

# MHD Waves

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Consider a uniform, stationary medium.

Ideal MHD equations:

$$\frac{\partial \vec{v}}{\partial t} + \nabla \cdot (\vec{v} \vec{v}) = 0$$

$$\frac{\partial \vec{v}}{\partial t} + \underbrace{\vec{v} \cdot \nabla \vec{v}}_{\text{in}} = -\frac{\nabla P}{\rho} - \nabla \Phi + \frac{1}{4\pi\rho} \underbrace{(\nabla \times \vec{B}) \times \vec{B}}_{\text{in}}$$

$$(\nabla \times \vec{v}) \times \vec{v} + \nabla \frac{1}{2} v^2 - \vec{B} \nabla \vec{B} - \nabla \frac{1}{2} B^2$$

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{v} \times \vec{B}), \quad \nabla \cdot \vec{B} = 0$$

With an energy eq., and the eq. of state

let's assume isothermal or adiabatic EOS

$$\Rightarrow P = K \rho T \quad K = P_0 \rho_0^{-\gamma}$$

Let's linearize the eqs. and assume sinusoidal perturb.

$$-\omega \rho_1 + i \vec{k} \cdot \vec{v}_1 \rho_0 = 0$$

$$-i\omega \vec{v}_1 = -C_s^2 i \frac{\vec{k} \rho_1}{\rho_0} - \frac{1}{4\pi\rho_0} (i \vec{k} \times \vec{B}_1) \times \vec{B}_0$$

$$-i\omega \vec{B}_1 = i \vec{k} \times (\vec{v}_1 \times \vec{B}_0) \quad C_s^2 \triangleq \gamma \frac{P_0}{\rho_0}$$

Take  $\vec{B}_0 = B_0 \hat{z}$ , define  $\vec{b}_1 = \vec{B}_1 / B_0$ ,  $\alpha_1 \equiv \rho_1 / \rho_0$ ,  $v_A^2 \equiv \frac{B_0^2}{4\pi\rho_0}$

$$\Rightarrow \alpha_1 = \frac{\vec{k} \cdot \vec{v}_1}{\omega} \quad (1)$$

$$\vec{v}_1 = \frac{C_s^2}{\omega} \vec{k} \alpha_1 + \frac{v_A^2}{\omega} (\vec{k} \times \vec{b}_1) \times \hat{z} \quad (2)$$

$$\vec{b}_1 = \frac{\vec{k} \times (\vec{v}_1 \times \hat{z})}{\omega} = \frac{\vec{v}_1 \vec{k} \cdot \hat{z} - \hat{z} \vec{k} \cdot \vec{v}_1}{\omega} \quad (3)$$

(2)

Substitute (1) and (3) into (2)

$$\vec{\omega}_1 = \frac{c_s^2}{\omega^2} \vec{k} \cdot \vec{k} \cdot \vec{\omega}_1 + \frac{v_A^2}{\omega^2} (\vec{k} \times (\vec{k} \times (\vec{\omega}_1 \times \hat{z}))) \times \hat{z}$$

$$\Rightarrow \omega^2 \vec{\omega}_1 = c_s^2 \vec{k} \cdot \vec{k} \cdot \vec{\omega}_1 - v_A^2 \hat{z} \times (\vec{k} \times (\vec{\omega}_1 \cdot k_z - \hat{z} \vec{k} \cdot \vec{\omega}_1))$$

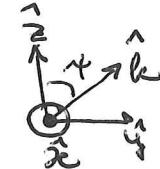
$$= c_s^2 \vec{k} \cdot \vec{k} \cdot \vec{\omega}_1 - v_A^2 (\vec{k} \cdot v_{iz} k_z - \vec{\omega}_1 \cdot k_z^2 - \vec{k} \cdot \vec{k} \cdot \vec{\omega}_1 + k_z \vec{k} \cdot \vec{\omega}_1 \hat{z})$$

$$(\omega^2 - v_A^2 k_z^2) \vec{\omega}_1 = (c_s^2 \vec{k} \cdot \vec{\omega}_1 + v_A^2 \vec{k} \cdot \vec{\omega}_1 - v_A^2 v_{iz} k_z \hat{z}) \vec{k} - v_A^2 k_z \vec{k} \cdot \vec{\omega}_1 \hat{z}$$

$$= ((c_s^2 + v_A^2) \vec{k} - v_A^2 k_z \hat{z}) \vec{k} \cdot \vec{\omega}_1 - v_A^2 v_{iz} k_z \vec{k} \quad (A)$$

Define coordinates with  $\hat{x} \perp \hat{z}, \hat{k}$

$$\Rightarrow \hat{k} \times \hat{z} = \hat{x}$$



$\Rightarrow$  Solutions:

1) If  $v_{ix} \neq 0 \Rightarrow$  must have  $\omega^2 = v_A^2 k_z^2$ ,  $\omega = \pm \frac{\vec{k} \cdot \vec{B}_0 \hat{z}}{\sqrt{4\pi\rho}}$   
we'll see that

This has  $\vec{k} \cdot \vec{\omega}_1 = 0$  and  $v_{iz} k_z = 0 \Rightarrow v_{iy} = 0$  for general  $\vec{k}$   
~~unless~~

$$((c_s^2 + v_A^2) \vec{k} - v_A^2 k_z \hat{z}) \cdot \vec{k} \cdot \vec{\omega}_1 - v_A^2 v_{iz} k_z \vec{k} = 0$$

Dot with  $\vec{\omega}_1 \Rightarrow$

$$(c_s^2 + v_A^2)(\vec{k} \cdot \vec{\omega}_1)^2 - 2v_A^2 k_z v_{iz} (\vec{k} \cdot \vec{\omega}_1) = 0$$

$$\Rightarrow \vec{k} \cdot \vec{\omega}_1 = 0 \text{ or } \vec{k} \cdot \vec{\omega}_1 = \frac{2v_A^2 k_z v_{iz}}{c_s^2 + v_A^2}$$

If  $\vec{k} \cdot \vec{\omega}_1 = 0 \Rightarrow k_z v_{iz} = 0$

$$\text{If } \vec{k} \cdot \vec{\omega}_1 \neq 0 \Rightarrow ((c_s^2 + v_A^2) \vec{k} - v_A^2 k_z \hat{z}) \cdot \frac{2v_A^2 k_z v_{iz}}{c_s^2 + v_A^2} - v_A^2 v_{iz} k_z \vec{k} = 0$$

$$\Rightarrow \vec{k} = \frac{2v_A^2 k_z \hat{z}}{c_s^2 + v_A^2} \Rightarrow \hat{k} = \hat{z} \text{ and } 2v_A^2 = c_s^2 + v_A^2 \Rightarrow c_s^2 = v_A^2 \quad (3)$$

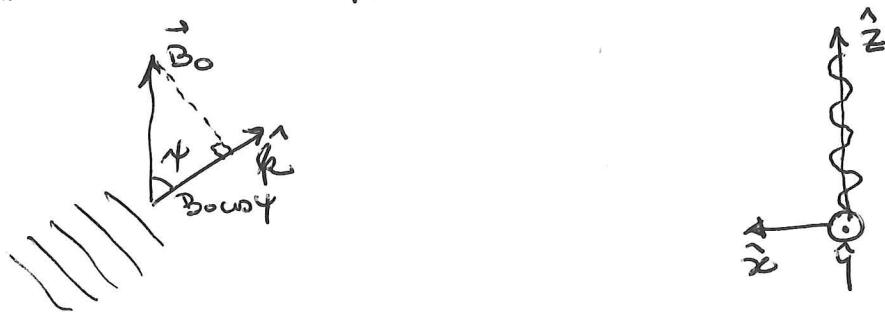
So if  $v_{ix} \neq 0$  either  $\vec{k} \cdot \vec{v}_i = 0$  and  $v_{iz} k_z = 0 \Rightarrow v_{iy}, v_{iz} = 0$   
 or  $\hat{k} = \hat{z}$  and  $v_A^2 = c_s^2$  (special case)

These modes are called Alfvén waves

From (1),  $\vec{k} \cdot \vec{v}_i = 0 \Rightarrow p_i = 0$

$$\text{From (3), } \vec{b}_i = \frac{\vec{v}_i k_z}{\omega} = \frac{\vec{v}_i k_z}{\pm k_z v_A} \Rightarrow \vec{b}_i = \pm \frac{\vec{v}_i}{v_A}$$

The perturbations ~~in~~  $\vec{v}_i$  and  $\vec{B}$  are purely transverse  
 These are Tension waves. Only the component of  $\vec{B}_0 \parallel \vec{k}$   
 contributes to tension if displacements have waveforms  
 as shown. This is why  $\omega = k \cdot v_A$  for Alfvén waves



$$2) v_{ix} = 0 \Rightarrow \text{Let } \vec{k} = k(\cos \theta \hat{z} + \sin \theta \hat{y})$$

$$(v_A^2 - k^2 c_s^2 \cos^2 \theta)(v_{iz} \hat{z} + v_{iy} \hat{y}) = [(c_s^2 + v_A^2)(\cos \theta \hat{z} + \sin \theta \hat{y}) - v_A^2 \hat{z} \cos \theta] k^2 (v_{iz} \cos \theta + v_{iy} \sin \theta) - v_A^2 v_{iz} \cos \theta k^2 (\cos \theta \hat{z} + \sin \theta \hat{y})$$

$$= \hat{z} [v_{iz}(c_s^2 - v_A^2) \cos^2 \varphi + v_{iy} \cos \varphi \sin \varphi c_s^2] k^2 \\ + \hat{y} [v_{iz}(c_s^2 \sin^2 \varphi) + v_{iy} \sin^2 \varphi (c_s^2 + v_A^2)] k^2$$

$$\Rightarrow \left. \begin{aligned} (\omega^2 - v_A^2 k^2 \cos^2 \varphi) v_{iz} - v_{iz}(c_s^2 - v_A^2) k^2 \cos^2 \varphi - v_{iy} \cos \varphi \sin \varphi c_s^2 k^2 &= 0 \\ (\omega^2 - v_A^2 k^2 \cos^2 \varphi) v_{iy} - v_{iz} k^2 c_s^2 \sin^2 \varphi - v_{iy} \sin^2 \varphi (c_s^2 + v_A^2) k^2 &= 0 \end{aligned} \right\}$$

$$\underbrace{\begin{bmatrix} -\cos \varphi \sin \varphi c_s^2 k^2 & \omega^2 - c_s^2 k^2 \cos^2 \varphi \\ \omega^2 - v_A^2 k^2 - c_s^2 k^2 \sin^2 \varphi & -c_s^2 k^2 \sin \varphi \cos \varphi \end{bmatrix}}_M \begin{bmatrix} v_{iy} \\ v_{iz} \end{bmatrix} = 0$$

$$\det(M) = 0 \Leftrightarrow (\omega^2)^2 - \omega^2(v_A^2 + c_s^2)k^2 + c_s^2 v_A^2 k^4 \cos^2 \varphi = 0$$

$$\Rightarrow \left( \frac{\omega}{k} \right)^2 = \frac{1}{2} \left\{ (v_A^2 + c_s^2) \pm \left[ (v_A^2 + c_s^2)^2 - 4c_s^2 v_A^2 \cos^2 \varphi \right]^{1/2} \right\}$$

- + is the fast MHD wave
- is the slow MHD wave

Let's look at special cases:  $\hat{k} \parallel \hat{B}_0 = \hat{z} \Rightarrow \sin \varphi = 0, \cos \varphi = 1$

$$\Rightarrow (\omega^2 - v_A^2 k^2 - (c_s^2 - v_A^2) k^2) v_{iz} = 0 \Rightarrow (\omega^2 - c_s^2 k^2) v_{iz} = 0 \\ (\omega^2 - v_A^2 k^2) v_{iy} = 0$$

$$\Rightarrow v_{iz} = 0 \text{ and } v_{iy} \neq 0 \text{ with } \omega^2 = v_A^2 k^2$$

or

$$v_{iz} \neq 0 \text{ and } v_{iy} = 0 \text{ with } \omega^2 = c_s^2 k^2$$

The case with  $v_{iz} \neq 0$  and  $v_{iy} \neq 0$  is Alfvén-like

$v_{iy} = 0$        $v_{iz} \neq 0$  is a sound wave

$$\left(\frac{\omega}{k}\right)^2 = \frac{1}{2} \left\{ (v_A^2 + c_s^2) \pm \sqrt{(v_A^2 - c_s^2)^2} \right\}$$

$$= \frac{1}{2} \left\{ \frac{2v_A^2}{2c_s^2} = \begin{cases} v_A^2 \\ c_s^2 \end{cases} \right. \Rightarrow \text{fast wave is either } v_A \text{ or } c_s, \\ \text{whichever is larger} \\ \text{slow wave is the other}$$

$$\hat{k} \perp \hat{B}_0 = \hat{z} \Rightarrow \sin \varphi = 1, \cos \varphi = 0$$

$$\omega^2 v_{iz} = 0$$

$$(\omega^2 - (c_s^2 + v_A^2) k^2) v_{iy} = 0 \Rightarrow v_{iz} = 0 \text{ and } v_{iy} \neq 0 \text{ with} \\ \omega^2 = (c_s^2 + v_A^2) k^2$$

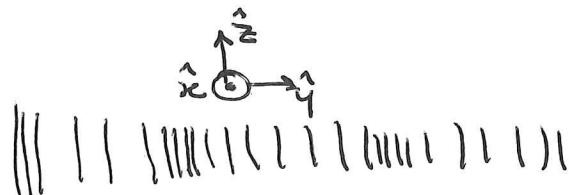
This is the fast magnetosonic wave

Recall:  $\alpha_1 = \frac{k v_{iy}}{\omega}$  from (1)

$$v_{iy} = \frac{c_s^2}{\omega} k \alpha_1 - \frac{v_A^2}{\omega} b_{iz} k \text{ from (2)}$$

$$b_{iz} = -\frac{v_i k}{\omega} \text{ from (3)}$$

$$v_{iz} = b_{ix} = 0$$



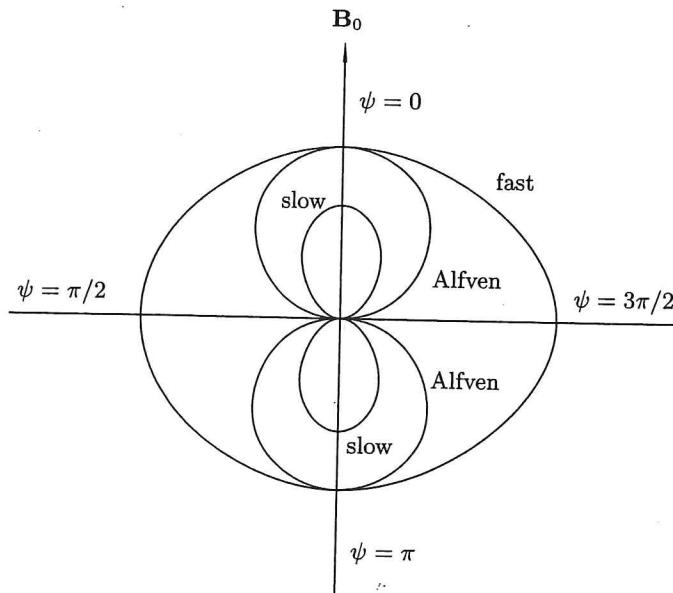


FIGURE 22.4

Wave-propagation diagram displayed in a polar plot for the three waves of classical magnetohydrodynamics: fast, slow, and Alfvén. The plot is drawn for the case when the Alfvén speed exceeds the sound speed.

$\gamma_{\text{mag}} = 2$  for the  
This identification  
compression implies  
see that  $\gamma_{\text{mag}}$  effects  
in three dimensions,  
mass that can be  
magnetic fields.

th  $\mathbf{u}_1 \propto \mathbf{k} \times \hat{\mathbf{n}}$ ; and  
phase (and group)  
ces dominate those

tation. Imagine an  
in a region where  
direction. At  $t = 0$ ,  
ie, the fronts of the  
oci indicated in the  
netic field ( $\psi = 0$  or  
eed of propagation,  
perturbed magnetic  
etosonic speed—the  
disturbance—while  
igate away from the

origin in these directions. At intermediate angles of wave propagation, the information carried by the fast mode always reaches the observer first; that carried by the slow mode, last.

### EIGENVECTORS

Given the satisfaction of the dispersion relation, we can recover the eigenvectors of the disturbance in the usual manner. We have already discussed in some detail the properties of a pure Alfvén wave. The following forms a summary for the fast and slow modes. Equation (22.6) shows  $\mathbf{b}_1$  to be orthogonal to  $\mathbf{k}$ :  $\mathbf{k} \cdot \mathbf{b}_1 = 0$ . Hence, for the fast and slow modes, we may take  $\mathbf{k} = k\mathbf{e}_x$ ,  $\mathbf{b}_1 = b_1\mathbf{e}_y$ ,  $\mathbf{u}_1 = u_{1x}\mathbf{e}_x + u_{1y}\mathbf{e}_y$ . The eigenvectors then work out to be

$$\alpha_1 : u_{1x} : u_{1y} : b_1 = \cos \psi \sin \psi : \frac{\omega}{k} \cos \psi \sin \psi : \frac{\omega}{k} \left[ \sin^2 \psi + \frac{a_s^2 - \omega^2/k^2}{v_A^2} \right] : -\frac{(a_s^2 - \omega^2/k^2)}{v_A^2} \cos \psi.$$