ASTR 340: Origin of the Universe

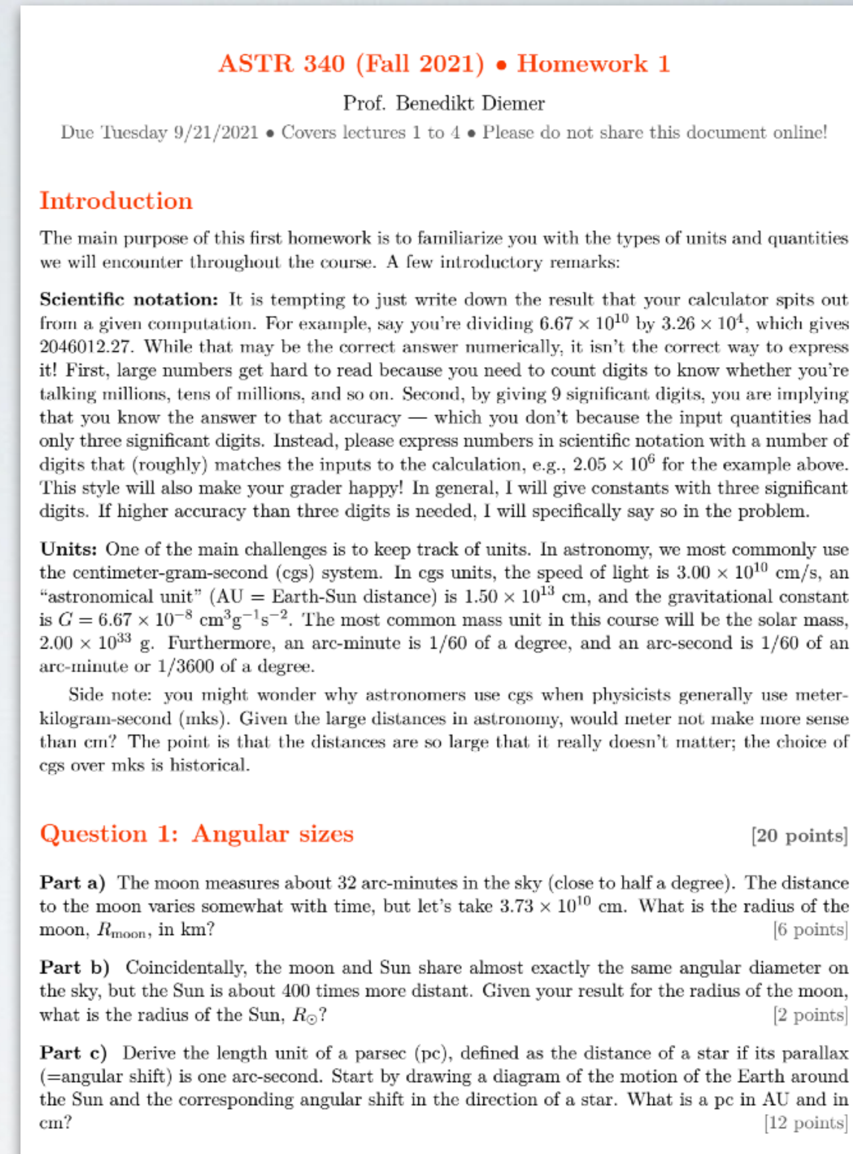
Prof. Benedikt Diemer

Lecture 7 • Special Relativity I

09/21/2021

Logistics

- **Homework #1**
  - Due Thursday night!
  - Grades within 2 weeks (roughly)
- **Homework #2**
  - Assigned Thursday
  - Due Thu 10/07
- **Google etc**
  - You are allowed to use books and the internet!
  - Avoid googling answers except for verification — it'll hurt you in the end



ASTR 340 (Fall 2021) • Homework 1

Prof. Benedikt Diemer

Due Tuesday 9/21/2021 • Covers lectures 1 to 4 • Please do not share this document online!

Introduction

The main purpose of this first homework is to familiarize you with the types of units and quantities we will encounter throughout the course. A few introductory remarks:

Scientific notation: It is tempting to just write down the result that your calculator spits out from a given computation. For example, say you're dividing 6.67×10^{10} by 3.26×10^4, which gives 2046012.27. While that may be the correct answer numerically, it isn't the correct way to express it! First, large numbers get hard to read because you need to count digits to know whether you're talking millions, tens of millions, and so on. Second, by giving 9 significant digits, you are implying that you know the answer to that accuracy — which you don't because the input quantities had only three significant digits. Instead, please express numbers in scientific notation with a number of digits that (roughly) matches the inputs to the calculation, e.g., 2.05×10^6 for the example above. This style will also make your grader happy! In general, I will give constants with three significant digits. If higher accuracy than three digits is needed, I will specifically say so in the problem.

Units: One of the main challenges is to keep track of units. In astronomy, we most commonly use the centimeter-gram-second (cgs) system. In cgs units, the speed of light is 3.00×10^{10} cm/s, an "astronomical unit" (AU = Earth-Sun distance) is 1.50×10^{13} cm, and the gravitational constant is $G = 6.67 \times 10^{-8}\ \mathrm{cm^3 g^{-1} s^{-2}}$. The most common mass unit in this course will be the solar mass, 2.00×10^{33} g. Furthermore, an arc-minute is 1/60 of a degree, and an arc-second is 1/60 of an arc-minute or 1/3600 of a degree.

Side note: you might wonder why astronomers use cgs when physicists generally use meter-kilogram-second (mks). Given the large distances in astronomy, would meter not make more sense than cm? The point is that the distances are so large that it really doesn't matter; the choice of cgs over mks is historical.

Question 1: Angular sizes [20 points]

Part a) The moon measures about 32 arc-minutes in the sky (close to half a degree). The distance to the moon varies somewhat with time, but let's take 3.73×10^{10} cm. What is the radius of the moon, R_{moon}, in km? [6 points]

Part b) Coincidentally, the moon and Sun share almost exactly the same angular diameter on the sky, but the Sun is about 400 times more distant. Given your result for the radius of the moon, what is the radius of the Sun, $R_{\odot}$? [2 points]

Part c) Derive the length unit of a parsec (pc), defined as the distance of a star if its parallax (=angular shift) is one arc-second. Start by drawing a diagram of the motion of the Earth around the Sun and the corresponding angular shift in the direction of a star. What is a pc in AU and in cm? [12 points]

	Notes	Book (opt.)	Internet	Collaborate	Group
Participation (group work, quizzes)	✓	✓	✗	✓	✓
Comprehension quizzes	✓	✓	✓	✗	✗
Homework Assignments	✓	✓	✓	✗	✗
Midterm and final exams	✗	✗	✗	✗	✗

Metacognition free-write

- Weak equivalence principle (gravity as an acceleration)
- History of Greek cosmology
- Galilean relativity
- Scientific method
- Symmetry and Noether's theorem
- Inertial frames and fictitious forces
- Age of the Earth and cosmic time
- The Copernican principle (no center to the Universe)

Participation: Recap

Respond to the poll on TurningPoint

Session ID: diemer

30 seconds

Einstein's solution

Postulate 1: The **laws of nature** are the same in all inertial frames of reference

Postulate 2: The **speed of light** in a vacuum is the same in all inertial frames of reference

~~**Galilean relativity:** relative motions of frames add, there is an absolute time~~

Einstein's solution

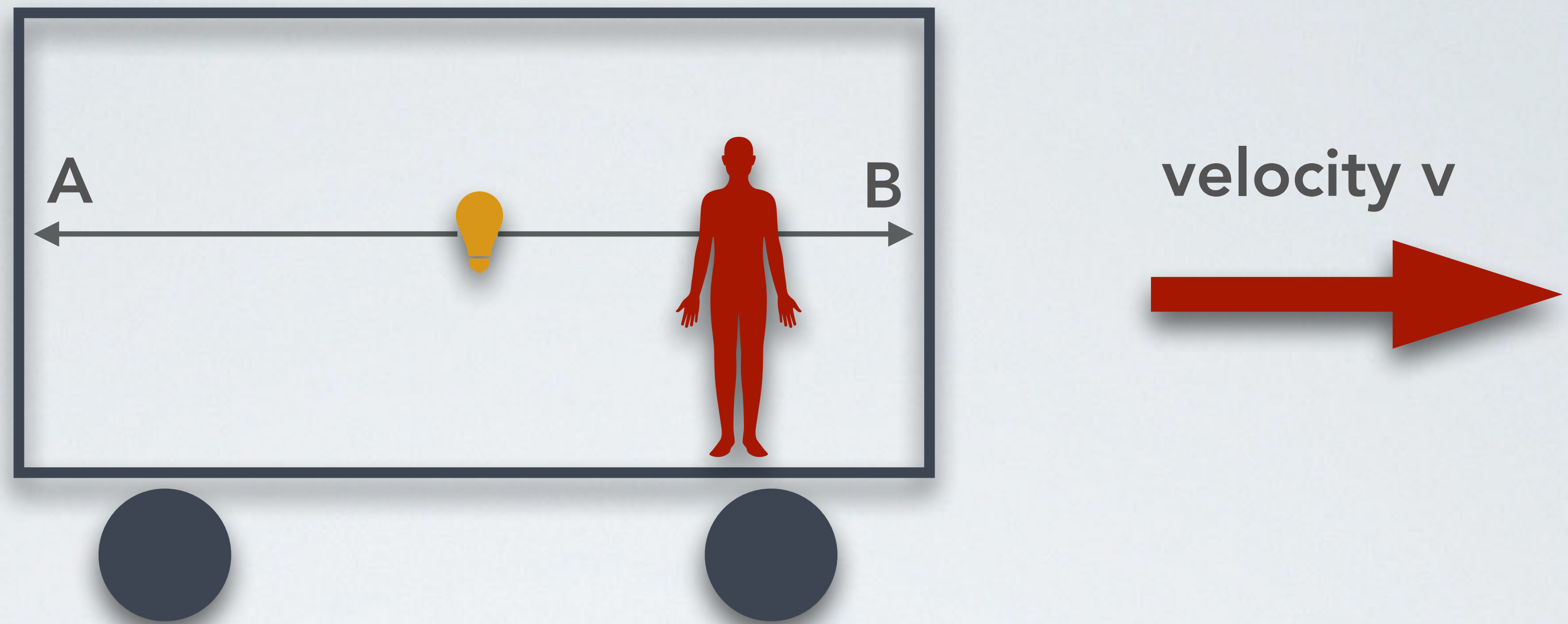
- Today we explore the **consequences** of these postulates
- We perform **thought experiments** (Gedankenexperimente) to find out what observers moving at different speeds perceive
- For now, we will **ignore gravity**: we suppose we are in the middle of deep space (or in free fall)

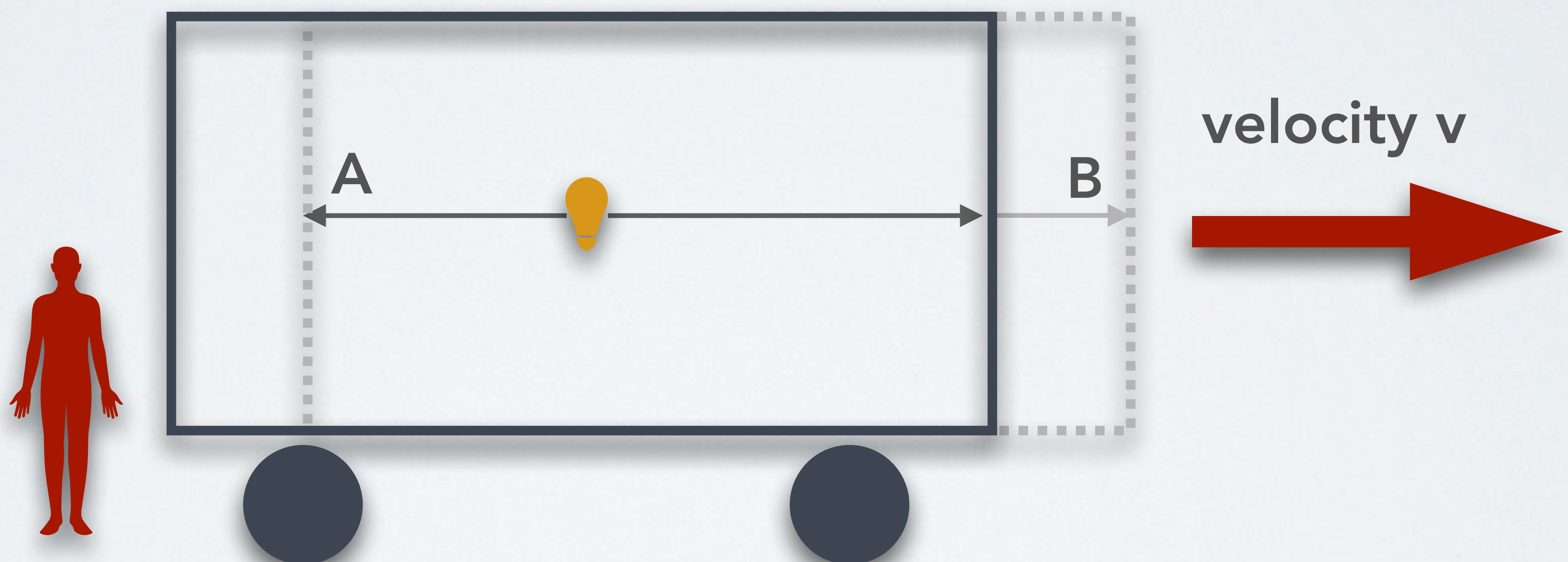
Today

- Simultaneity & Time dilation
- Length contraction
- Lorentz transformation
- Velocity addition

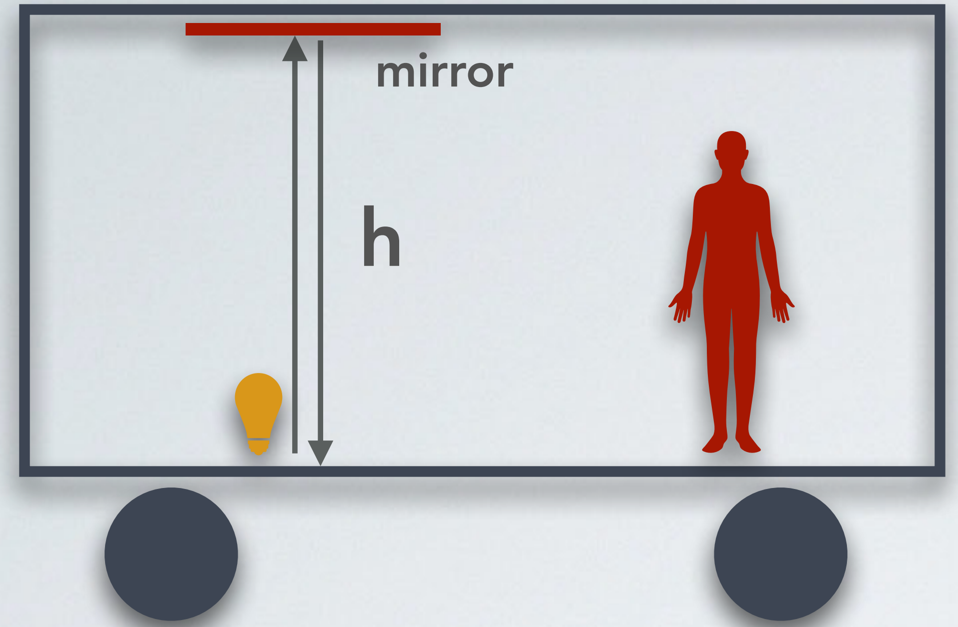
Part 1: Simultaneity & Time dilation

Simultaneity





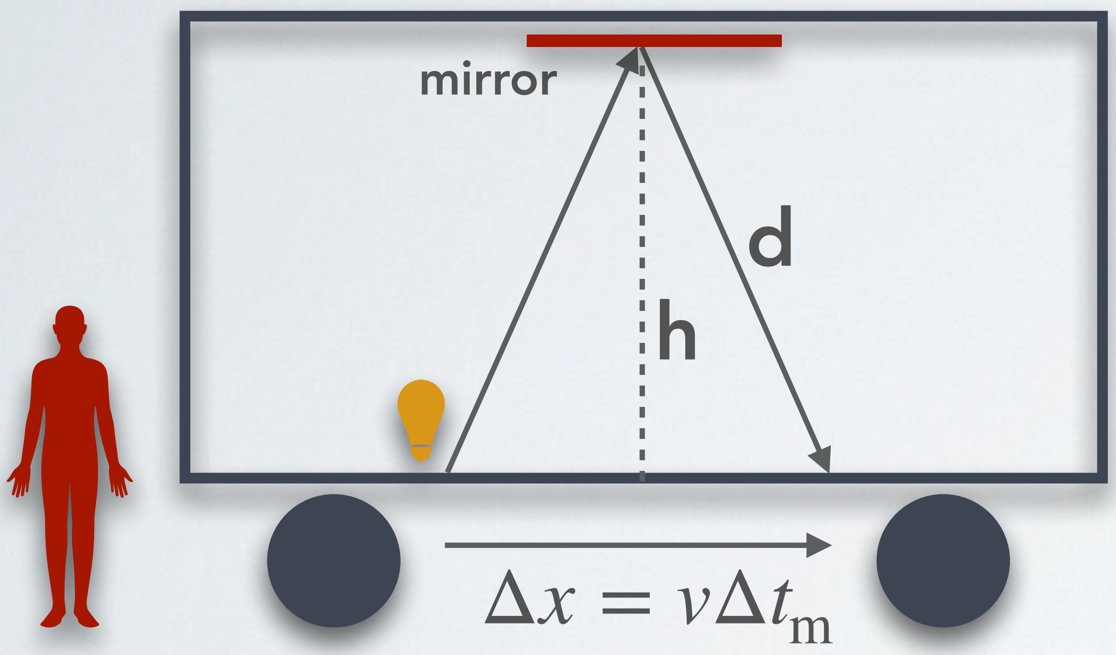
Simultaneity in one inertial frame does not imply simultaneity in another.



velocity v

Interval in stationary frame
(proper time):

$$\Delta t_{\mathrm{p}} = \frac{2h}{c} \implies h = \frac{c}{2}\Delta t_{\mathrm{p}}$$



velocity v

Interval in moving frame
(with respect to experiment):

$$\Delta t_{\mathrm{m}} = \frac{2d}{c} \implies d = \frac{c}{2}\Delta t_{\mathrm{m}}$$

$$d^2 = h^2 + \left(\frac{\Delta x}{2}\right)^2$$

$$\implies \frac{c^2}{4}\Delta t_{\mathrm{m}}^2 = \frac{c^2}{4}\Delta t_{\mathrm{p}}^2 + \frac{v^2}{4}\Delta t_{\mathrm{m}}^2$$

$$\implies \Delta t_{\mathrm{p}}^2 = \Delta t_{\mathrm{m}}^2 - \frac{v^2}{c^2}\Delta t_{\mathrm{m}}^2 \implies$$

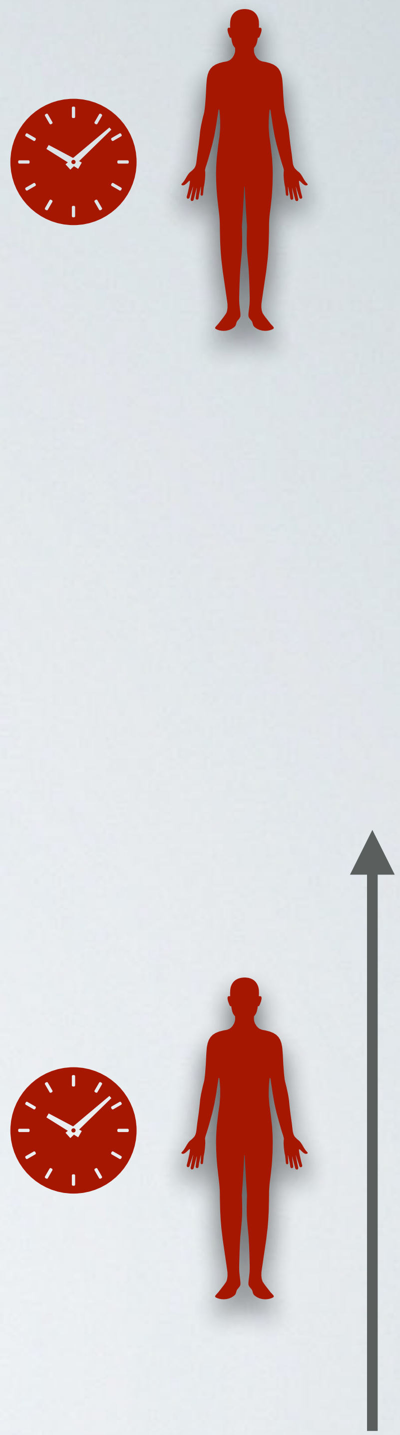
Time dilation:

$$\boxed{\Delta t_{\mathrm{m}} = \Delta t_{\mathrm{p}} \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}}$$

Time is seen to progress more slowly in moving frame compared to proper time (rest frame)

Time dilation

- Is it about **clocks or time?**
- **Time is not absolute!**
  - Depends on observer / reference frame
  - "**Proper time**" is time measured in rest frame
  - Clocks always tick **most rapidly** when measured by observer **in its own rest frame**
  - Clock **slows** (ticks take longer) from perspective of **other observers** (people on spaceship seem to age slower)
- **Reciprocity:** we see clocks in moving frame running more slowly than local observers do, they see our clocks running more slowly



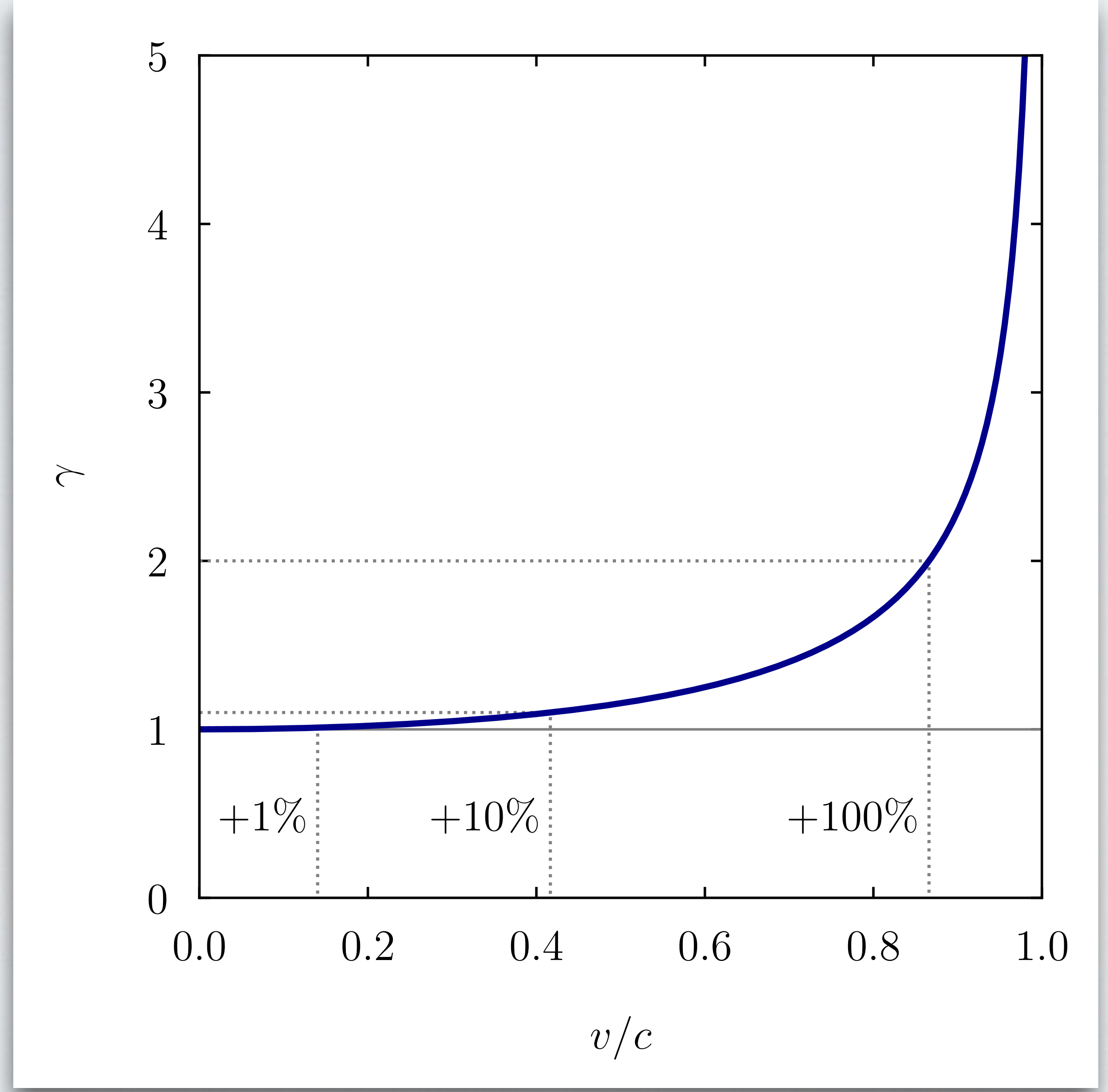
$$\Delta t_{\mathrm{m}} = \Delta t_{\mathrm{p}} \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Lorentz factor

$$\gamma \equiv \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\Longrightarrow$$

$$\Delta t_{\mathrm{m}} = \gamma \Delta t_{\mathrm{p}}$$



Examples: Human experience

- **1 m/s** a typical human walking speed
- **12.4 m/s** for the fastest 20 meters of the 100 m sprint by Usain Bolt
- **28 m/s** a car travelling at 60 miles per hour (mi/h or mph) or 100 kilometres per hour (km/h)
- **341 m/s** the current land speed record, which was was set by ThrustSSC in 1997
- **343 m/s** the speed of sound under standard conditions (varies according to air temperature)
- **464 m/s** Earth’s rotation speed at the equator
- **559 m/s** the average speed of Concorde's record Atlantic crossing (1996)
- **1000 m/s** the speed of a typical rifle bullet
- **1400 m/s** the speed of the Space Shuttle when the solid rocket boosters separate
- **8000 m/s** the speed of the Space Shuttle just before it enters orbit
- **11,082 m/s** high speed record for manned vehicle (set by Apollo 10)
- **29,800 m/s** speed of the Earth in orbit around the Sun (about 30 km/s)
- **36,666 m/s** speed of Pioneer 10, the fastest probe (36.6 km/s)

- **42,000,000 m/s** the speed where time is dilated by 1%
- **299,792,458 m/s** the speed of light (about 300,000 km/s)

Examples: Muons

- Created in upper atmosphere from cosmic ray hits
- Typical travel speeds are 0.99995×c

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{1}{\sqrt{1 - 0.99995^2}} \approx 100$$

- **Half-life** of muons in their own rest frame (measured in lab) is t_h= 2μs = 0.000002s
- Traveling at 0.99995×c for t_h= 2μs, the muons would go only **600 m**
- Traveling for $\gamma \times t_h$ = 200μs, the muons can go **60 km**
- They reach the Earth's surface, and are detected!
- Half-life can be measured by comparing muon flux on a mountain and at sea level; result agrees with $\gamma \times t_h$

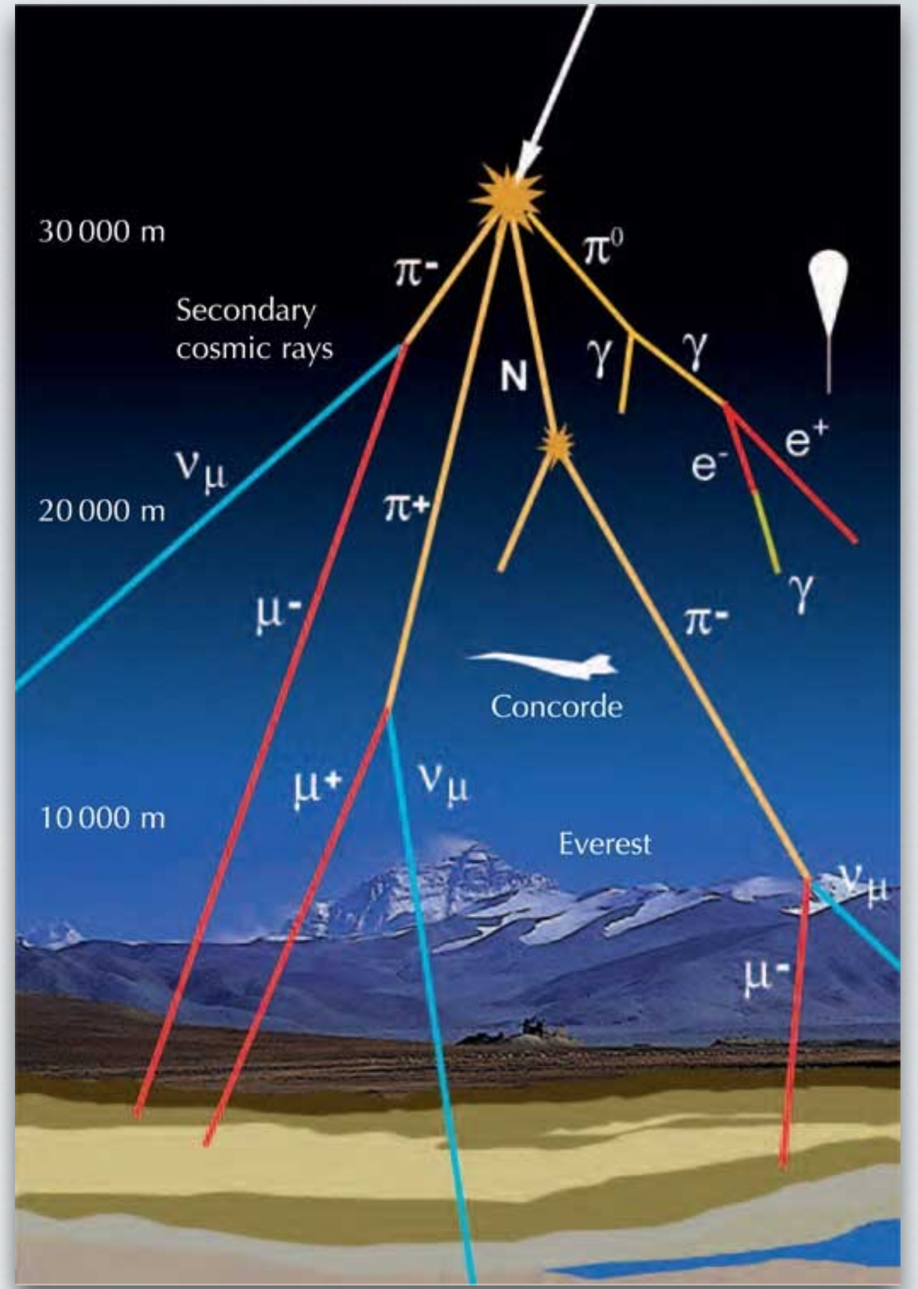
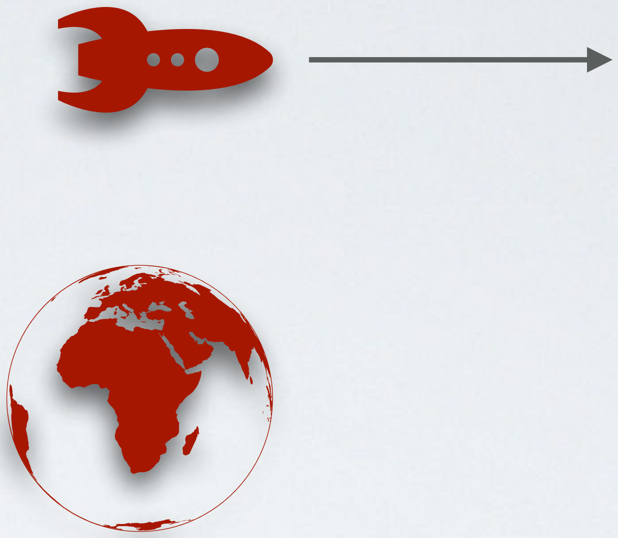


Image: PhysicsOpenLab

Participation: Time dilation #1



We see a spaceship fly past earth at a speed of 0.87c (or $\gamma = 2$).
After one Earth year, how much seem the astronauts to have aged?

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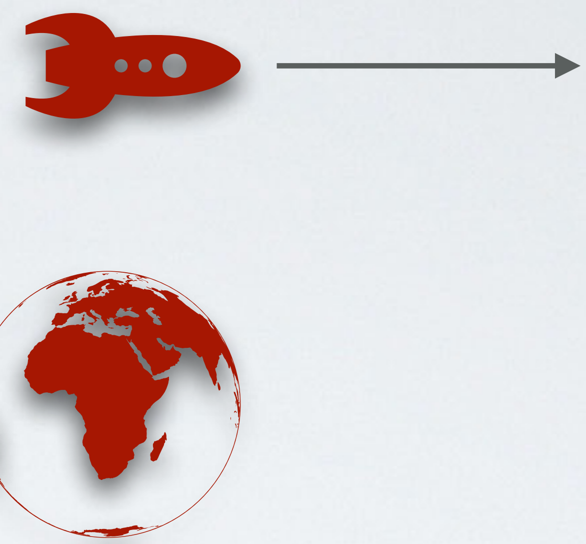
30 seconds

Participation: Time dilation #2



The astronauts also have clocks on board. After **one year of their time**, how much do **people on Earth** seem to have aged according to them?

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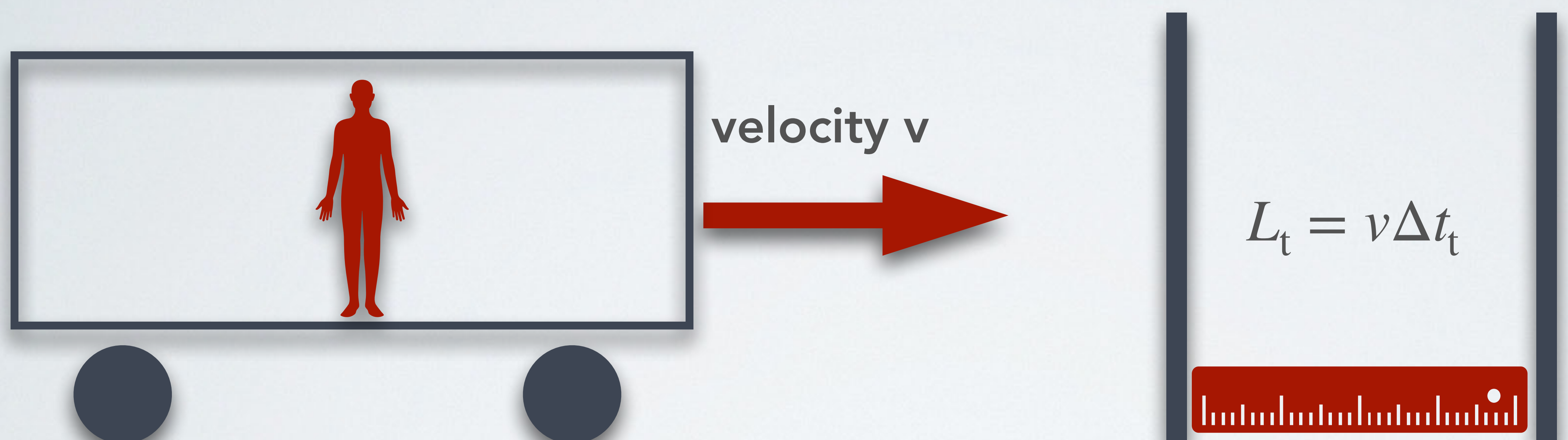
30 seconds

Part 2: Length contraction



Observer on train sees time on ground as dilated:

$$\Delta t_g = \gamma \Delta t_t$$

$$\frac{L_t}{L_g} = \frac{\Delta t_t}{\Delta t_g} = \frac{1}{\gamma}$$

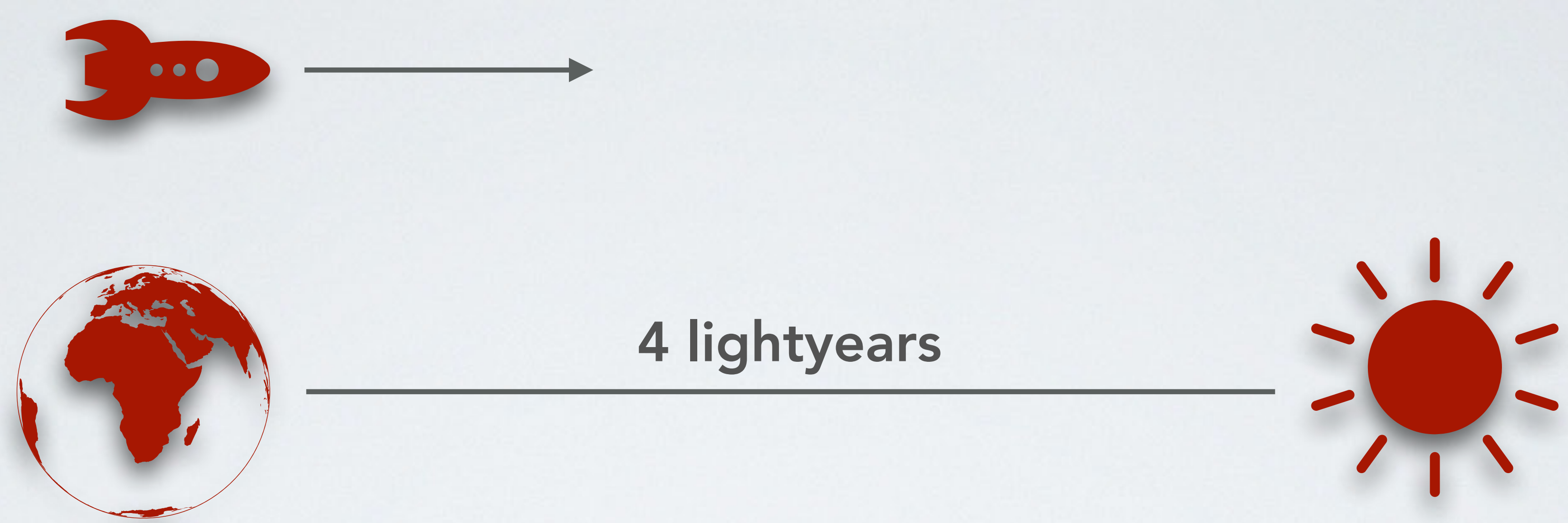
$$\Longrightarrow \quad L_t = \frac{1}{\gamma} L_g$$

Length contraction:

$$\boxed{L_m = \frac{1}{\gamma} L_p}$$

In moving frame, objects are contracted compared to rest (along direction of motion)

Participation: Time dilation #3



We see a spaceship fly past earth at a speed of 0.97c (or $\gamma = 4$). They are going to Alpha Centauri, 4 lightyears away. **How long does the trip take according to the astronauts?**

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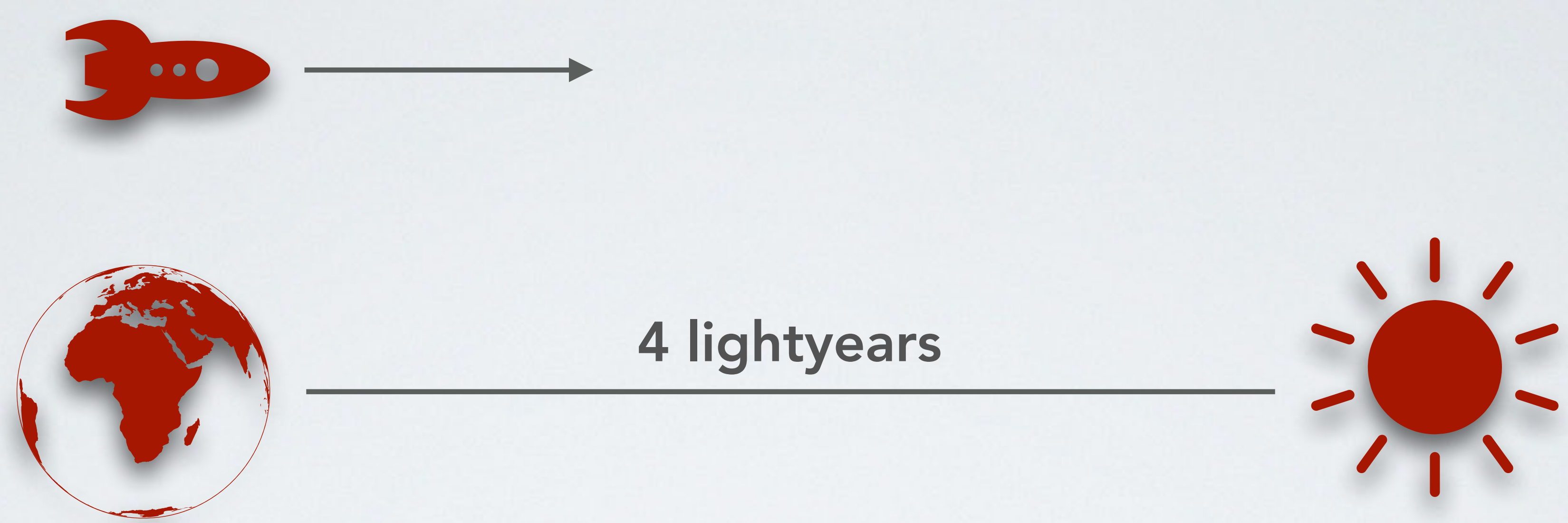
Participation: Length contraction #1



We see a spaceship fly past earth at a speed of 0.97c (or $\gamma = 4$). They are going to Alpha Centauri, 4 lightyears away. **How long does the distance appear to the astronauts?**

Session ID: diemer

30 seconds

Example: Muons (again)

- Consider **perspective of the muons**
- Decay time in this frame is 2μs
- From point of view of muon, the **atmosphere's height contracts by factor of γ = 100**
- Muons can then **travel reduced distance** (at almost speed of light) before decaying

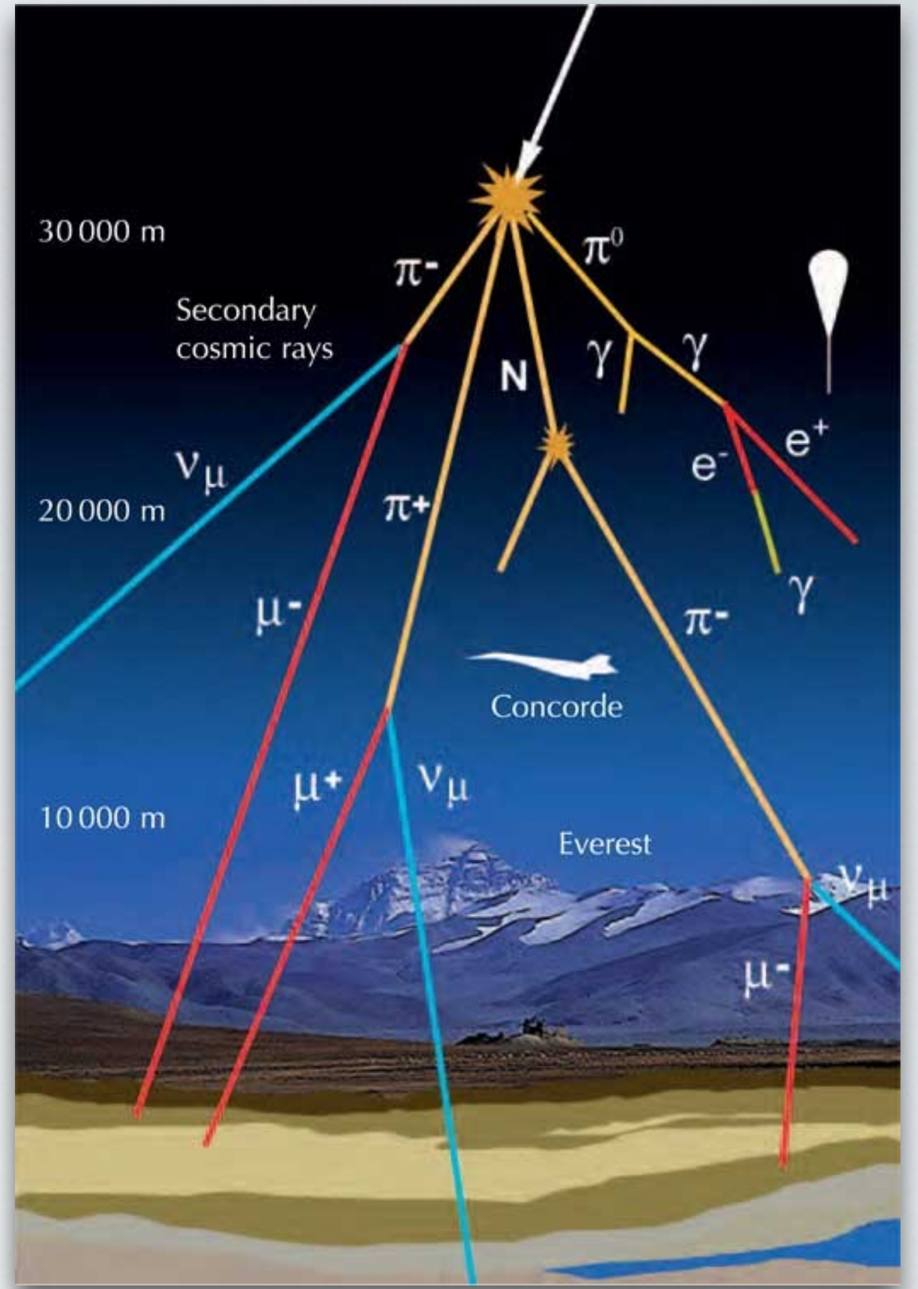
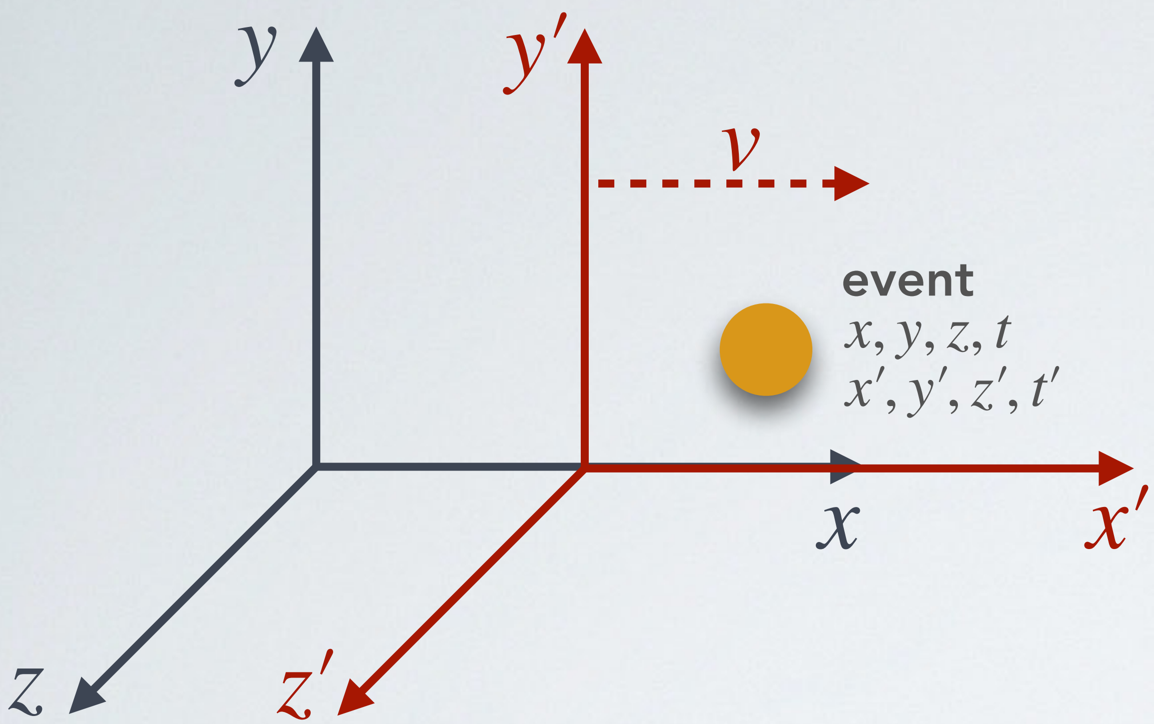


Image: PhysicsOpenLab

Part 3: Lorentz transformation & velocity addition

Lorentz transformation



x = x′ = 0 at t = t′ = 0

$$x = vt + \frac{x'}{\gamma} \implies x' = \gamma(x - vt)$$

By reciprocity:

$$x' = -vt' + \frac{x}{\gamma} \implies x = \gamma(x' + vt)$$

$$x' = \gamma(x - vt)$$

$$x = \gamma(x' + vt')$$

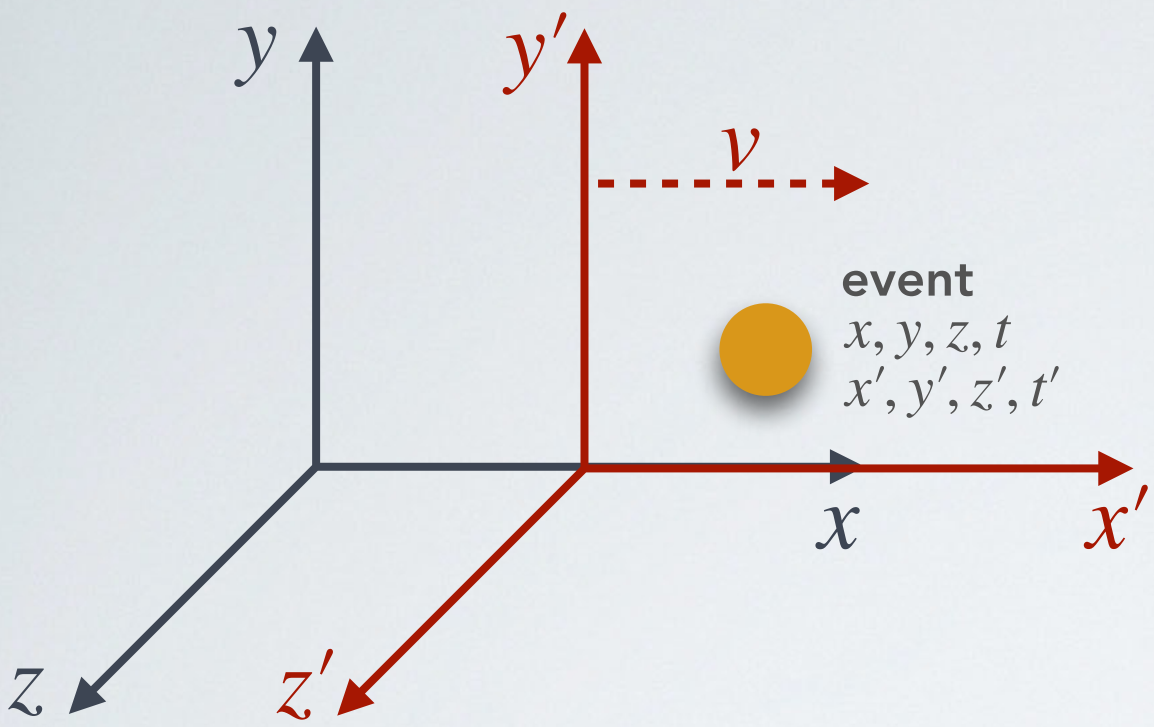
$$y' = y\,,\ z' = z$$

$$y = y'\,,\ z = z'$$

$$t' = \gamma\left(t - \frac{vx}{c^2}\right)$$

$$t = \gamma\left(t' + \frac{vx'}{c^2}\right)$$

Lorentz transformation



$$x' = \gamma(x - vt)$$

$$t' = \gamma\left(t - \frac{vx}{c^2}\right)$$

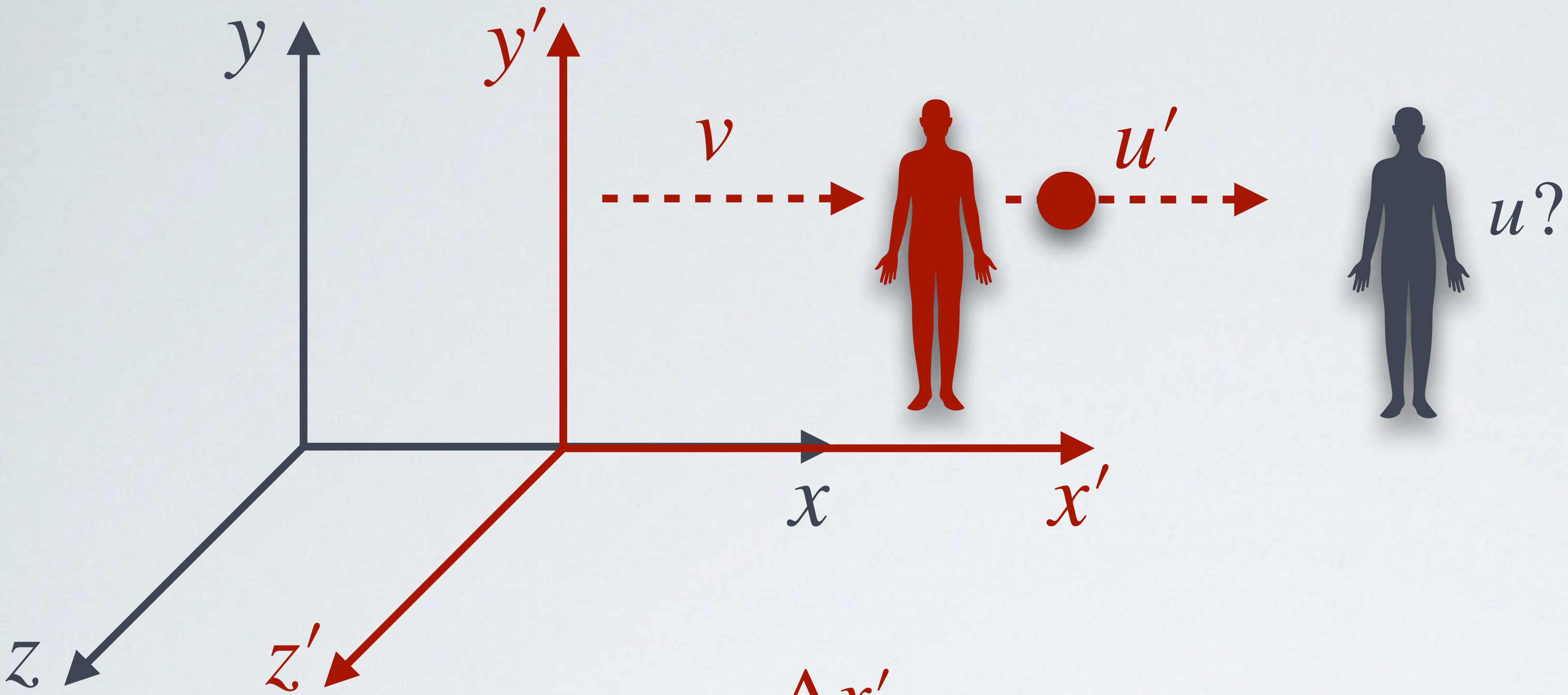
In Galilean limit:

$$v \ll c \implies \gamma \approx 1 \qquad \implies x' = x - vt \qquad \implies t' = t$$

$$\implies \frac{vx}{c^2} \approx 0$$

etc.

Velocity addition



$$u' = \frac{\Delta x'}{\Delta t'}$$

$$u = \frac{\Delta x}{\Delta t} = \frac{\gamma(\Delta x' + v\Delta t')}{\gamma\left(\Delta t' + \frac{v\Delta x'}{c^2}\right)}$$

$$= \frac{\Delta t'\left(\frac{\Delta x'}{\Delta t'} + v\right)}{\Delta t'\left(1 + \frac{v\Delta x'}{c^2\Delta t'}\right)} = \frac{u' + v}{1 + \frac{u'v}{c^2}}$$

Lorentz transformation:

$$x = \gamma(x' + vt')$$

$$t = \gamma\left(t' + \frac{vx'}{c^2}\right)$$

Velocity addition (or subtraction) formula in relativity:

$$v_{\text{tot}} = \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}}$$

Velocity addition: small and large

$$v_{\text{tot}} = \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}}$$

$$v_1 \ll c\,,\ v_2 \ll c$$

$$\implies v_1 v_2 \ll c^2$$

$$\implies v_{\text{tot}} \approx v_1 + v_2$$

If both velocities are much smaller than c, we recover the Galilean transformation.

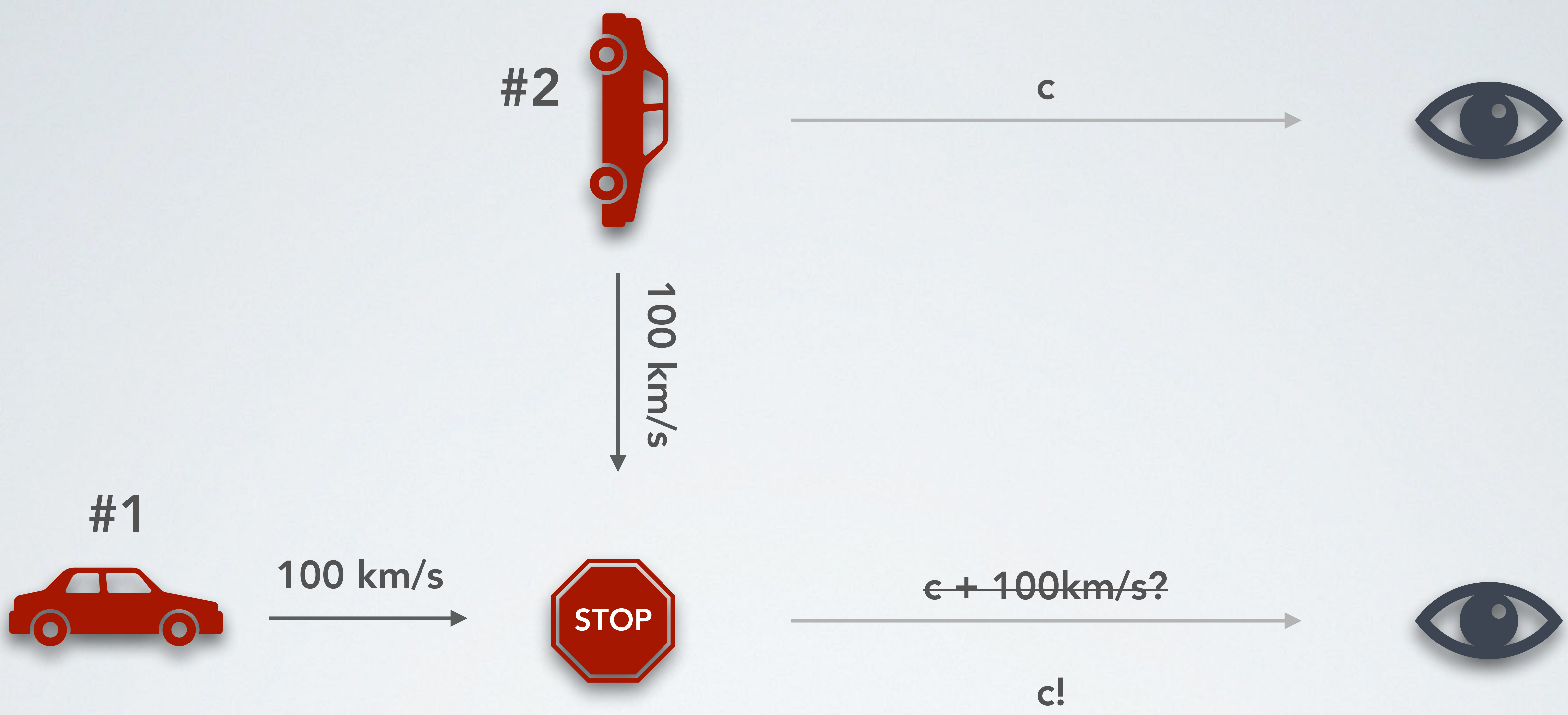
$$v_1 = c$$

$$\implies v_{\text{tot}} = \frac{c + v_2}{1 + \frac{v_2}{c}}$$

$$= c\frac{1 + \frac{v_2}{c}}{1 + \frac{v_2}{c}} = c$$

Adding any velocity to c gives c, meaning the speed of light is constant in any frame

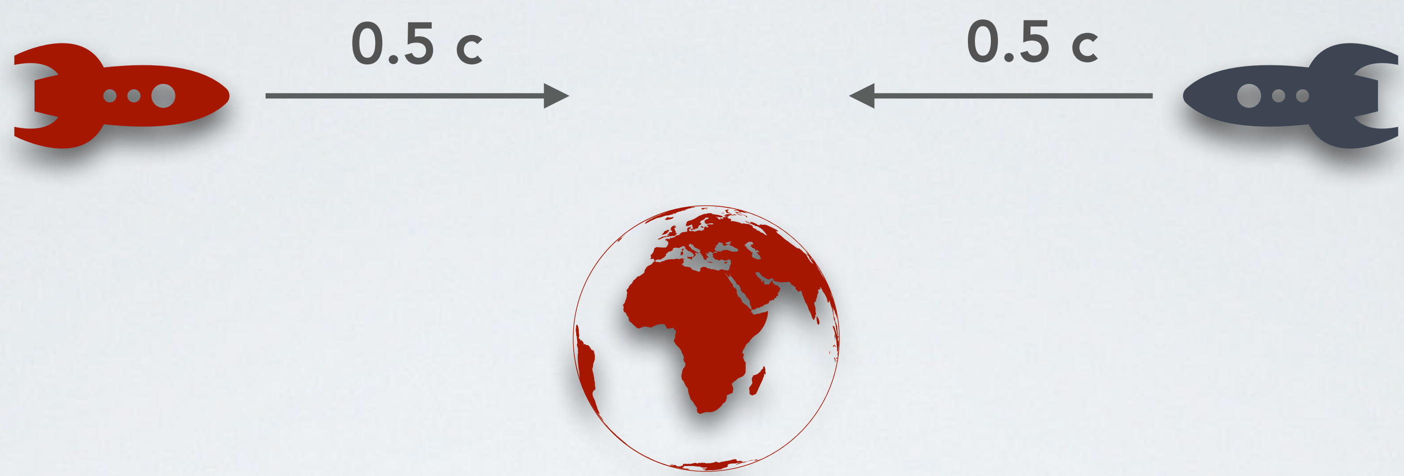
Thought experiment: Collision or not?



Velocity addition: small and large

- Speed of light is **speed limit**: nothing can move faster in any frame of reference
  - It's not just light though: any massless particle
- Thought experiment: if the speed of light was infinite…
  - Galilean relativity would be correct
  - Newton's forces and electromagnetism would "act at a distance" instantaneously

Participation: Velocity addition #1



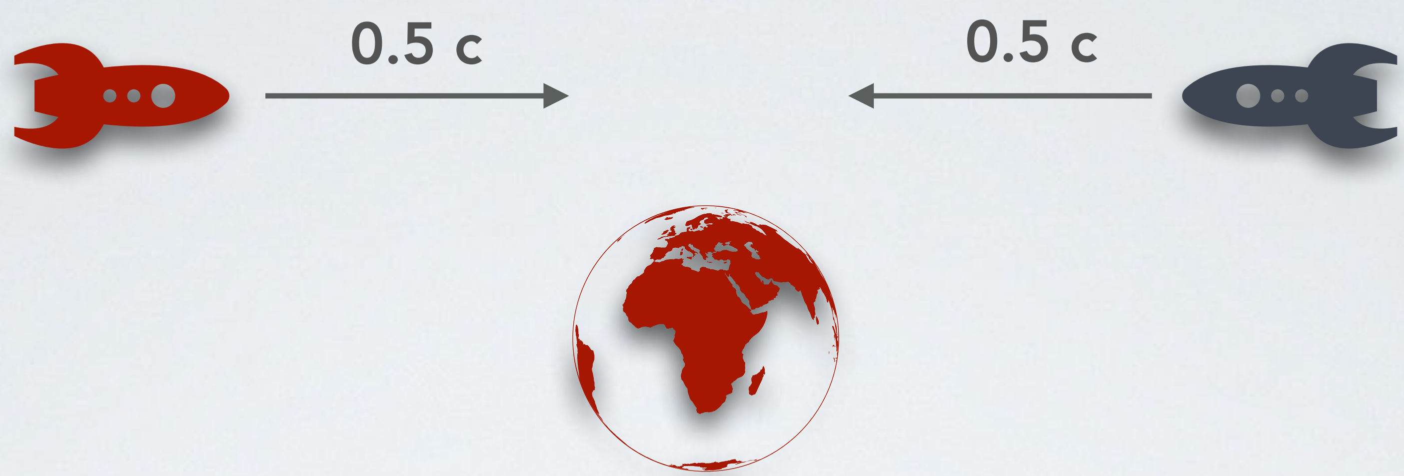
Two spaceships are going 0.5c each towards each other, as seen from Earth. One sends a radio signal. **What is the speed of the signal as seen on the other spaceship?**

$$v_{\text{tot}} = \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}}$$

Session ID: diemer

30 seconds

Participation: Velocity addition #2



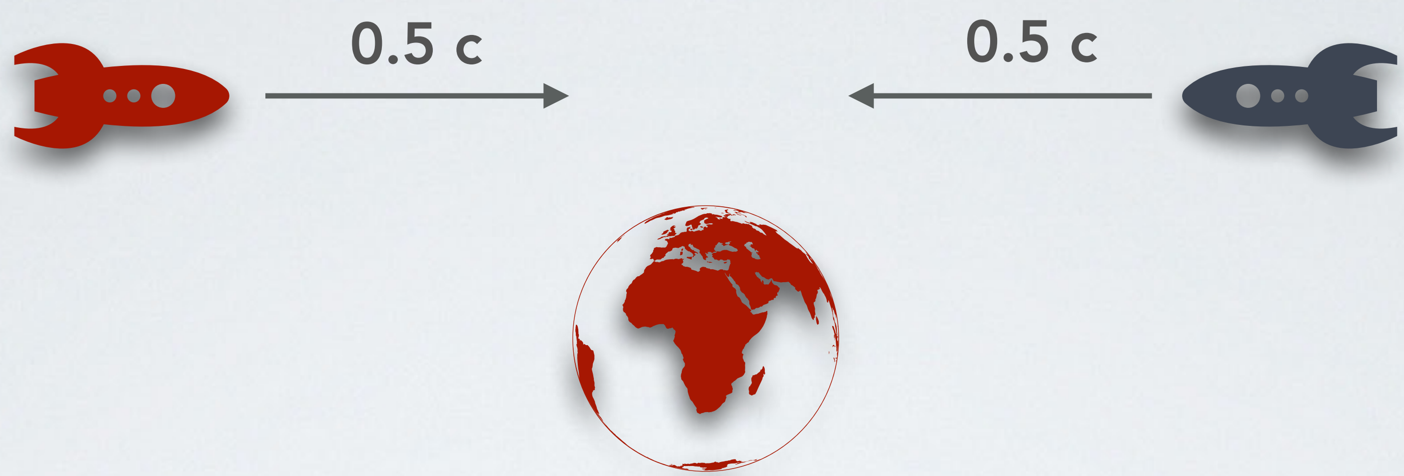
Two spaceships are going 0.5c each towards each other, as seen from Earth. **How fast do they see each other going?**

$$v_{\text{tot}} = \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}}$$

Session ID: diemer

120 seconds

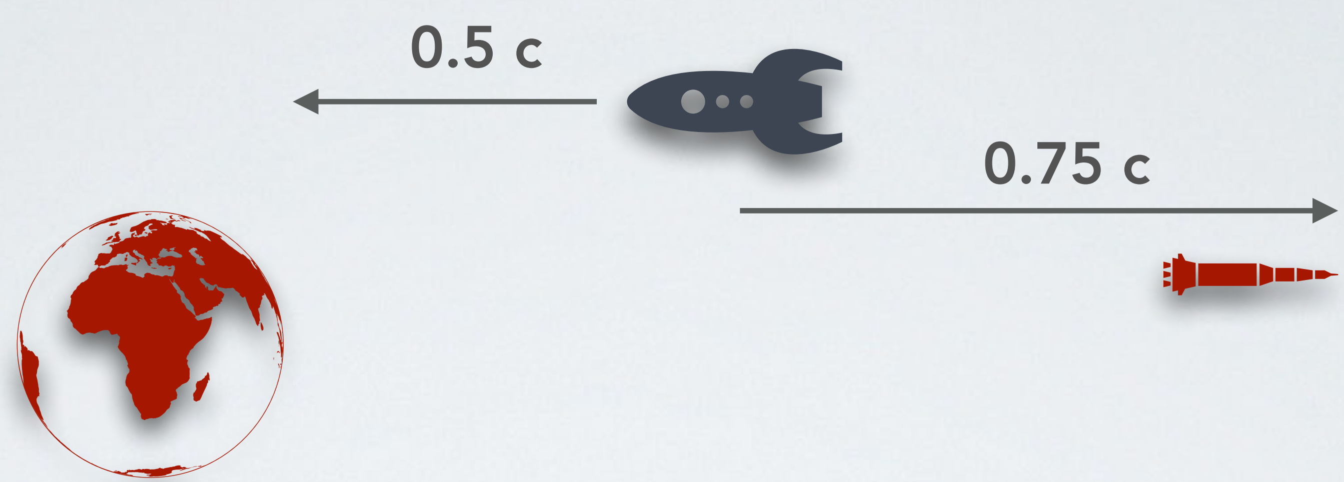
Participation: Velocity addition #2



Two spaceships are going 0.5c each towards each other, as seen from Earth. **How fast do they see each other going?**

$$v_{\text{tot}} = \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}} = \frac{\frac{1}{2}c + \frac{1}{2}c}{1 + \frac{\frac{1}{2} \times \frac{1}{2}c^2}{c^2}} = c\frac{1}{1 + \frac{1}{4}} = c\frac{1}{\frac{5}{4}} = \frac{4}{5}c = 0.8c$$

Participation: Velocity addition #3



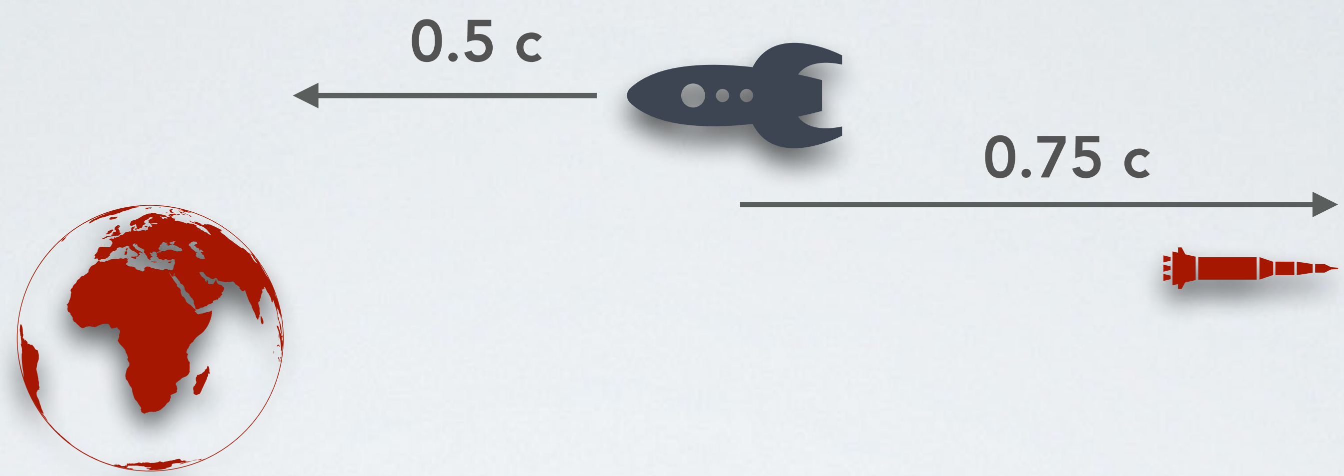
A spaceship is going 0.5c towards Earth and shoots a rocket at 0.75c in the opposite direction. **How fast does the rocket travel as seen from Earth?**

$$v_{\text{tot}} = \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}}$$

Session ID: diemer

120 seconds

Participation: Velocity addition #3



A spaceship is going 0.5c towards Earth and shoots a rocket at 0.75c in the opposite direction. **How fast does the rocket travel as seen from Earth?**

$$v_{\text{tot}} = \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}} = \frac{\frac{1}{2}c - \frac{3}{4}c}{1 + \frac{\frac{1}{2} \times -\frac{3}{4}c^2}{c^2}} = c\frac{-\frac{1}{4}}{1 - \frac{3}{8}} = c\frac{-\frac{1}{4}}{\frac{5}{8}} = -\frac{8}{20}c = -0.4c$$

Take-aways

- Events that are **simultaneous** in one frame may not be in another
- **Time** is seen to progress slower by a factor of γ in moving frames (as compared to proper time in rest frame)
- **Lengths** are seen as contracted by a factor of γ in moving frames (compared to proper length in rest frame)
- **Velocities** are not simply added, and the combined velocity can **never exceed the speed of light**

Next time…

We’ll talk about:

- Causality, twin paradox, energy

Assignments

- Post-lecture quiz (by tomorrow night)
- Homework #1 (Thursday tonight)

Reading:

- H&H Chapter 7