left(1+\gamma\frac{\rho_1}{\rho_0}\right)$

$\Rightarrow P_1 = \frac{\gamma P_0}{\rho_0}\rho_1 \triangleq c_0^2\rho_1$, $c_0^2 = \frac{\gamma P_0}{\rho_0}$ square of sound speed

Mass continuity:

① $\frac{\partial \rho_1}{\partial t} + \nabla \cdot (\rho_0 \vec{v}_1) = 0$

Momentum conservation:

② $\frac{\partial \vec{v}_1}{\partial t} = -\frac{\nabla P_1}{\rho_0} - \nabla \Phi_1$

Gravity:

③ $\nabla^2 \Phi_1 = 4\pi G \rho_1$ (Jeans cheat: doesn't have $\nabla^2 \Phi_0 = 4\pi G \rho_0$)

Eq. of state:

④ $P_1 = c_0^2 \rho_1$

Ansatz is $\propto e^{i(\vec{k}\cdot\vec{x} - \omega t)}$

$$\Rightarrow \left.\begin{aligned} &-i\omega\rho_1 + i\vec{k}\cdot\vec{v}_1 \rho_0 = 0 && (1)\\ &-i\omega\vec{v}_1 = -i\frac{\vec{k}P_1}{\rho_0} - i\vec{k}\Phi_1 && (2)\\ &-k^2\Phi_1 = 4\pi G\rho_1 && (3)\\ &P_1 = c_0^2\rho_1 && (4)\end{aligned}\right\}$$

$\Rightarrow$ (1) $\Rightarrow \frac{\rho_1}{\rho_0} = \frac{\vec{k}\cdot\vec{v}_1}{\omega}$; (4) $\Rightarrow \frac{P_1}{\rho_0} = \frac{c_0^2 \vec{k}\cdot\vec{v}_1}{\omega}$ ← as for sound waves

(3) $\Rightarrow \Phi_1 = -\frac{4\pi G \rho_0}{k^2} \frac{\vec{k}\cdot\vec{v}_1}{\omega}$

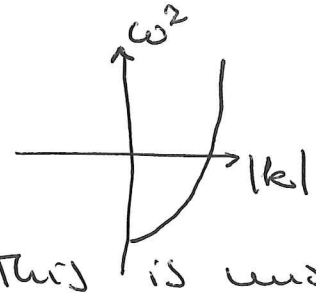
$\Rightarrow$ Plugging them into (2) $\Rightarrow -i\omega\vec{v}_1 = -i\vec{k}\frac{c_0^2}{\omega}\vec{k}\cdot\vec{v}_1 - i\vec{k}\cdot\left(\frac{-4\pi G\rho_0}{k^2}\cdot\frac{\vec{k}\cdot\vec{v}_1}{\omega}\right)$

Multiply by $i\omega \Rightarrow$

$$\omega^2 \vec{v}_1 = \vec{k} c_0^2 \vec{k}\cdot\vec{v}_1 - \frac{4\pi G \rho_0}{k^2} \vec{k}\, \vec{k}\cdot\vec{v}_1$$

Dot-product with $\vec{k} \Rightarrow$

$$\omega^2 = k^2 c_0^2 - 4\pi G \rho_0$$



$\Rightarrow$ For $|k| < \left(\frac{4\pi G \rho_0}{c_0^2}\right)^{1/2} \triangleq k_J$ This is <u>unstable</u>

Jeans wavelength $\lambda_J \triangleq \frac{2\pi}{k_J} = 2\pi\left(\frac{c_0^2}{4\pi G \rho_0}\right)^{1/2} = c_0\left(\frac{\pi}{G\rho_0}\right)^{1/2}$

If $\lambda > \lambda_J \Rightarrow$ Those scales are <u>unstable</u>

$$\omega^2 = c_0^2(k^2 - k_J^2)$$

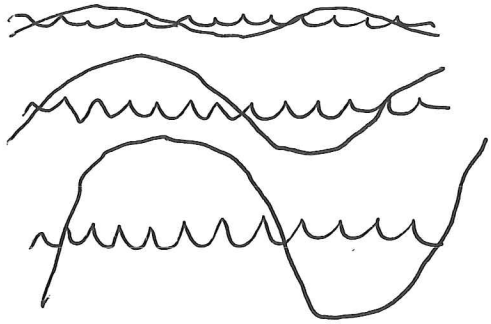
Growth rate $\gamma \equiv |\omega| = c_0 (k_J^2 - k^2)^{1/2}$ for $k < k_J$

$\cancel{\Rightarrow \gamma = c_0 k_J \left(1 - \left(\frac{\lambda_J}{\lambda}\right)^2\right)^{1/2}}$

$\Rightarrow$ Growth rate $\uparrow$ for $\lambda \uparrow$

$\Rightarrow$ Small scale perturbations inside large scale pertur. end up getting swallowed-up in the large scale collapse (unless their amplitude is initially larger $\rightarrow$ not usually the case)

big one grows faster than small one



Most unstable wavelength is that of the whole medium, if $L > \lambda_J$, which had unphysical unperturbed state. The actual growth rate won't be accurate, but the question of instability is still ok.

Mass inside λ_J^3 box is $M_J \triangleq \rho_0 c_0^3 \left(\frac{\pi}{G\rho_0}\right)^{3/2}$

Physical interpretation: Consider its grav. potential

$$\Phi \sim -\frac{GM}{d} \sim -\frac{G d^3 \rho_0}{d} \sim G d^2 \rho_0$$

Specific thermal energy of fluid element

$$\frac{3}{2}\frac{kT}{\mu} \sim c_s^2$$

If $c_s^2 > -\Phi_g \Rightarrow$ expect unbound

$c_s^2 < -\Phi_g \Rightarrow$ expect bound

$$\Rightarrow c_s^2 = G d^2 \rho_0 \qquad c_s\left(\frac{1}{G\rho}\right)^{1/2} = \left(\frac{c_s^2}{G\rho}\right)^{1/2} = d$$

$$\Rightarrow \lambda_J \sim c_s\left(\frac{\pi}{G\rho}\right)^{1/2}$$

Density waves and the grav. instab. of rotating gas disks

We looked @: rotating media w/o pressure
self-gravitating media w/ pressure but w/o rot.

Stability of galactic or protostellar disks?

We'll consider only axisymmetric $\left(\frac{\partial}{\partial\varphi}=0\right)$, since non-axisymmetric disturbances don't have modes $\propto e^{i(\vec{k}\cdot\vec{x}-\omega t)}$

Eqs. of motion: Treat as locally uniform pressure and density with adiabatic/isothermal eos.

$d \ll R \Rightarrow$ neglect curvature terms, using $\frac{1}{R} \ll k$

z-integrate $\Rightarrow$ $\rho \to \Sigma = \rho H$

$P \to \Pi = PH$

Perturbations are $\propto e^{i(\vec{k}(R-R_0)\hat{R}-\omega t)}$
radial

$\nabla = \left(\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial\varphi}, \frac{\partial}{\partial z}\right)$

$\nabla_0 \vec{v} = \frac{\partial(r\cdot v_R)}{\partial r} + \frac{1}{r}\frac{\partial v_\varphi}{\partial\varphi}$

$$\frac{\partial\Sigma_1}{\partial t} + \nabla_0(\Sigma_0\vec{v}_1) = 0 \Rightarrow \frac{\partial\Sigma_1}{\partial t} = -\Sigma_0 \overset{\shortparallel}{\nabla\cdot\vec{v}_1} \quad \left(\underset{\text{uniform}}{\nabla\Sigma_0 = 0}\right)$$

$$\Rightarrow \boxed{-i\omega\Sigma_1 = -\Sigma_0 ik v_{1R}} \quad (\text{drop } \tfrac{v_R}{R} \text{ term})$$

$$\frac{\partial\vec{v}_1}{\partial t} + \vec{v}_1\cdot\nabla\vec{v}_0 + \vec{v}_0\nabla\vec{v}_1 = -\frac{\nabla\Pi_1}{\Sigma_0} - \nabla\Phi_1$$

$$\frac{\partial\vec{v}_1}{\partial t} = \underbrace{-\left(\frac{v_{1\varphi}}{R}\frac{\partial}{\partial\varphi} + v_{1R}\frac{\partial}{\partial R}\right)(\Omega R\hat{\varphi})}_{\vec{v}_1\cdot\nabla\vec{v}_0}$$

$$\underbrace{-\left(\frac{\Omega R}{R}\frac{\partial}{\partial\varphi}\right)(v_{1\varphi}\hat{\varphi} + v_{1R}\hat{R})}_{\vec{v}_0\cdot\nabla\vec{v}_1} - \frac{\nabla\Pi_1}{\Sigma_0} - \nabla\Phi_1$$

$$= 2\Omega v_{1\varphi}\hat{R} - v_{1R}\frac{\partial}{\partial R}\left[(\Omega R)\right]\hat{\varphi} - \Omega v_{1R}\hat{\varphi} - \frac{\nabla \Pi_1}{\Sigma_0} - \nabla\Phi_1$$

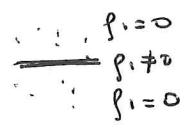
$$\boxed{\frac{\partial \hat{R}}{\partial \varphi} = \hat{\varphi}, \quad \frac{\partial \hat{\varphi}}{\partial \varphi} = -\hat{R}}$$

(Take $\nabla \Pi_0 = 0$)

$$\left.\begin{aligned} -i\omega v_{1R} &= 2\Omega v_{1\varphi} - \frac{ik\Pi_1}{\Sigma_0} - ik\Phi_1 \\ -i\omega v_{1\varphi} &= -v_{1R}\frac{\partial}{\partial R}(\Omega R) - \Omega v_{1R} \end{aligned}\right\}$$

$\Pi_1 = c_0^2 \Sigma_1 \leftarrow$ adiabatic eos

Poisson: $\nabla^2\Phi_1 = 4\pi G \rho_1$ $\Rightarrow$ assume thin disk



Above and below disk $\rho_1 = 0 \Rightarrow \left(\frac{\partial^2}{\partial z^2} + \nabla^2_{plane}\right)\Phi_1 = 0$

$\Phi_1 = e^{ikx} f(z)$ by assumption $(x \equiv R - R_0)$

$$-k^2 f + \frac{\partial^2 f}{\partial z^2} = 0 \Rightarrow f \propto e^{\pm |k| z}$$

want to have $f \to 0$ for $z \to \infty \Rightarrow f \propto e^{-|kz|}$

$$\Rightarrow \Phi_1(x,z) = \Phi_k e^{ikx} e^{-|kz|}$$

Take $\nabla^2\Phi_1 = 4\pi G\rho_1$ and integrate from below to above the plane

$$\int dz \left(\frac{\partial^2 \Phi_1}{\partial z^2}\right) = \int 4\pi G \rho_1 \, dz$$

$$\Rightarrow 2\left.\frac{\partial \Phi_1}{\partial z}\right|_{z=0^+} = 4\pi G \Sigma_1$$

$\Phi_1 \propto e^{-|k|z}$ for $z > 0$ $\Rightarrow -2|k|\Phi_1 = 4\pi G \Sigma_1$

$$\Rightarrow \boxed{\Phi_1 = -\frac{2\pi G \Sigma_1}{|k|}}$$

$$\Sigma_1 = \frac{\Sigma_0 k v_{1R}}{\omega}$$

$$\Rightarrow v_{1\varphi} = \frac{-i}{\omega}\left[\frac{1}{R}\frac{d}{dR}(\Omega R^2)\right] v_{1R}$$

$$\Phi_1 = \frac{-2\pi G\,\Sigma_1}{|k|} = \frac{-2\pi G}{|k|}\frac{\Sigma_0 k}{\omega} v_{1R}$$

$$-i\omega v_{1R} = 2\Omega\left(\frac{-i}{\omega R}\frac{d(\Omega R^2)}{dR}\right) v_{1R} - \frac{ik}{\Sigma_0} c_0^2 \frac{\Sigma_0 k}{\omega} v_{1R} - ik\left(\frac{-2\pi G \Sigma_0 k}{\omega |k|}\right) \cdot v_{1R}$$

$\Rightarrow$ multiplying by $i\omega$:

$$\omega^2 = \frac{2\Omega}{R}\frac{d}{dR}(\Omega R^2) + c_0^2 k^2 - \frac{k^2}{|k|} 2\pi G\,\Sigma_0$$

$$= \frac{1}{R^3}\frac{d}{dR}\left((\Omega R^2)^2\right) + c_0^2 k^2 - |k|\, 2\pi G\,\Sigma_0$$

$$\Rightarrow \boxed{\omega^2 = c_0^2 k^2 + \kappa^2 - 2\pi G\,\Sigma_0 |k|}$$ density wave dispersion relation

For $\begin{cases} P=0 \\ \text{self-gravity}=0 \end{cases}$ $(c_0^2=0)$ we get back $\omega^2 = \kappa^2$ in the "fluid particle" treatment of rotation

For no rotation and no self gravity $\Rightarrow$ $\omega^2 = c_0^2 k^2$ sound waves

More generally:
- restoring force from rotation (κ^2)
- " " " pressure $(c_0^2 k^2)$
- destabilizing " " gravity $(2\pi G \Sigma |k|)$

(Just like Jean's, but with $2\pi G \Sigma |k|$ instead of $4\pi G \rho_0$)

for $k=0$

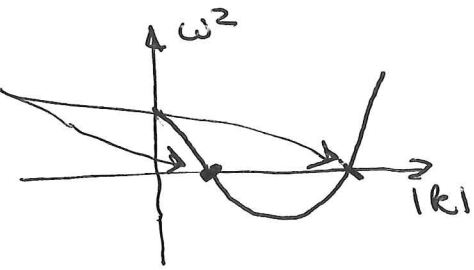
with $\kappa = 0$, $\omega^2 = (c_0^2|k| - 2\pi G\Sigma)|k|$

$\Rightarrow$ unstable for $|k| < \frac{2\pi G\Sigma}{c_0^2} \equiv k_J$ equiv. to Jeans wave # for thin disk

Stability with $\kappa \neq 0$

Galaxy is unstable to run-away fragmentation if min. of $\omega^2 < 0$

$\Rightarrow$ for $|k| = \frac{\pi G\Sigma}{c_0^2} \pm \left(\left(\frac{\pi G\Sigma}{c_0^2}\right)^2 - 4\frac{\kappa^2}{c_0^2}\right)^{1/2}$



$$\frac{d\omega^2}{d|k|} = 2c_0^2|k| - 2\pi G\Sigma_0$$

$$\Rightarrow (|k|)_{min} = \frac{\pi G\Sigma_0}{c_0^2} = \frac{k_J}{2}$$

At $|k| = k_J/2$, $\omega^2 = -\left(\frac{\pi G\Sigma}{c_0}\right)^2 + \kappa^2$

So $\omega^2 < 0$ if $\kappa^2 < \left(\frac{\pi G\Sigma}{c_0}\right)^2$, i.e. $\frac{\kappa c_0}{\pi G\Sigma} < 1$

$\frac{\kappa c_0}{\pi G\Sigma}$ is known as "Toomre's Q"

$\Rightarrow$ Gas in a galaxy disk is unstable/stable to "ring like" fragmentation if $Q < 1$ / $Q > 1$

For non-axisymmetric perturb. the ~~collapse~~ analysis is more complicated. A process known as "swing amplification" leads to growth of perturbations. Woo-Tae Kim's thesis shows that $Q \lesssim 1.5$ for instability

This value allows for 3D, $\vec{B}$, and turbulence in initial conditions. (20)

No $\vec{B}$, critical Q is lower (maybe even < 1).

This is smaller than axisymmetric result because finite thickness dilutes gravity: Q_{crit} for thick disk is $< Q_{Toomre}$.