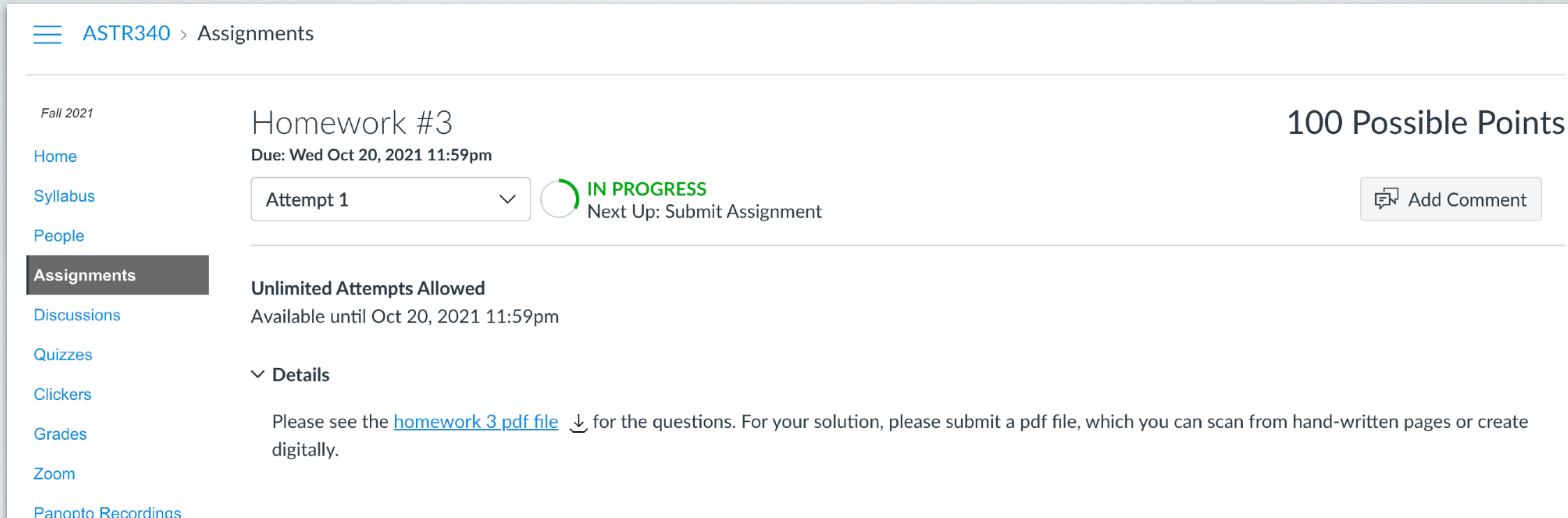
ASTR 340: Origin of the Universe

Prof. Benedikt Diemer

Lecture 13 • Density is destiny

10/12/2021

Homework 3



- Due Wednesday 10/20!

Homework 3

Question 1: Redshift, in all kinds of flavors [28 points]

Part a) Redshift is a somewhat generic term for changes in the frequency of light. In your own words, explain the causes for, and the differences between, classical (Doppler) redshift, relativistic redshift, gravitational redshift, and cosmological redshift. [12 points]

Part b) You are speeding down the highway at 0.6 c, so fast that you don't even notice the lurking police cruiser on the side of the road. The cops (now behind you) turn on their blue lights, which have a wavelength of 450 nanometers (where 1 nm $= 10^{-9}$ m $= 10^{-7}$ cm). What wavelength do you perceive if you look back? Explain why you (unfortunately) fail to notice the cops' lights. (Hint: convert wavelength to frequency to use the formula we learned in class.) [8 points]

Office Hours (Wednesday)

Zoom only this Wednesday!

Midterm

- **Logistics**
  - Thursday 10/21 in class
  - 1:15
  - No collaboration!!!
  - No plagiarism, of course
  - Express yourself in your own words
- **How to practice**
  - Come to review lecture (Tuesday 10/19)
  - Make sure you understand all the homework problems and quiz questions

Part 0: Recap

Connecting redshift and scale factor

Definition of redshift: $$z = \frac{\lambda_{\rm obs} - \lambda_{\rm em}}{\lambda_{\rm em}}$$

Cosmological expansion: $$\lambda_{\rm obs} = \frac{a_{\rm obs}}{a_{\rm em}} \lambda_{\rm em}$$

$$\implies z = \frac{a_{\rm obs}}{a_{\rm em}} - 1$$

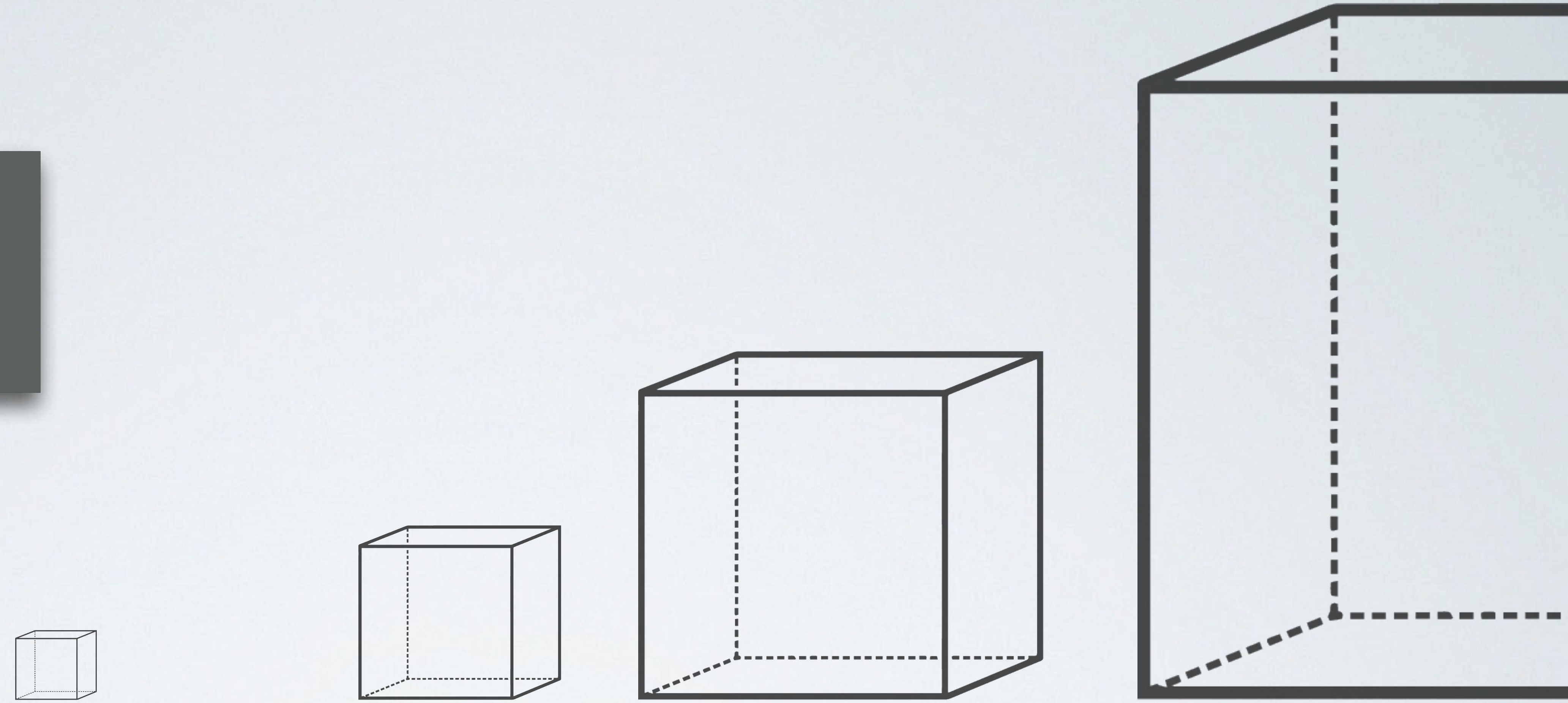
Define scale of Universe today: $R_{\rm obs} = R_0 = R(t_0) = R(\text{today})$

Define a relative to today: $a(t) = R(t)/R(t_0)$

$$\implies z = \frac{1}{a} - 1 \implies a = \frac{1}{1+z}$$

Connecting redshift and scale factor

$$z = \frac{1}{a} - 1$$



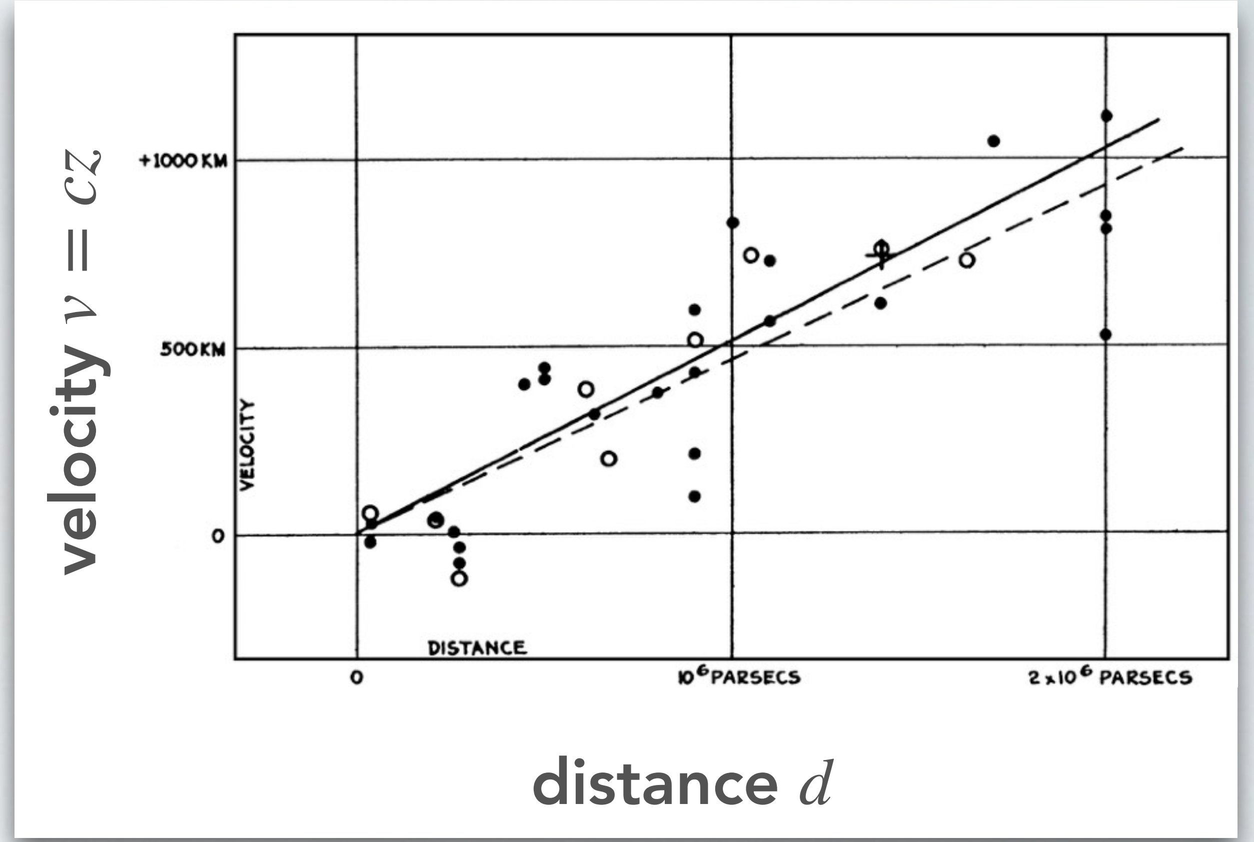
$a = 0$	$a = 0.2$	$a = 0.5$	$a = 1$	$a = 2$
$z = \infty$	$z = 4$	$z = 1$	$z = 0$	$z = -0.5$

Big Bang ← today → future

Connection to Hubble rate

$$H(t) = \frac{1}{a(t)} \frac{\mathrm{d}a(t)}{\mathrm{d}t}$$

- The Hubble rate is the **fractional rate** at which the Universe expands **per time!** The **Hubble constant** is $H_0 = H(t_0)$, the **expansion rate today**



$$v = H_0 \times d$$

Participation: Recap #1

TurningPoint:
What is the Hubble time, given a measured Hubble rate H?

Session ID: diemer

30 seconds

How old is the Universe?

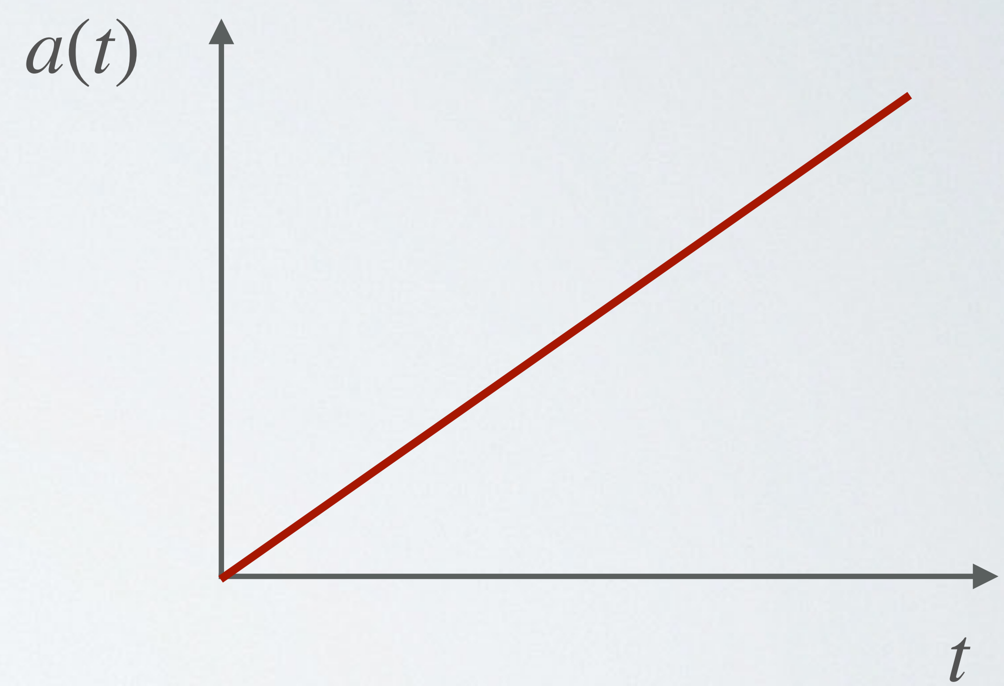
$$H(t) = \frac{1}{a(t)}\frac{\mathrm{d}a(t)}{\mathrm{d}t}$$

- Imagine a Universe that is expanding at a constant rate
- $a(t) = t \times da/dt$

$$\Longrightarrow \quad H(t) = \frac{1}{t}$$

- With this constant expansion, the **relative rate** of expansion **decreases**
- We define the Hubble time, the time it would take to get to a certain expansion rate H if the rate is constant:

$$t_{\mathrm{H}} = \frac{1}{H(t)}$$



- What is this time today?
- If the Universe had been expanding at the same rate for its entire life, it would be about 13.8 Gyr old

$$t_{\mathrm{H},0} = \frac{1}{H_0} \approx \frac{1}{0.07/\mathrm{Gyr}} \approx 13.8\ \mathrm{Gyr}$$

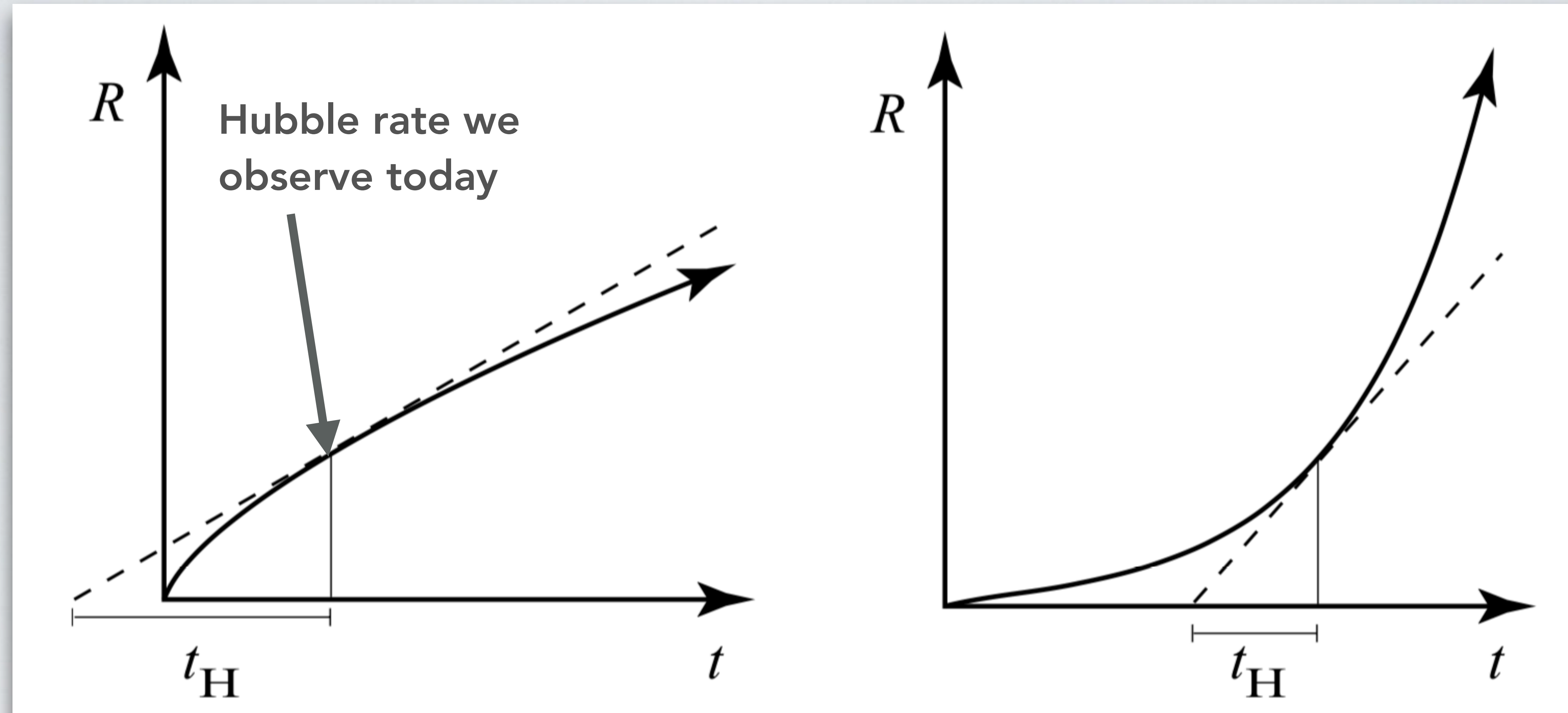
Participation: Recap #2

TurningPoint:
Imagine we observe the distance to some galaxy to be 10 Mpc one year, and 10.1 Mpc the next year. How old would we think the Universe is, assuming the expansion rate is constant?

Session ID: diemer

60 seconds

How old is the Universe?



Hawley & Holcomb

Participation: Recap #3

TurningPoint:
What is the Friedmann-Lemaitre-Robertson-Walker metric?

Session ID: diemer

30 seconds

Participation: Recap #4

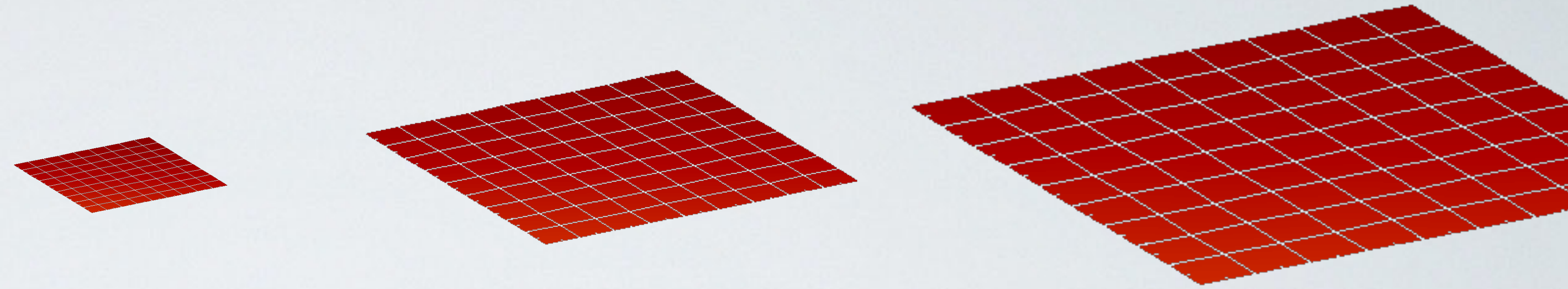
TurningPoint:
Which geometries are allowed in an FLRW metric?

Session ID: diemer

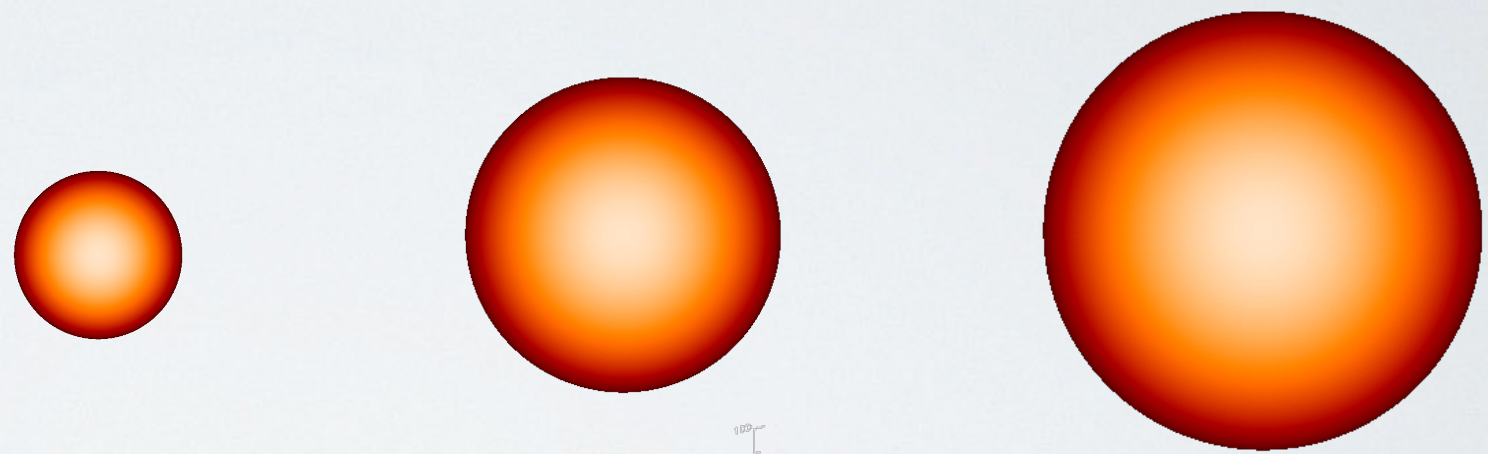
30 seconds

Geometry of space

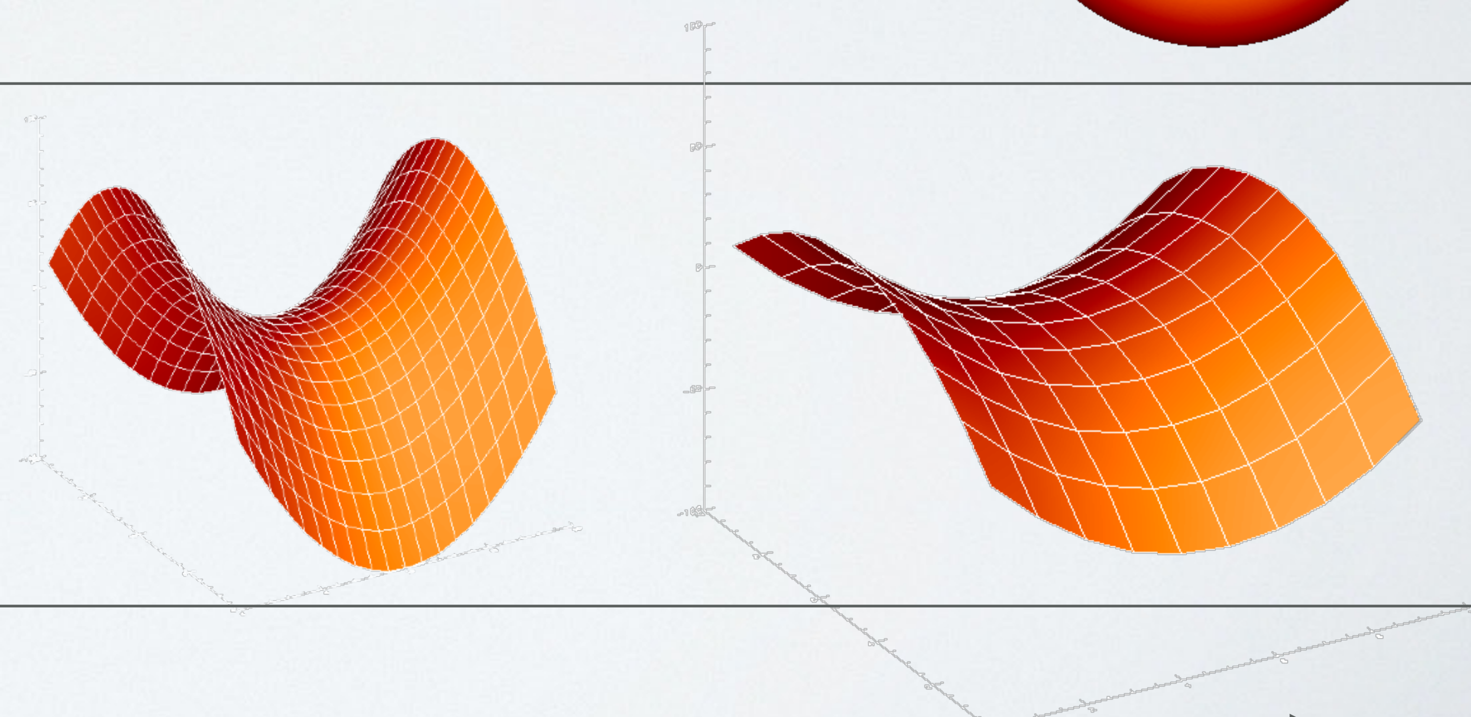
$k = 0$
Flat space



$k > 0$ ($k = +1/R^2$)
Positively curved



$k < 0$ ($k = -1/R^2$)
Negatively curved (hyperbolic)



small R → large R

Today

- Big Bang Cosmology
- Density is Destiny
- Velocities

Part 1: Big Bang Cosmology

The big question

- The big question:
  - **What is the expansion history, a(t)**, of the Universe?
  - What does it **depend** on?
  - What is the **geometry**? Flat, positively, or negatively curved?
- Answer: **Solve the equations of General Relativity** for the Universe!

Einstein field equation

Newton's gravitational constant

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

"Curvature tensor" that describes the curvature of 4D space-time

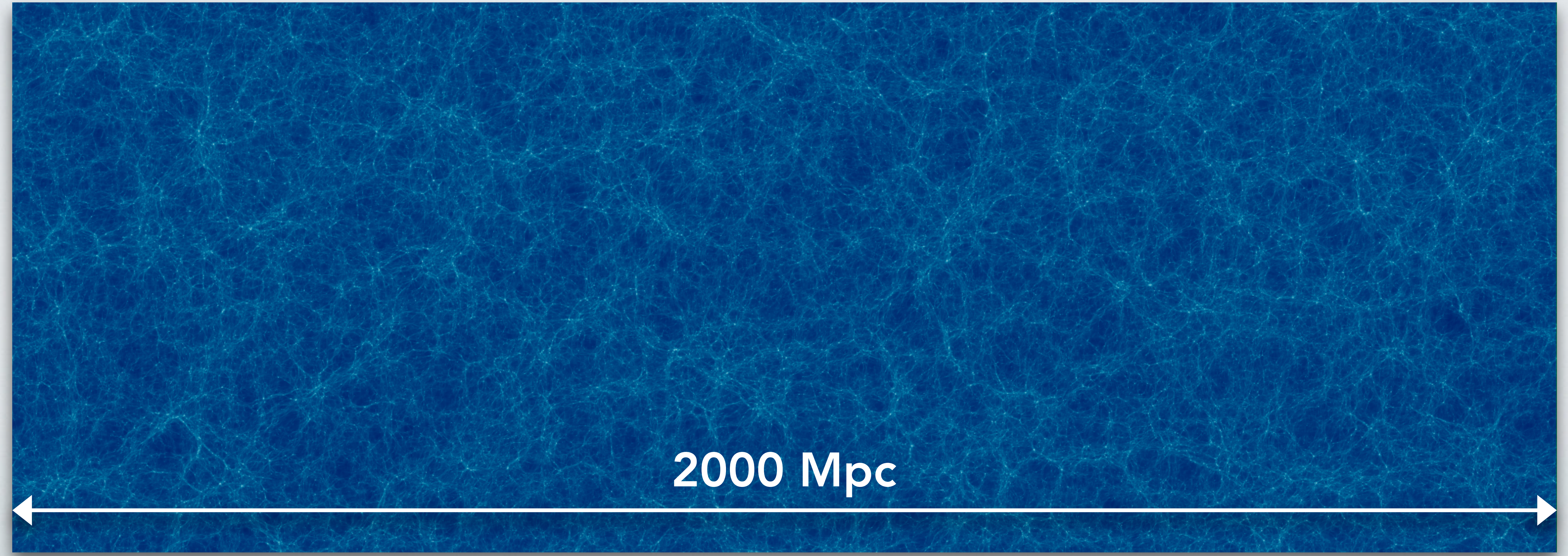
speed of light

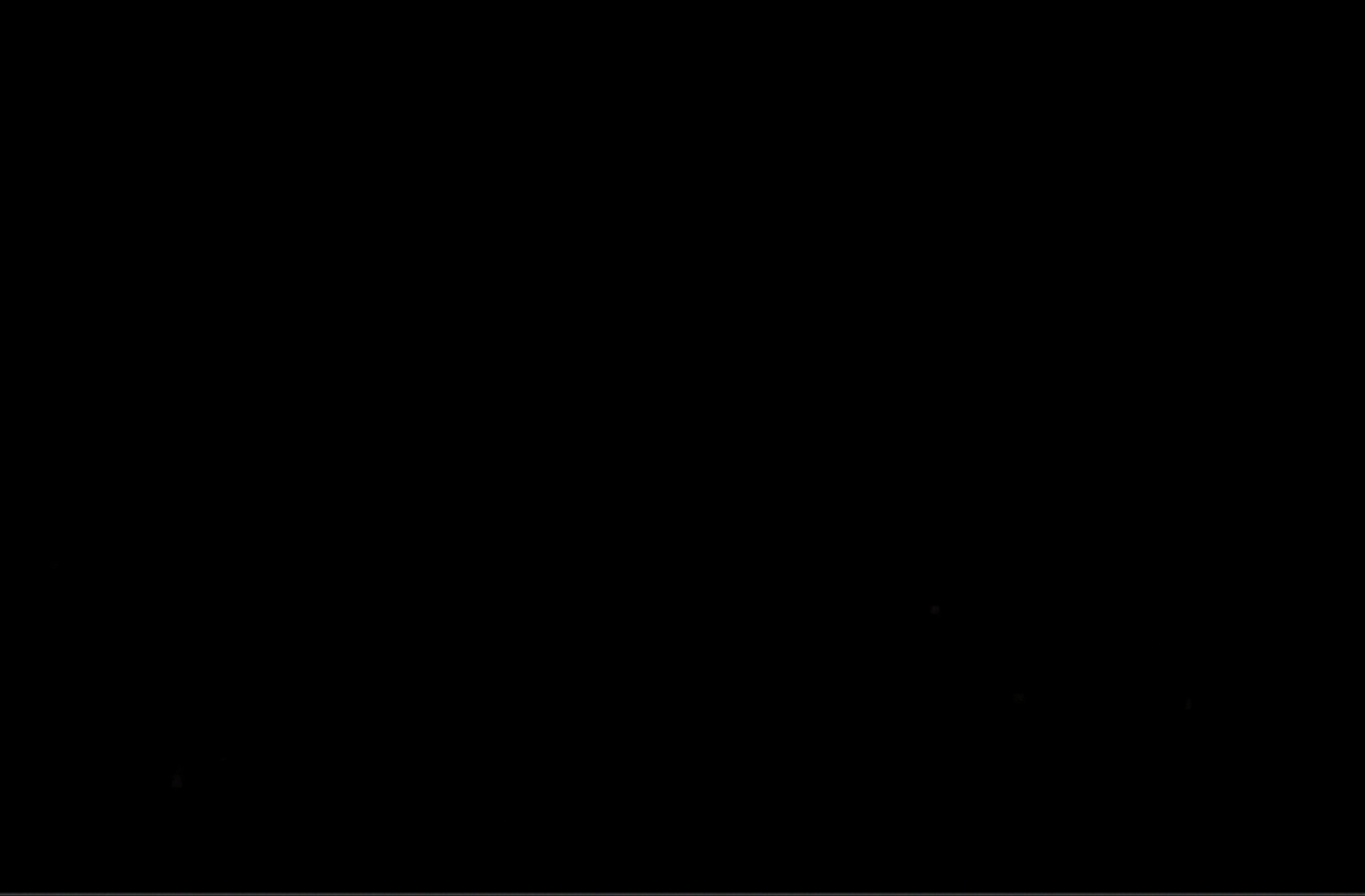
"Stress-energy tensor" that describes distribution of mass and energy

- Geometry = constant * (matter + energy)
  - $G_{\mu\nu}$ and $T_{\mu\nu}$ can be written in terms of components, similar to a matrix
  - Horrendous set of 10 coupled equations!
- $G_{\mu\nu}$ must correspond to **FLRW metric** (with or without curvature)
- $T_{\mu\nu}$ represents all the **mass and energy in the Universe**

Symmetry in the cosmos

- **Cosmological principle**:
  - the Universe **has no center** and is thus….
  - **isotropic**: roughly the same in all directions
  - **homogeneous**: roughly the same in all locations

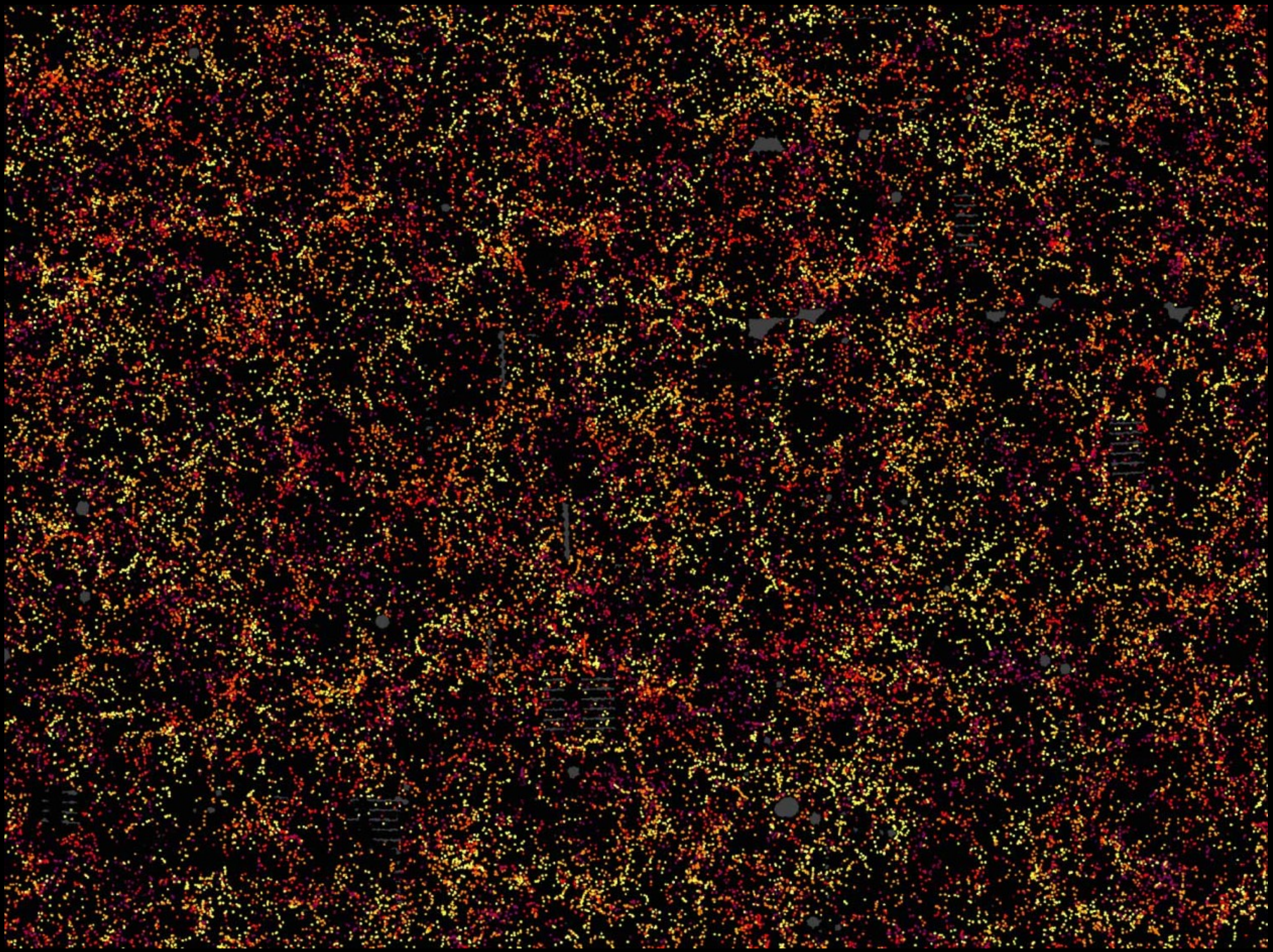




SDSS / WMAP collaborations, Dinoj Surendran, Mark SubbaRao

Assumptions

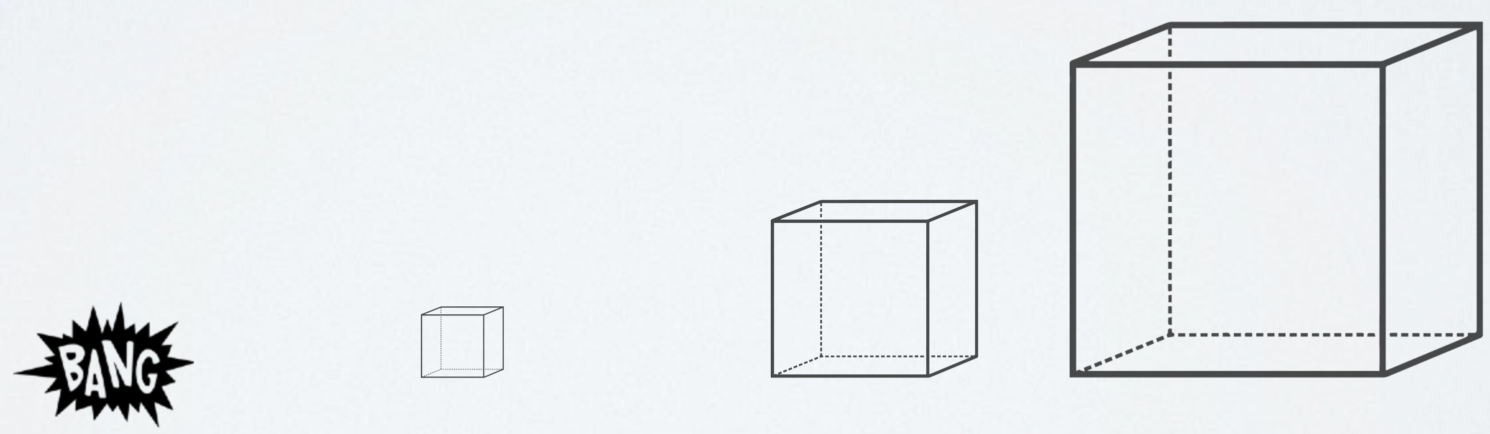
- Assume that Universe is **homogeneous and isotropic**
- Ignore the details (galaxies, structure…)



Sloan Digital Sky Survey / BOSS

Big Bang Cosmology

- Assume that the Universe is smooth on large scales (**homogeneous & isotropic**)
- Assume that the Universe starts from a **Big Bang**
  - The scale factor starts at **a(0) = 0**
  - This happens **everywhere in space**: there is **no "location"** to the Big Bang
  - There is **no space or time before the Big Bang**



$a = 0$	$a = 0.2$	$a = 0.5$	$a = 1$
$z = \infty$	$z = 4$	$z = 1$	$z = 0$
$t = 0$			$t = t_0$

Part 2: Density is Destiny

Participation: Escape velocity

TurningPoint:
When you throw a ball up in the air, is there a speed where it would totally escape the Earth's gravity?

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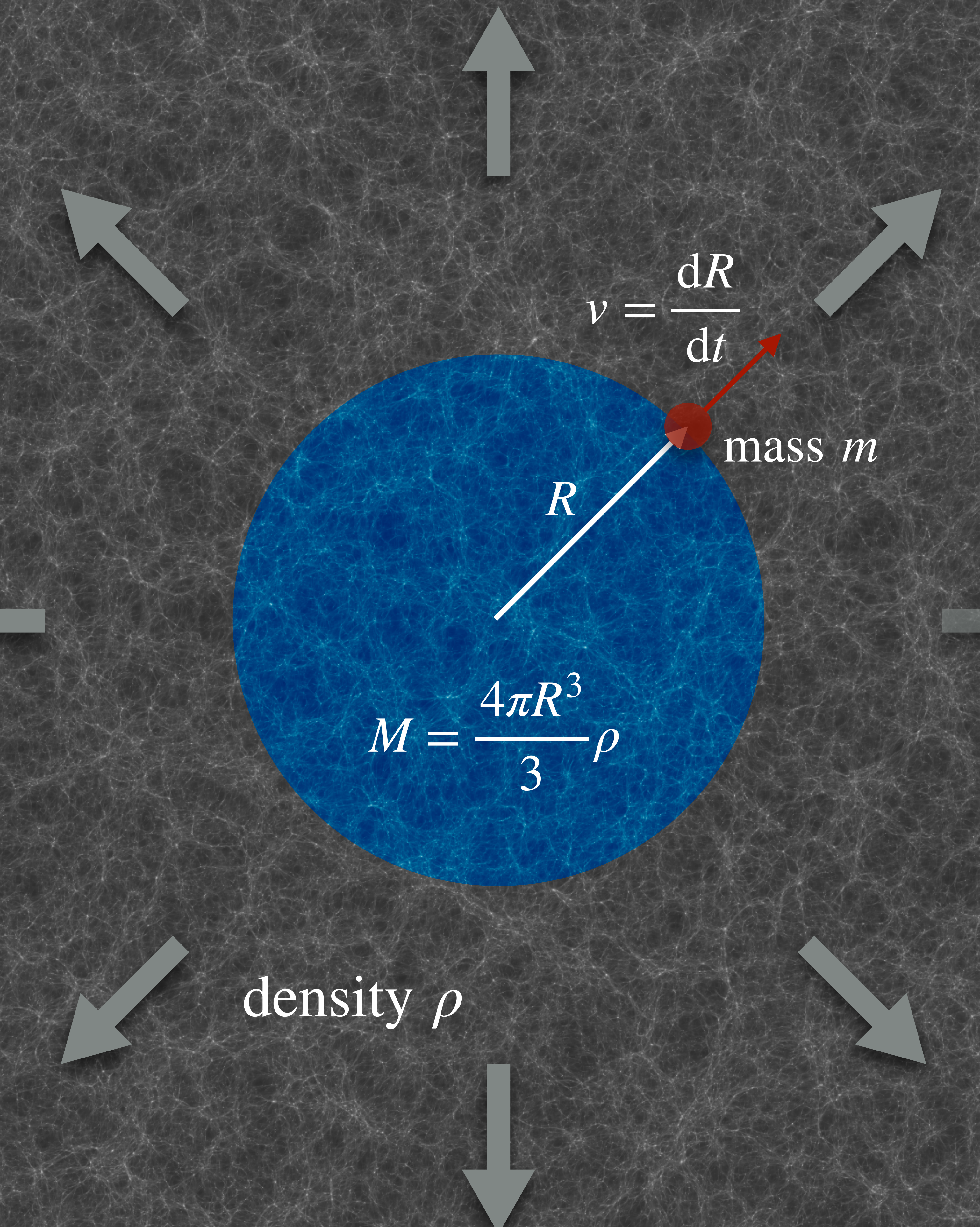
30 seconds



$$E_{\text{kin}} = \frac{1}{2}mv^2$$

$$E_{\text{grav}} = \frac{GMm}{R}$$

$$\implies v_{\text{esc}} = \sqrt{\frac{2GM}{R}}$$

We can think of the velocity as the escape velocity +/- some constant: either the particle escapes (positive constant) or it falls back (negative constant)

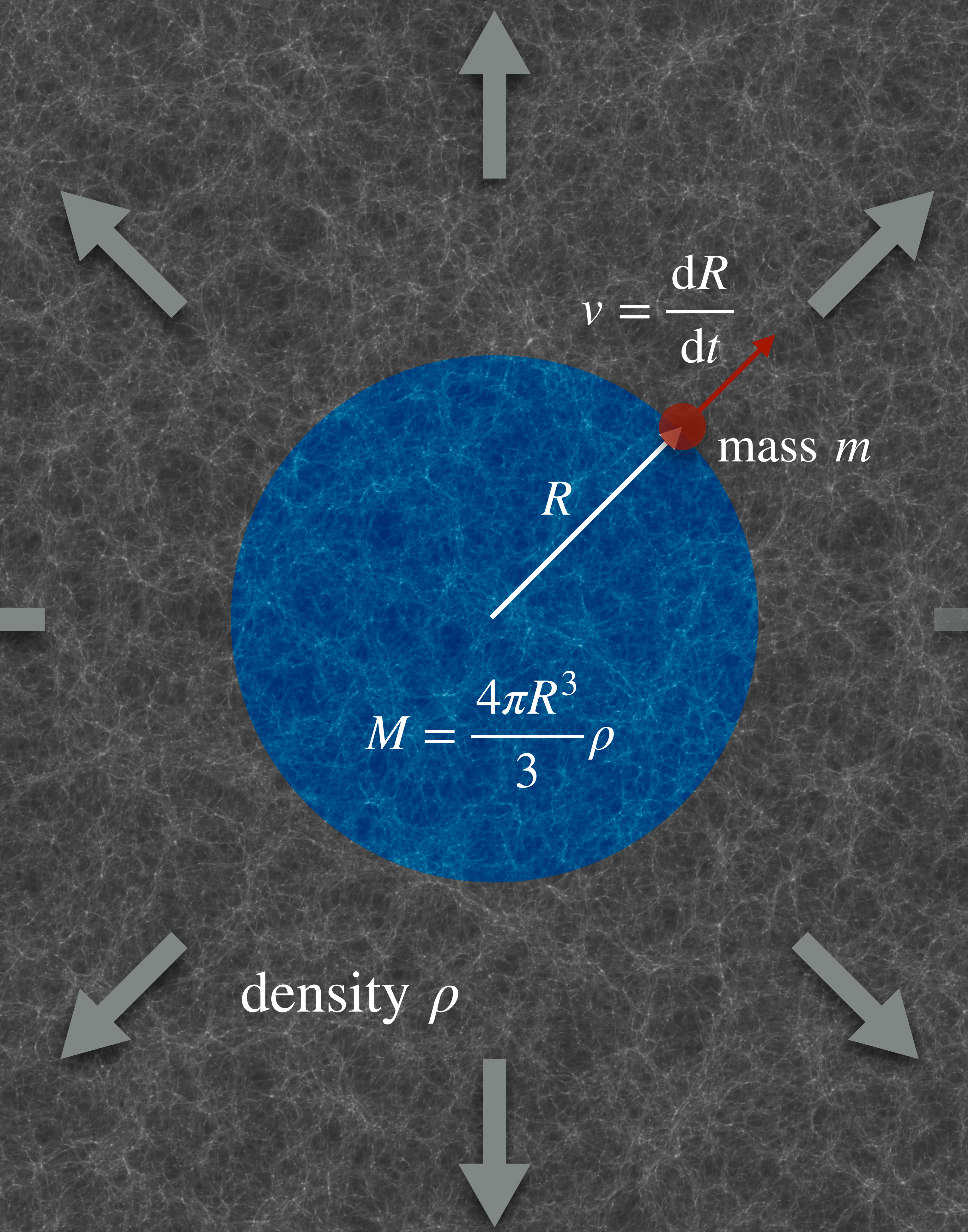
$$v^2 = \left(\frac{\mathrm{d}R}{\mathrm{d}t}\right)^2 = \frac{2GM}{R} + \text{const}$$

$$\implies \left(\frac{\mathrm{d}R}{\mathrm{d}t}\right)^2 = \frac{8\pi G R^3 \rho}{3R} + \text{const}$$

$$\implies \boxed{\left(\frac{1}{R}\frac{\mathrm{d}R}{\mathrm{d}t}\right)^2 = H^2 = \frac{8\pi G}{3}\rho + \frac{\text{const}}{R^2}}$$

Note: here, R is a radius! But it cancels out in the end, so we can think of it as unitless

$$\left(\frac{1}{R}\frac{\mathrm{d}R}{\mathrm{d}t}\right)^2 = H^2 = \frac{8\pi G}{3}\rho + \text{const}$$



Similarly, we can work out the acceleration of the test mass:

$$F = ma = m\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{GMm}{R^2}$$

$$\implies \frac{1}{R}\frac{\mathrm{d}^2R}{\mathrm{d}t^2} = -\frac{1}{R}\frac{GM}{R^2} = -\frac{4\pi G}{3}\rho$$

Interpretation:

- The **more matter** (density) there is, the more the Universe wants to **collapse**
- The **more density** there is, the **faster** the **Hubble rate** needs to be to keep the Universe expanding
- The fate of the Universe depends on the **balance between density and expansion**

Solving the Universe in GR

"geometry = constant * (matter + energy)"

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

Homogeneous distribution of matter + energy

FLRW metric

$$\Delta s_{\text{FLRW,curved}} = \sqrt{(c\Delta t)^2 - a^2(t) \left[\frac{\Delta r^2}{1 - kr^2} + r^2(\Delta\theta^2 + \sin^2(\theta)\Delta\phi^2)\right]}$$

- Curvature tensor (geometry) must correspond to FLRW metric
- Assume that matter is uniformly distributed (homogeneous)
- This makes the Einstein equation solvable! We get the **Friedmann equation:**

$$\left(\frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}t}\right)^2 = H^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2}$$

- This version describes a Universe with **only matter and curvature**
- Recall that $k = \pm 1/R^2$, where R is the radius of curvature

Understanding the Friedmann equation

Constant term depends on gravitational constant; the more gravity, the faster the expansion has to be

Curvature also acts like an "energy density": positive curvature ($k=+1/R^2$) means slower expansion, negative curvature ($k=-1/R^2$) faster expansion (at fixed density)

Hubble rate = fractional expansion per time

$$\left(\frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}t}\right)^2 = H^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2}$$

Mean density of the Universe is a key number

This looks like curvature becomes less important as the Universe expands — but wait! Density changes too…

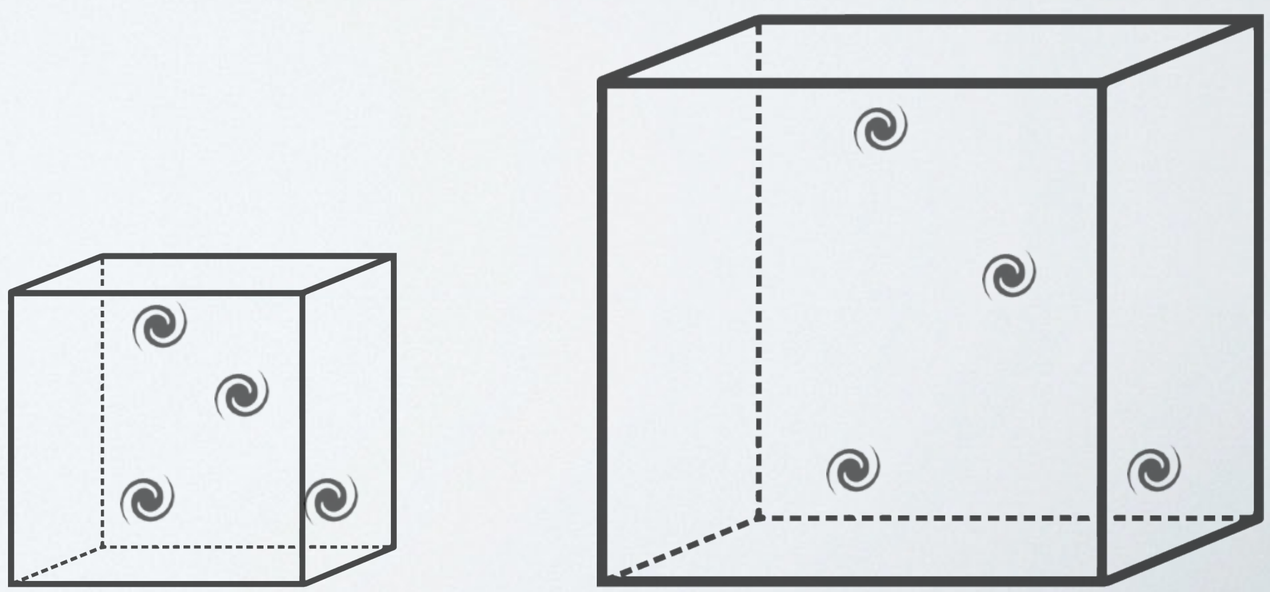
Understanding the Friedmann equation

Some terms change with time!

$$H(t)^2 = \frac{8\pi G}{3}\rho(t) - \frac{kc^2}{a(t)^2}$$

$$\rho(t) = \frac{\rho_0}{a^3(t)}$$

- Assuming that no new matter is created or destroyed, the **density of matter decreases as space expands.**
- $\rho_0 = \rho(t_0)$ is the density today



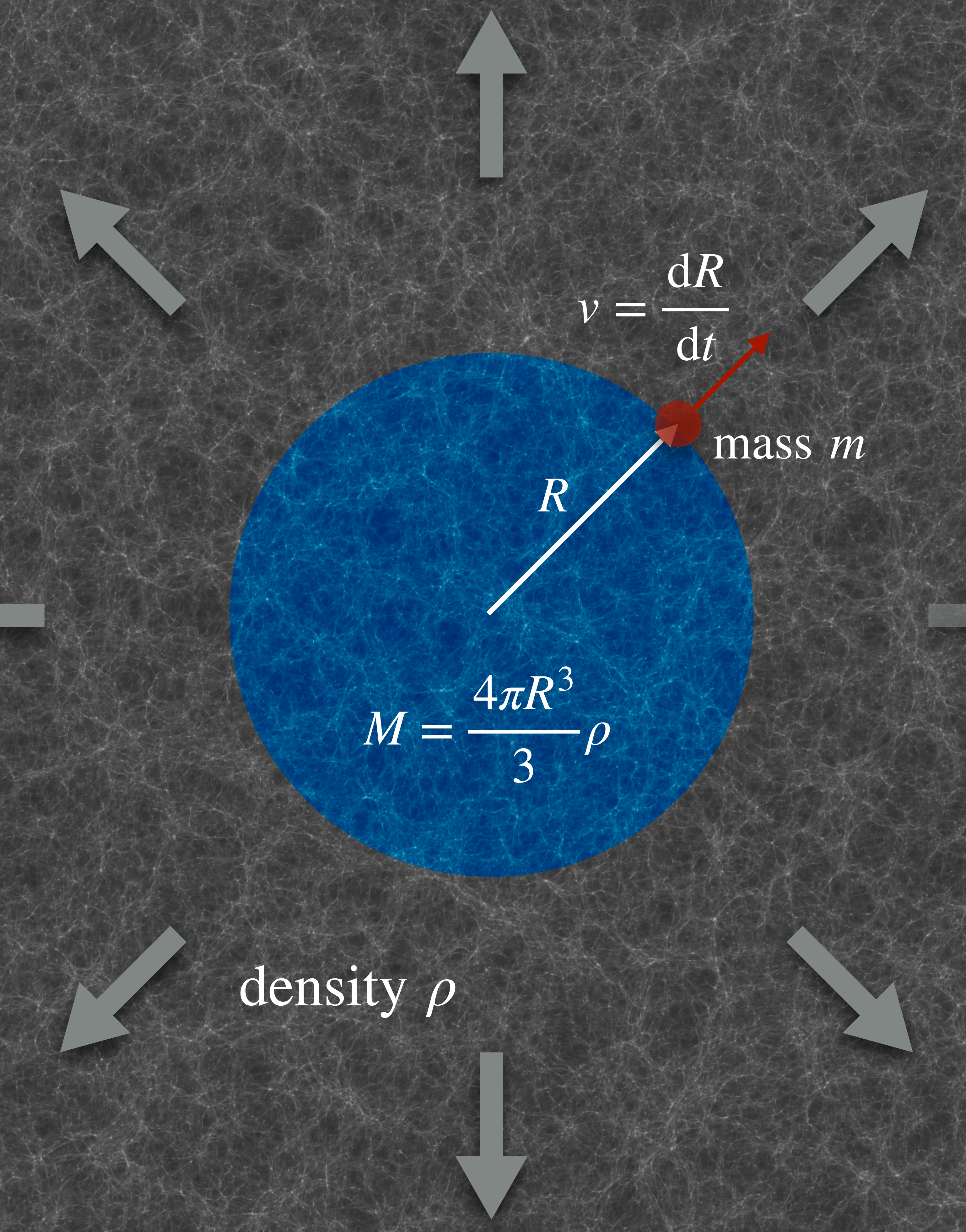
Understanding the Friedmann equation

If density is zero, curvature must be negative!

Squared, so must be positive!

$$H^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2}$$

There must be enough matter to “afford” positive curvature



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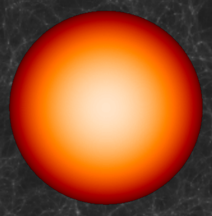
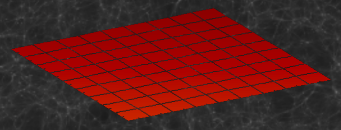
$$H^2 = \frac{8\pi G}{3}\rho + \text{const}$$

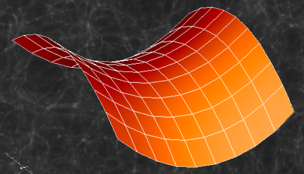
$$\Longrightarrow \quad \boxed{\text{const} = -\frac{kc^2}{a^2}}$$



positive $k \Longrightarrow v < v_{\rm esc}$



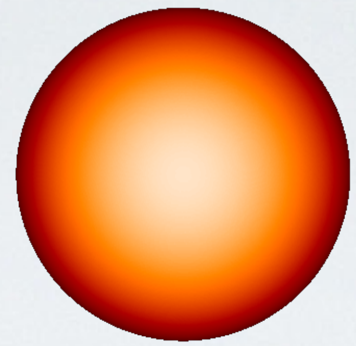
$k = 0 \Longrightarrow v = v_{\rm esc}$



negative $k \Longrightarrow v > v_{\rm esc}$

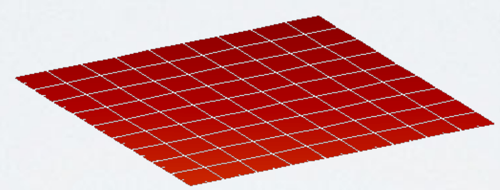
Curvature

positive $k \implies v < v_{\text{esc}}$



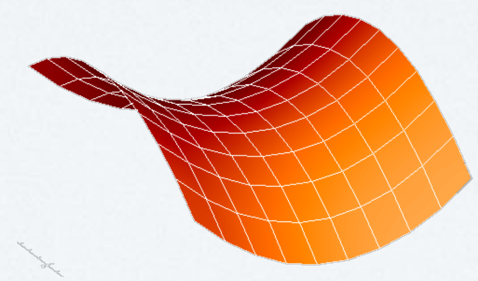
Universe collapses eventually?

$k = 0 \implies v = v_{\text{esc}}$



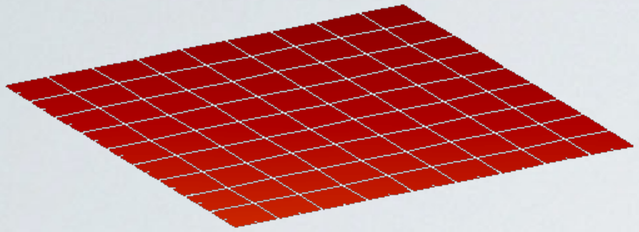
Universe slowly grinds to a halt?

negative $k \implies v > v_{\text{esc}}$



Universe expands forever?

Flat Universe



$$H^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} \quad \overset{k=0}{\Longrightarrow} \quad H^2 = \frac{8\pi G}{3}\rho$$

$$\rho_c(t) \equiv \frac{3H^2(t)}{8\pi G}$$

- The Universe is **flat** if the density is the **critical density,** ρ_c
- In a flat Universe, the Hubble rate is fully determined by the density and vice versa
- It makes sense to express the density as the Hubble rate because that is easiest to observe

Critical density

$$H^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2}$$

$$\rho_c(t) = \frac{3H^2(t)}{8\pi G}$$

1) Divide by H²:

$$1 = \frac{8\pi G}{3H^2}\rho - \frac{kc^2}{a^2H^2}$$

$$= \frac{\rho}{\rho_c} - \frac{kc^2}{a^2H^2}$$

2) Define density parameter as fraction of critical density:

$$\Omega_m(t) \equiv \frac{\rho(t)}{\rho_c(t)}$$

3) Similarly define "density equivalent" for curvature:

$$\Omega_k(t) \equiv -\frac{kc^2}{a^2H^2}$$

4) Write total content of the Universe as:

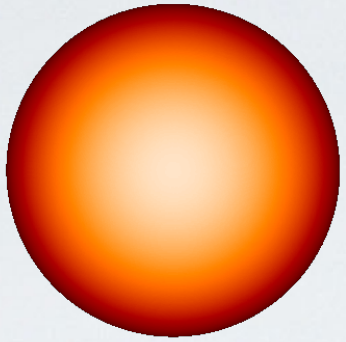
$$\Omega_{tot} \equiv \Omega_m + \Omega_k = 1$$

In a Universe with only matter, density determines the geometry!

The fate of the Universe (with matter and curvature)

Case 1: Closed Universe

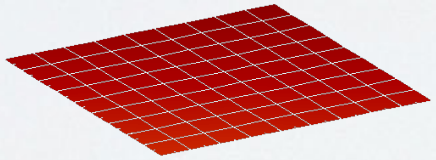
$$\Omega_m > 1 \implies \Omega_k < 0$$
$$\implies k > 0$$



Collapses eventually (big crunch)

Case 2: Flat Universe

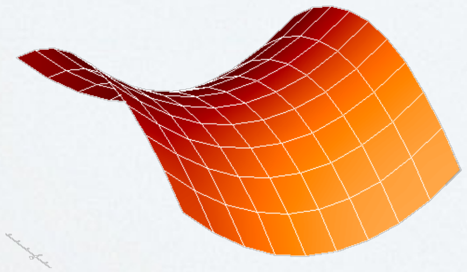
$$\Omega_m = 1 \implies \Omega_k = 0$$
$$\implies k = 0$$



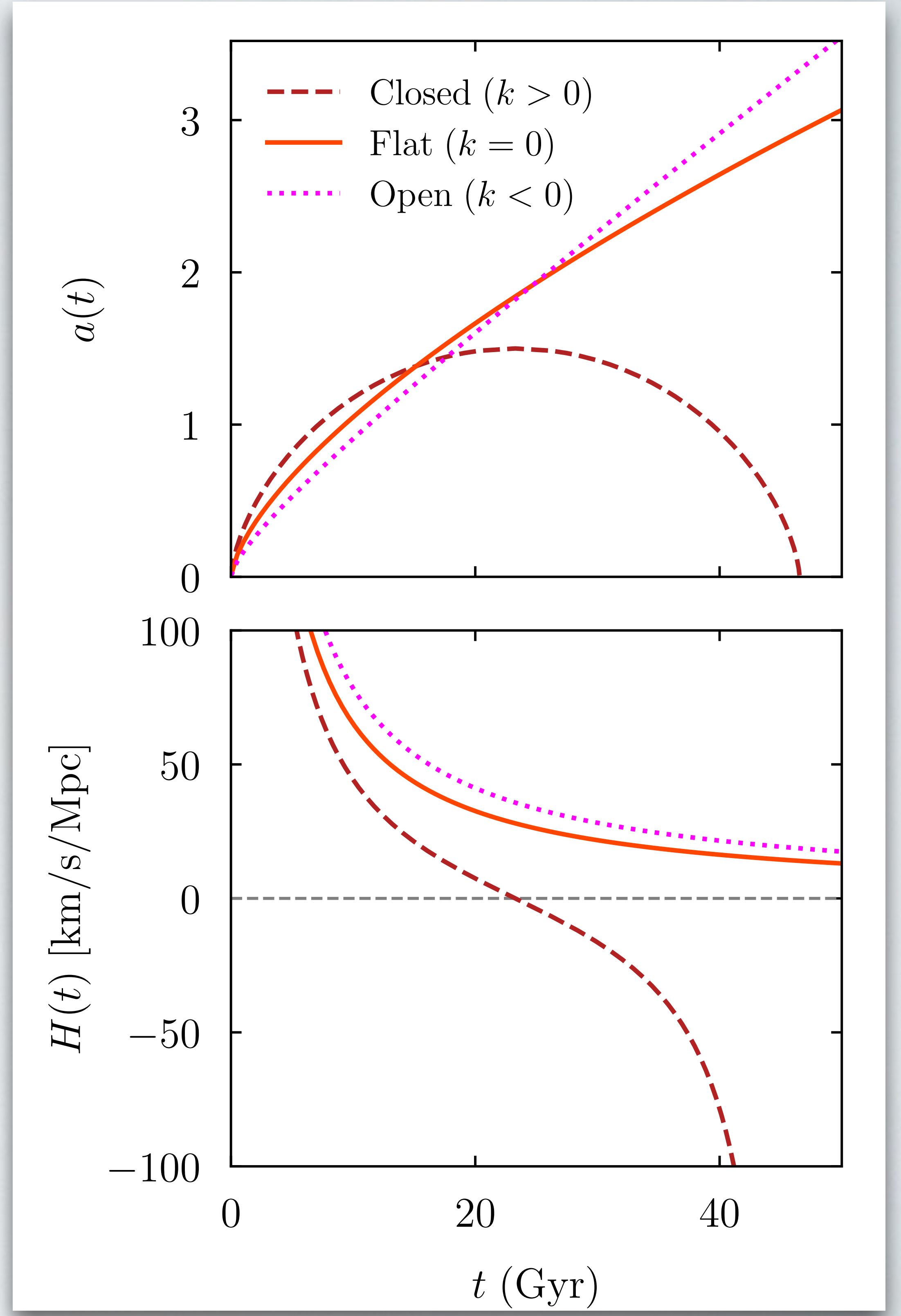
Slowly grinds to a halt

Case 3: Open Universe

$$\Omega_m < 1 \implies \Omega_k > 0$$
$$\implies k < 0$$



Expands forever



Density is Destiny

Critical density

What is the **critical density today?**

$$H_0 \approx \frac{70 \text{ km}}{3.1 \times 10^{19} \text{ km}} \frac{1}{\text{s}} \approx \frac{2.3 \times 10^{-18}}{\text{s}}$$

$$\rho_{\text{c},0} = \frac{3H_0^2}{8\pi G} = \frac{3 \times (2.3 \times 10^{-18}/\text{s})^2}{8\pi \times 6.7 \times 10^{-8} \text{ cm}^3/g/\text{s}^2} \approx 10^{-29} \frac{g}{\text{cm}^3}$$

$$\approx \frac{6 \text{ protons}}{\text{m}^3}$$

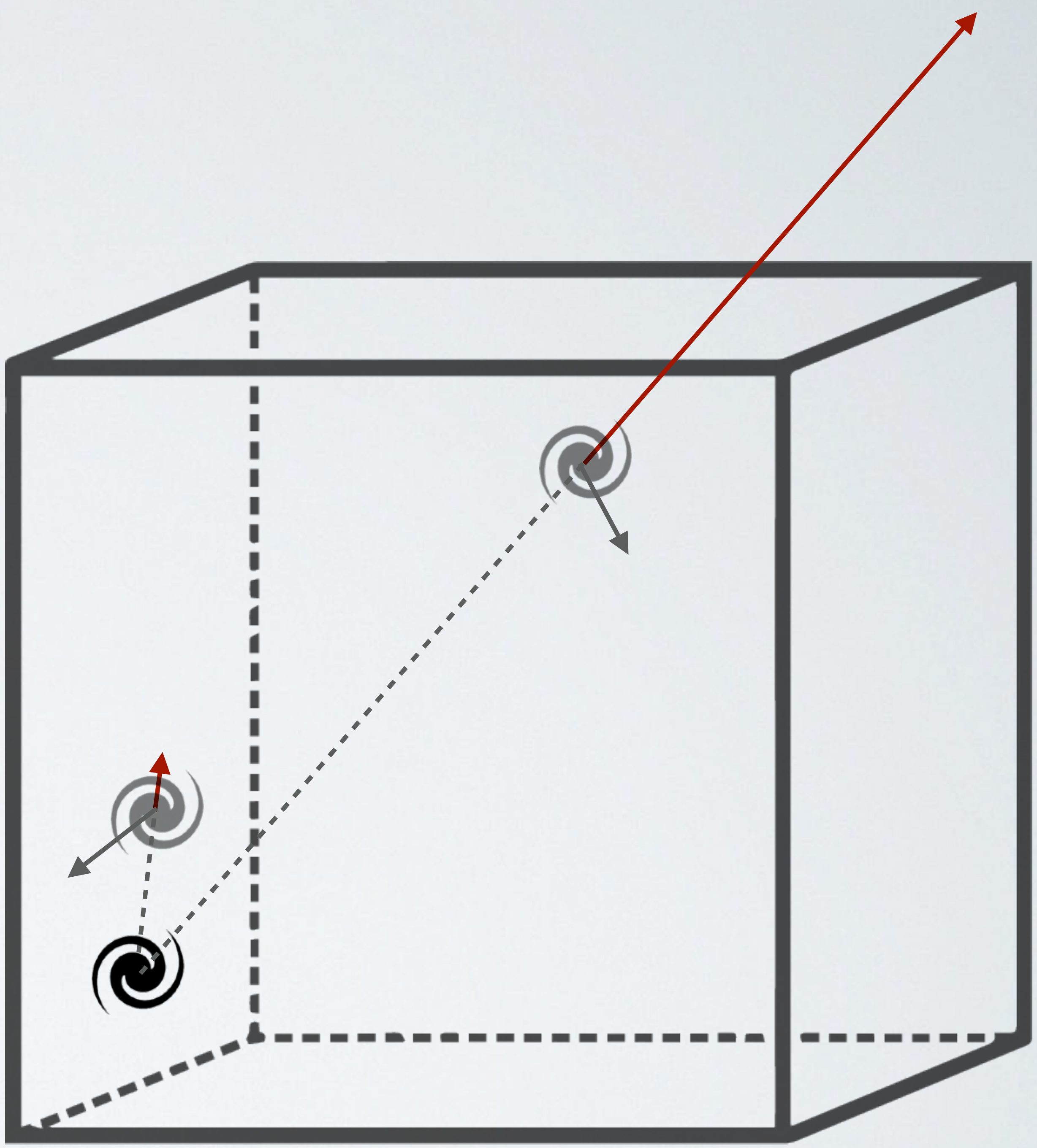
$$\approx \frac{30000 \text{ tons}}{\text{AU}^3}$$

In reality, the mean matter density of the Universe is less than the critical density — we will see that later.

Part 3: Can galaxies break the speed of light?

Peculiar velocities

- Of course, galaxies are not precisely fixed in space
- They have local random motions, called **peculiar velocities**
- This is the reason that observational Hubble law is not exact straight line but has scatter
- Random velocities do not increase with separation
- Cosmological (Hubble) recession velocity does increase with separation
- Thus, **recession velocity dominates at large distances** (or high redshifts)



Participation: Recession velocity

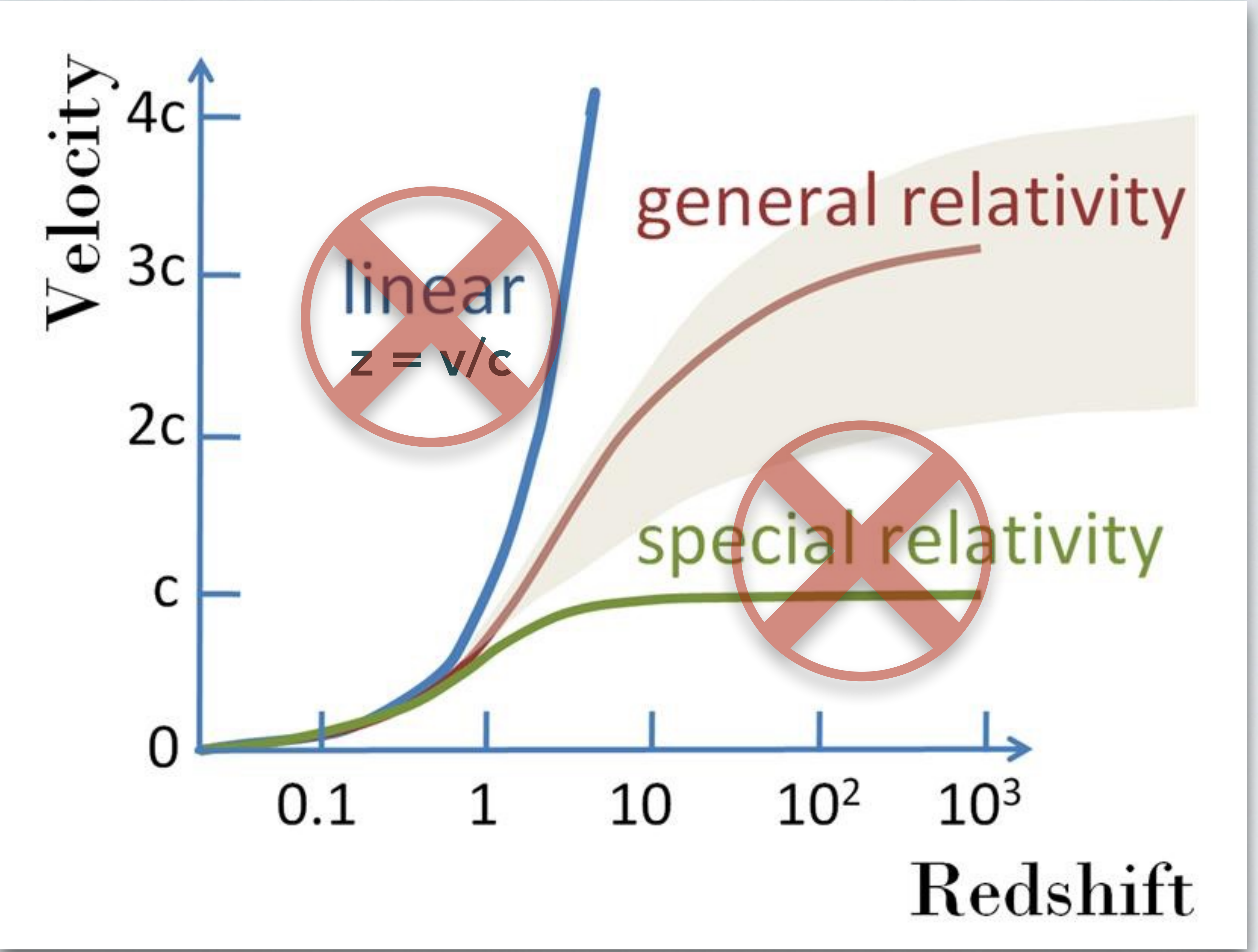
TurningPoint:

Can galaxies recede faster than the speed of light?

Session ID: diemer

30 seconds

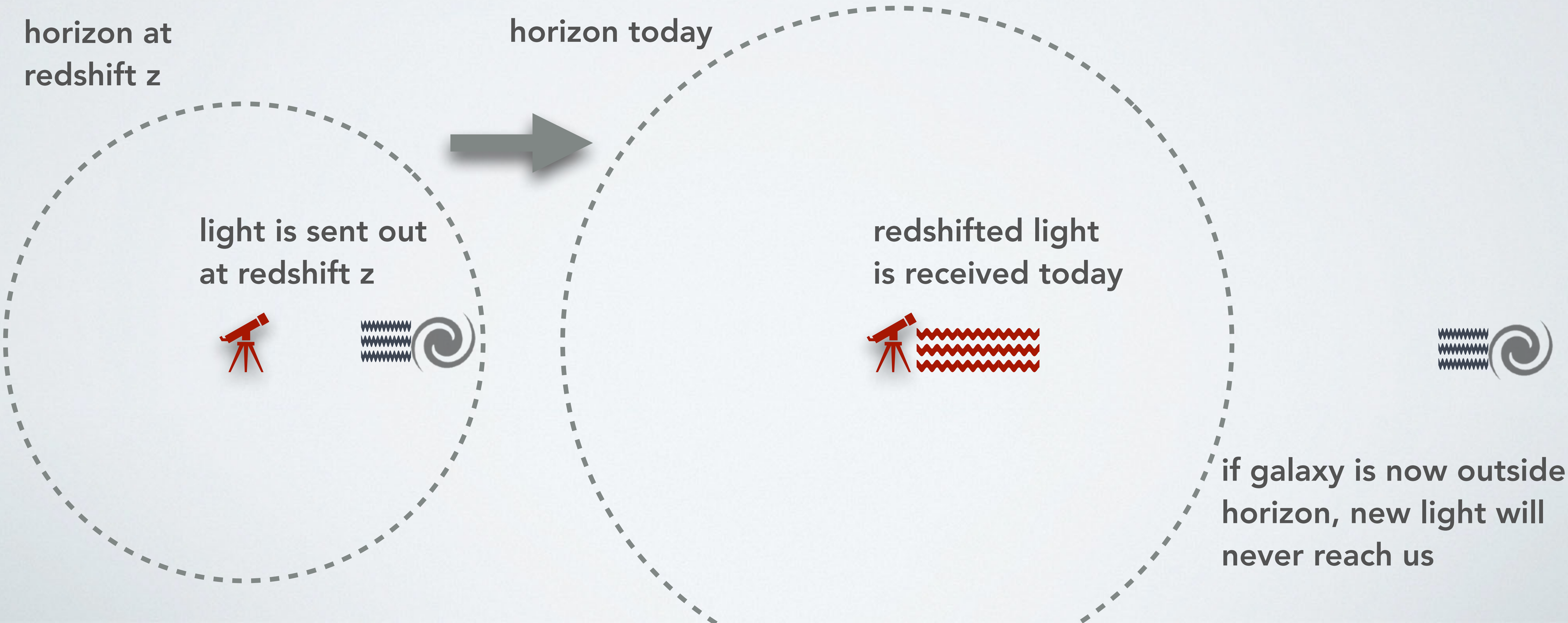
Super-light speed recession?



Davis & Lineweaver 2000

Horizon

- How can light from distant galaxies (z > 2) reach us if they are receding faster than the speed of light?
- The **light is very old!** We see the galaxies as they were at redshift z
- What matters is whether they were inside our **horizon (visible Universe)** when the light was sent out
- Horizon **depends on the expansion history** of the Universe between z and now
- In our Universe, the horizon today is **about 14 Gigaparsec (Gpc)**



Take-aways

- **Big Bang cosmology** means that the Universe started infinitesimally small, at a = 0

- In a Universe with only matter, the **expansion rate** and **geometry** are determined by the density of matter

- There are three possibilities: **closed** (density higher than critical, collapses), **flat** (critical density, expands but more and more slowly), and **open** (density less than critical density, expands forever)

Next time…

We'll talk about:

- Dark Energy

Assignments

- Post-lecture quiz (by tomorrow night)
- Homework #3 (by 10/20)

Reading:

- H&H Chapter 13