

Shocks

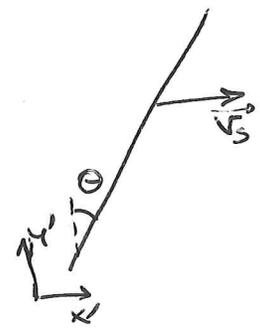
(1)

Shock front; nearly discontinuous (very steep) change in velocity, density, and pressure in a gas, that propagates through space.

Forms through nonlinear steepening of sound waves or external forces that accelerate material to supersonic speeds.

Despite ρ, \vec{v}, T being discontinuous @ a shock, conservation principles still hold and allow us to relate their values on one and the other side of the shock.

Assume shock front is a steady, straight front that propagates @ speed \vec{v}_s not necessarily \perp to shock



$\Rightarrow \vec{v}_s' = v_s \hat{x}' \Rightarrow$ Transform to frame that travels with shock and where shock is \parallel to \hat{y}

$$\vec{v} = \vec{v}' - v_s \hat{x}'$$

$$\hat{x}' = \hat{x} \cos \theta + \hat{y} \sin \theta$$

$$\hat{y}' = -\hat{x} \sin \theta + \hat{y} \cos \theta$$

The corresponding velocities are:

$$v_x = \hat{x} \left[\underbrace{(\hat{x} \cos \theta + \hat{y} \sin \theta)}_{\hat{x}'} (v_x' - v_s) + \underbrace{(-\hat{x} \sin \theta + \hat{y} \cos \theta)}_{\hat{y}'} v_y' \right]$$

$$= \cos \theta (v_x' - v_s) - \sin \theta v_y'$$

Similarly

$$v_y = \hat{j} \cdot [(\hat{x} \cos \theta + \hat{y} \sin \theta)(v_x' - v_s) + (-\hat{x} \sin \theta + \hat{y} \cos \theta) v_y'] \quad (2)$$

$$= \sin \theta (v_x' - v_s) + \cos \theta v_y'$$

$$\Rightarrow v_y' = v_y \cos \theta - v_x \sin \theta$$

$$v_x' = v_s + v_x \cos \theta + v_y \sin \theta$$

To transform back

(a) Consider mass conservation across the shock front in the shock frame. Let's call the upstream side (1) and the downstream side (2). The mass flow rate must be the same on both sides

$$\Rightarrow \rho_1 v_{1x} = \rho_2 v_{2x}$$

(b) Now consider \hat{y} comp. of the momentum eq. $\rho \frac{\partial \vec{v}}{\partial t} + \rho \vec{v} \cdot \nabla \vec{v} = -\nabla P$

The y -momentum per time per area in the \hat{x} dir. is

$$\rho v_x v_y$$

Since ρv_x is conserved $\Rightarrow v_{1y} = v_{2y}$

$$\nabla(\rho v^2) + \nabla P = 0$$

(c) Now consider the \hat{x} comp. of the momentum eq.

The x -mom / time / area in the \hat{x} dir. is

$$\rho v_x v_x + P$$

This is conserved across the

$$\text{shock front} \Rightarrow \rho_1 v_{1x} v_{1x} + P_1 = \rho_2 v_{2x} v_{2x} + P_2$$

(d) Internal energy eq.

$$\frac{\partial}{\partial t} \left(\rho \left(\epsilon + \frac{1}{2} v^2 \right) \right) + \nabla \cdot \left(\rho \vec{v} \left(\epsilon + \frac{1}{2} v^2 \right) + P \vec{v} \right) = 0$$

Flux in the \hat{x} dir. is preserved across the shock \Rightarrow

$$\rho v_x \left(\epsilon + \frac{1}{2} v^2 \right) + P v_x = \text{constant}$$

$\epsilon = \text{internal energy / mass} = \frac{1}{\gamma-1} \frac{P}{\rho}$ for perfect gas

(3)

$$\Rightarrow \epsilon + \frac{P}{\rho} = \frac{\gamma}{\gamma-1} \frac{P}{\rho}$$

$$\rho v_x (\epsilon + \frac{1}{2} v^2) + P v_x = \rho v_x (\epsilon + \frac{1}{2} v^2 + \frac{P}{\rho}) = \rho v_x (\frac{\gamma}{\gamma-1} \frac{P}{\rho} + \frac{1}{2} v^2)$$

$$\Rightarrow \rho_1 v_{1x} (\frac{\gamma}{\gamma-1} \frac{P_1}{\rho_1} + \frac{1}{2} v_1^2) = \rho_2 v_{2x} (\frac{\gamma}{\gamma-1} \frac{P_2}{\rho_2} + \frac{1}{2} v_2^2)$$

$$\text{and since } \rho_1 v_{1x} = \rho_2 v_{2x} \Rightarrow \frac{\gamma}{\gamma-1} \frac{P_1}{\rho_1} + \frac{1}{2} v_1^2 = \frac{\gamma}{\gamma-1} \frac{P_2}{\rho_2} + \frac{1}{2} v_2^2$$

$$\text{or more in general } \epsilon_1 + \frac{P_1}{\rho_1} + \frac{1}{2} v_1^2 = \epsilon_2 + \frac{P_2}{\rho_2} + \frac{1}{2} v_2^2$$

These equations are known as the Rankine-Hugoniot shock jump conditions

We'll like to determine the downstream conditions based on the upstream basic condition. We'll assume a perfect gas, with same γ upstream and downstream.

$$\text{Since } v_{iy} = v_{2y} \Rightarrow \text{(a) becomes } \epsilon_1 + \frac{1}{2} v_{1x}^2 + \frac{P_1}{\rho_1} = \epsilon_2 + \frac{1}{2} v_{2x}^2 + \frac{P_2}{\rho_2}$$

\Rightarrow Now only the $v \perp$ to the shock front (v_x) is what matters. We'll call it u .

$$\Rightarrow \frac{\rho_1}{\rho_2} = \frac{u_2}{u_1} \quad \text{(A)}$$

$$\rho_1 (u_1^2 + \frac{P_1}{\rho_1}) = \rho_2 (u_2^2 + \frac{P_2}{\rho_2}) \quad \text{(B)}$$

$$u_1^2 + \frac{2\gamma}{\gamma-1} \frac{P_1}{\rho_1} = u_2^2 + \frac{2\gamma}{\gamma-1} \frac{P_2}{\rho_2} \quad \text{(C)}$$

Define upstream Mach # as $M_1 = u_1/c_1$

$$c_1 = \left(\frac{\gamma P_1}{\rho_1}\right)^{1/2} \Rightarrow u_1 = c_1 M_1 \Rightarrow \rho_1 u_1^2 = \gamma M_1^2 P_1$$

$$\Rightarrow (B) \rightarrow (\gamma M_1^2 + 1) P_1 = \rho_2 u_2^2 + P_2$$

$$\begin{aligned} (A) \rightarrow &= u_1^2 \left(\frac{\rho_2}{u_1}\right)^2 \rho_2 + P_2 \\ &= u_1^2 \left(\frac{\rho_1}{\rho_2}\right)^2 \rho_2 + P_2 \\ &= M_1 \gamma P_1 \frac{\rho_1}{\rho_2} + P_2 \end{aligned}$$

$$\Rightarrow \boxed{P_2 - P_1 = P_1 \gamma M_1^2 \left(1 - \rho_1/\rho_2\right)} \quad (*)$$

$$\text{From (c)} \rightarrow \rho_1 u_1^2 + \frac{2\gamma}{\gamma-1} P_1 = \rho_1 u_2^2 + \frac{2\gamma}{\gamma-1} P_2 \frac{\rho_1}{\rho_2}$$

$$= \rho_1 u_1^2 \left(\frac{\rho_1}{\rho_2}\right)^2 + \frac{2\gamma}{\gamma-1} P_2 \frac{\rho_1}{\rho_2}$$

$$\Rightarrow \gamma M_1^2 P_1 + \frac{2\gamma}{\gamma-1} P_1 = \gamma P_1 M_1^2 \left(\rho_1/\rho_2\right)^2 + \frac{2\gamma}{\gamma-1} P_2 \frac{\rho_1}{\rho_2}$$

$$P_1 \gamma M_1^2 \left(1 - \left(\rho_1/\rho_2\right)^2\right) = \frac{2\gamma}{\gamma-1} \left(P_2 \frac{\rho_1}{\rho_2} - P_1\right)$$

$$\Rightarrow \boxed{P_1 \gamma M_1^2 \left(1 - \left(\rho_1/\rho_2\right)^2\right) = \frac{2\gamma}{\gamma-1} \left(P_2 \left(\rho_1/\rho_2 - 1\right) + \left(P_2 - P_1\right)\right)} \quad (**)$$

$$\text{From (*)} \quad \frac{P_2}{P_1} = \gamma M_1^2 \left(1 - \rho_1/\rho_2\right) + 1$$

$$\text{From (**)} \quad \frac{P_2}{P_1} = \left[\frac{\gamma-1}{2} M_1^2 \left(1 - \left(\rho_1/\rho_2\right)^2\right) + 1\right] \frac{\rho_2}{\rho_1}$$

\Rightarrow After some algebra we find

⑤

$$\rho_1/\rho_2 = \frac{1 + \frac{\gamma-1}{2} \pi_1^2}{\gamma \pi_1^2 - (\frac{\gamma-1}{2}) \pi_1^2}$$

$$\rho_2/\rho_1 = \frac{\frac{\gamma+1}{2} \pi_1^2}{1 + (\frac{\gamma-1}{2}) \pi_1^2} = \frac{(\gamma+1) \pi_1^2}{2 + (\gamma-1) \pi_1^2} = \frac{\rho_2}{\rho_1}$$

$$\frac{P_2}{P_1} = 1 + \gamma \pi_1^2 (1 - \rho_1/\rho_2) = 1 + \gamma \pi_1^2 \left(1 - \frac{2 + (\gamma-1) \pi_1^2}{(\gamma+1) \pi_1^2} \right)$$

$$\Rightarrow \frac{P_2}{P_1} = \frac{1 - \gamma + 2\gamma \pi_1^2}{\gamma + 1}$$

$$\frac{T_2}{T_1} = \frac{P_2/\rho_2}{P_1/\rho_1} = \frac{P_2}{P_1} \cdot \frac{\rho_1}{\rho_2} = \frac{((1-\gamma) + 2\gamma \pi_1^2)/(\gamma+1)}{(\gamma+1) \pi_1^2 / (2 + (\gamma-1) \pi_1^2)}$$

$$\Rightarrow \frac{T_2}{T_1} = \frac{((1-\gamma) + 2\gamma \pi_1^2)(2 + (\gamma-1) \pi_1^2)}{(\gamma+1)^2 \pi_1^2}$$

Limiting cases:

1) Isothermal eq. of state ($\gamma \rightarrow 1$) $\Rightarrow \frac{P_2}{P_1} \rightarrow \frac{2M_1^2}{2} = M_1^2$

$$\frac{P_2}{P_1} \rightarrow \frac{2M_1^2}{2} = M_1^2 = \frac{P_2}{P_1}$$

$$\frac{T_2}{T_1} = \frac{2M_1^2 \cdot 2}{2^2 M_1^2} = 1$$

\Rightarrow If $T = \text{constant}$, post-shock pressure and density increase by the square of the Mach #

2) Non-isothermal very strong ($M_1 \gg 1$) shocks

$\frac{P_2}{P_1} \rightarrow \frac{\gamma+1}{\gamma-1}$ ← limited for non-isothermal

$\frac{P_2}{P_1} \rightarrow \frac{2\gamma}{\gamma+1} M_1^2$ ← similar to non-iso.

$\frac{T_2}{T_1} \rightarrow \frac{2\gamma(\gamma-1)}{(\gamma+1)^2} M_1^2$

eg: for $\gamma = 5/3$

$\frac{P_2}{P_1} \rightarrow 4, \frac{P_2}{P_1} \rightarrow \frac{5}{4} M_1^2, \frac{T_2}{T_1} \rightarrow \frac{5}{16} M_1^2$

i.e. $T_2 = \frac{5}{16} \frac{v_1^2}{\gamma k} \mu^2 = 3200k \left(\frac{v_1}{10 \text{ km/s}} \right)^2$

$(c_1 = \sqrt{\gamma \frac{P_1}{\rho_1}}, P = \frac{\rho_1 \cdot R T_1}{\mu})$

\Rightarrow few $\times 10 \text{ km/s} \rightarrow$ few 1,000K \rightarrow IR emitting
 $\times 100 \text{ km/s} \rightarrow 10^5 \text{ K} \rightarrow$ optical emission (UV emission)
Several $100 \text{ km/s} \rightarrow 10^6 \text{ K} \rightarrow$ X-ray emission

For $T = 10^4 \text{ K} \left(\frac{kT}{\mu} \right)^{1/2} = 8 \text{ km/s} \Rightarrow \left(\frac{\gamma kT}{\mu} \right)^{1/2} = 10 \text{ km/s} \Rightarrow$ a few 100 km/s
~~can~~ implies $T \sim 10^6 \text{ K}$.

~~3) No shock~~

3) No shock $\Rightarrow M_1 \rightarrow 1 \Rightarrow$

$$\frac{\rho_2}{\rho_1} \rightarrow 1$$

$$\frac{P_2}{P_1} \rightarrow 1$$

$$\frac{T_2}{T_1} \rightarrow 1$$

always good to check...

Entropy $S \propto \ln P \rho^{-\gamma}$, $S_2 - S_1 = \text{const.} \ln \left(\frac{P_2}{P_1} \right) \left(\frac{\rho_2}{\rho_1} \right)^{-\gamma}$

\Rightarrow For a strong shock, $S_2 - S_1 = \text{const.} \ln \left(\frac{2\gamma M_1^2}{\gamma+1} \right) \left(\frac{\gamma+1}{\gamma-1} \right)^{-\gamma} > 0$

entropy increases in a shock, b.c. ordered kinetic energy is converted into thermal energy.

This entropy increase is mediated by collisions for high-density gas.

Physical collisions occur within ~~mean~~ ^{mean} free path $\sim \ell = \frac{1}{n\sigma}$

For low densities, we can have a "collisionless shock" mediated by the interaction of particles with the magnetic field.

Consider expanding spherical wind, jet, etc, ^{our shock} with normal to shock $\theta = 0 \Rightarrow$

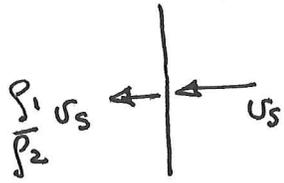
$$v_x = v_x' - v_s$$

$$v_y = v_y'$$

Assume pre-shocked gas is stationary $\Rightarrow v_{x1}' = v_{y1}' = 0$
 $\Rightarrow v_{x2} = 0, v_{x1} = v_s$

$$v_{x2} = \frac{\rho_1}{\rho_2} v_{x1} = -\frac{\rho_1}{\rho_2} v_s$$

$$\Rightarrow v_{x2}' = v_{x2} + v_s = v_s \left(1 - \frac{\rho_1}{\rho_2} \right)$$



with $\frac{p_2}{p_1} = \frac{(\gamma+1)M_1^2}{2+(\gamma-1)M_1^2}$,

$$v_{x2}' = v_s \left(1 - \left(\frac{2+(\gamma-1)M_1^2}{(\gamma+1)M_1^2} \right) \right) = v_s \left[\frac{2(M_1^2-1)}{(\gamma+1)M_1^2} \right]$$

$$\Rightarrow v_s = v_{x2}' \left(\frac{(\gamma+1)M_1^2}{2(M_1^2-1)} \right)$$

For a strong shock, $M_1^2 \gg 1$, $v_s = v_{x2}' \left(\frac{\gamma+1}{2} \right)$

\Rightarrow Shock speed is a factor $\frac{\gamma+1}{2}$ larger than the flow speed on the post-shock side, e.g. $\rightarrow \frac{4}{3}$ for $\gamma = 5/3$

The kinetic energy/vol on the post shock side is

$$E_k = \frac{1}{2} (v_{x2}')^2 p_2 \rightarrow \frac{1}{2} \left(\frac{2}{\gamma+1} \right)^2 v_s^2 \left(\frac{\gamma+1}{\gamma-1} \right) p_1 = \frac{2}{(\gamma-1)(\gamma+1)} v_s^2 p_1$$

The Thermal en./volume is

$$E_{th} = \frac{1}{\gamma-1} P_2 = \frac{1}{\gamma-1} \frac{2\gamma}{\gamma+1} \frac{v_s^2 p_1}{\gamma p_1} P_1 = \frac{2}{(\gamma+1)(\gamma-1)} v_s^2 p_1$$

\Rightarrow Kinetic energy = Thermal energy in post-shock region of a strong shock.

Radiative shocks

We computed the results for shock jump conditions for $\gamma=1$ perfect gas. But in fact a real gas doesn't have $\gamma=1$, This is just a model for strongly-cooling gas.

Real radiative shocks have structures such as these:

