Class 7 :
Cosmological Models with Curvature

- This class…
  - Recap of FRW metric
  - Density parameter Ω
  - Ω-curvature connection

0 : Recap : The Friedmann-Robertson-Walker (FRW) Metric

- What is the metric for these spacetimes? It can be shown that the appropriate metric is…

$$ds^2 = -c^2 dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right]$$

- Notes:
  - This covers all three geometries
    - $k>0$: Spherical
    - $k=0$: Flat
    - $k<0$: Hyperbolic
  - r, θ, ϕ are co–moving coordinates
  - Role of a(t) as scale factor is made clear
  - Substituting this metric into Einstein's Field Equations gives the Friedmann equation and the acceleration equation that we've already encountered (thus also encapsulates the fluid equation). The k above becomes the k of the Friedmann equation!

I : Behaviour of cosmological models with curvature

- Let's examine the Friedmann equation

$$\left(\frac{1}{a}\frac{da}{dt}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2}$$

- <u>**We've already examined the flat universe case (k=0)**</u>
  - Putting k=0 into Friedmann gives

$$\left(\frac{1}{a}\frac{da}{dt}\right)^2 = \frac{8\pi G}{3}\rho \quad \Rightarrow \quad \rho = \frac{3H^2}{8\pi G}$$

  - So, for a given Hubble parameter (H), there is a special density which gives a flat Universe – call this the **critical density** (ρ_{crit})

$$\boxed{\rho_{\rm crit} = \frac{3H^2}{8\pi G}}$$

- What about the general case with curvature? Let's manipulate Friedmann equation…

$$H^2 = \frac{8\pi G}{3}\rho - \frac{kc}{a^2}$$
$$\Rightarrow 1 = \frac{8\pi G}{3H^2}\rho - \frac{kc^2}{a^2H^2}$$
$$\Rightarrow 1 = \frac{\rho}{\rho_{crit}} - \frac{kc^2}{a^2H^2}$$

  - Define <u>**density parameter**</u> as ratio of density to critical density

$$\boxed{\Omega \equiv \frac{\rho}{\rho_{crit}}}$$

  - Then…

$$\Omega = 1 + \frac{kc^2}{a^2H^2}$$

so…

$$k < 0 \Rightarrow \Omega < 1$$
$$k = 0 \Rightarrow \Omega = 1$$
$$k > 0 \Rightarrow \Omega > 1$$

- What is the time evolution of models with curvature? We can figure out the basics without even solving the equations…
- **Firstly, let's look at spherical (k>0, Ω>1) universe**

$$\left(\frac{da}{dt}\right)^2 = \frac{8\pi G}{3}\rho a^2 - kc^2$$

- Recall the two cases we've been discussing…
  - Matter dominated universe… $\rho \propto 1/a^3 \Rightarrow \rho a^2 \propto 1/a$
  - Radiation dominated universe.. $\rho \propto 1/a^4 \Rightarrow \rho a^2 \propto 1/a^2$
  - For now, we will **ignore dark e**
- What happens for large a?
  - The matter/radiation term (first term on RHS) becomes smaller and smaller
  - Eventually, curvature term = matter term… then (da/dt)=0
  - **So, expansion of the universe stops, and it recollapses**.

- **Final case… hyperbolic (k<0, Ω<1) universe**

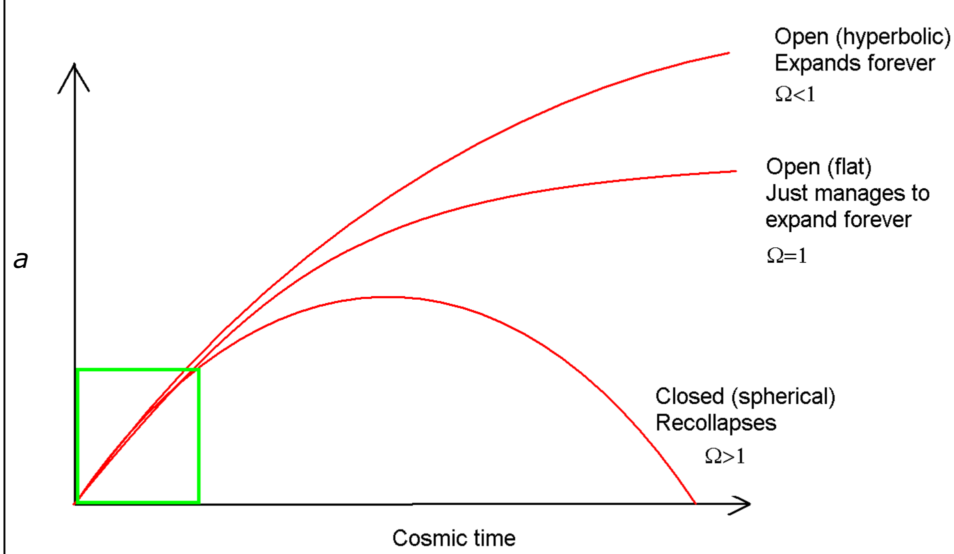
$$\left(\frac{da}{dt}\right)^2 = \frac{8\pi G}{3}\rho a^2 - kc^2$$

  - Clearly, for given scale factor, the k<0 universe will be expanding factor than the corresponding flat universe (since $-kc^2$ term is positive)
  - But we know that flat case expands forever ($a \sim t^{2/3}$ or $a \sim t^{1/2}$ for matter and radiation pressure dominated cases respectively)
  - So, the hyperbolic universe expands forever. As "a" becomes very large, the expansion rate reaches an asymptotic value

$$\left(\frac{da}{dt}\right)^2 \rightarrow -kc^2$$

III : Summary of the three simple (matter +radiation) cosmologies

- Hyperbolic Universe
  - Low density (Ω<1)
  - Spatially-infinite ("open" universe)
  - Expands forever, tending to finite da/dt
- Flat Universe
  - Critical density (Ω=1)
  - Spatially infinite (also "open" universe)
  - Expands forever, just.
- Spherical Universe
  - High density (Ω>1)
  - Spatially finite (also "closed" universe)
  - Expansion halts – then recollapses



IV : What's next??

- What have we done so far…
  - Used the cosmological principle + GR to determine basic equations that describe evolution of the universe
  - Found three cases (spherical, flat, hyperbolic)
  - Proven the connection between the density and the geometry (high ρ= spherical, critical ρ= flat, low ρ= hyperbolic)
  - Assuming universe contains matter and/or radiation, we've computed the evolution of Universe as function of time for these three cases
- What's next??
  - Determine which of these cases applies to our real universe!
    - What is the density of our Universe? How does it compare with the critical density?
    - What is the make up of the Universe? Matter, radiation, both?
  - Test the basic predictions of the model
    - E.g. compare the predicted age of the Universe with measured ages
    - Measure the change in scale factor over cosmic time