ASTR 670: Interstellar medium and gas dynamics

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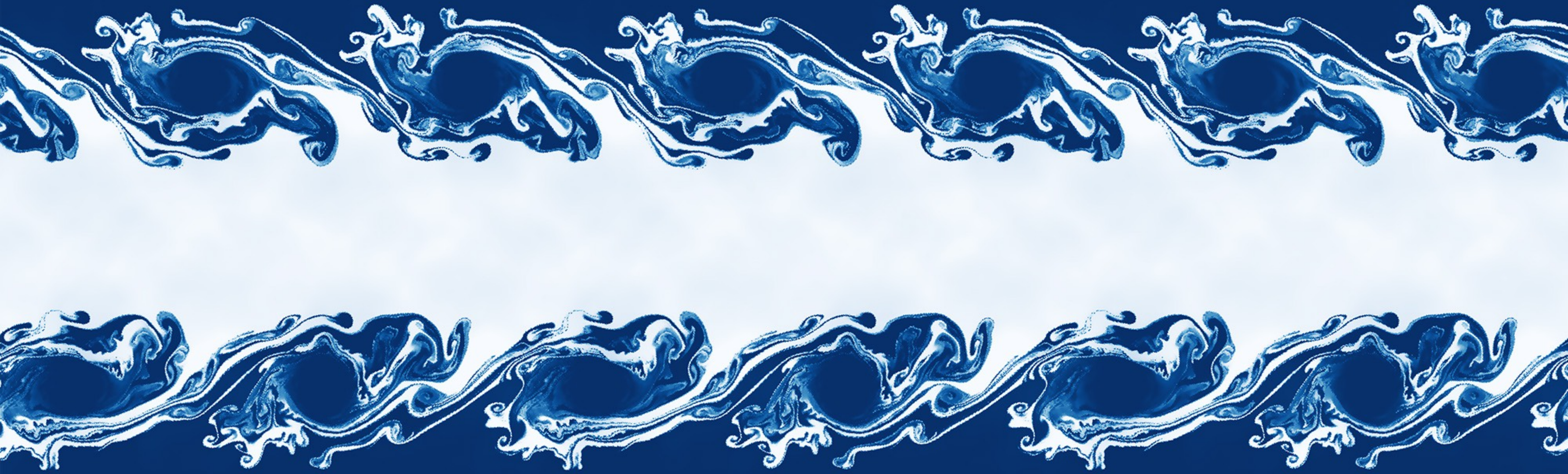
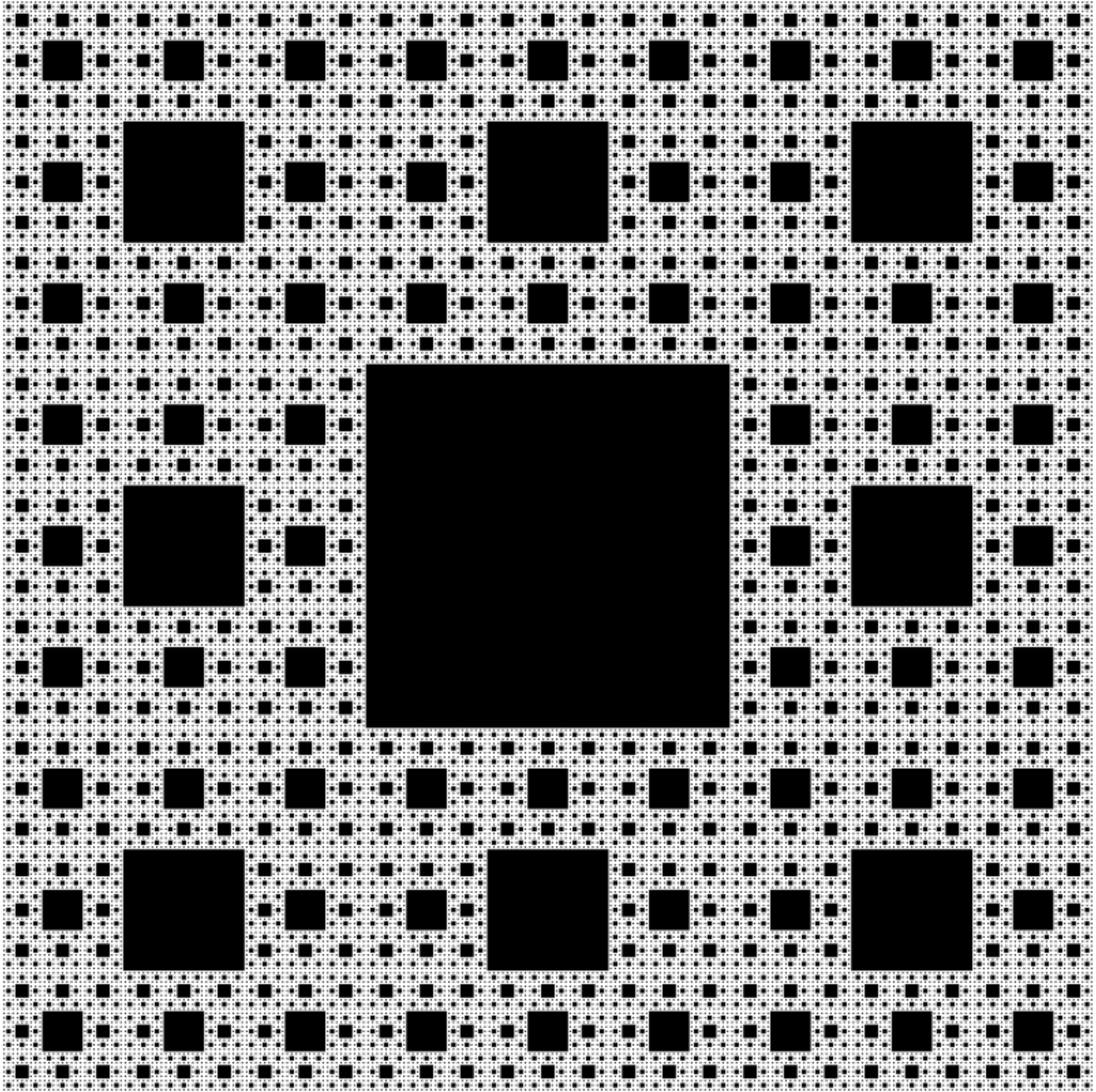


Chapter 5 • Computational hydro I: Theoretical background

Self-similarity



Sierpinski Carpet

Conservation laws

B.1 Conservation-law form of the Euler equations

The goal of this derivation is to convert the Eulerian equations for velocity (3.15) and internal energy (3.22) into conservation laws similar to the continuity equation (3.6). When we recast the equation for $\boldsymbol{u}$ as an equation for $\rho\boldsymbol{u}$, we get

$$\frac{\partial(\rho\boldsymbol{u})}{\partial t} = \rho\frac{\partial\boldsymbol{u}}{\partial t} + \boldsymbol{u}\frac{\partial\rho}{\partial t} = -\rho\boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u} - \boldsymbol{u}(\boldsymbol{u}\cdot\boldsymbol{\nabla}\rho) - \boldsymbol{u}\rho\boldsymbol{\nabla}\cdot\boldsymbol{u} - \boldsymbol{\nabla}P - \rho\boldsymbol{\nabla}\Phi\,, \tag{B.1}$$

where we have substituted the expressions from the continuity and "momentum" equations 3.6 and 3.15 and multiplied through. The $\boldsymbol{u}$-terms in this equation seem difficult to interpret until we write them in index notation,

$$\rho\boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u} + \boldsymbol{u}(\boldsymbol{u}\cdot\boldsymbol{\nabla}\rho) + \boldsymbol{u}\rho\boldsymbol{\nabla}\cdot\boldsymbol{u} = \rho u_i\frac{\partial u_j}{\partial x_i} + u_i u_j\frac{\partial\rho}{\partial x_i} + \rho u_j\frac{\partial u_i}{\partial x_i} = \frac{\partial}{\partial x_i}(\rho u_i u_j)\,. \tag{B.2}$$

The object on the right is a tensor, $\boldsymbol{u}\otimes\boldsymbol{u} = u_i u_j$ and the contraction of the derivative with the first index corresponds to the divergence, as before. We also include the pressure term into the so-called momentum flux density tensor (for ideal fluids),

$$\Pi_{ij} \equiv \rho u_i u_j + P\delta_{ij} = \rho\boldsymbol{u}\otimes\boldsymbol{u} + \boldsymbol{I}P \tag{B.3}$$

where $\boldsymbol{I}$ is the identity matrix (§A.2). We can now rewrite Equation B.1 as a conservation law,

$$\boxed{\frac{\partial(\rho\boldsymbol{u})}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\boldsymbol{u}\otimes\boldsymbol{u} + \boldsymbol{I}P) = -\rho\boldsymbol{\nabla}\Phi} \tag{B.4}$$

For the total energy equation, we remember the definition $E = \rho(|\boldsymbol{u}^2|/2 + \varepsilon + \Phi)$ and write

$$\frac{\mathrm{D}E}{\mathrm{D}t} = \frac{E}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\left(\boldsymbol{u}\cdot\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + \frac{\mathrm{D}\varepsilon}{\mathrm{D}t} + \frac{\mathrm{D}\Phi}{\mathrm{D}t}\right) = \frac{\partial E}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}E\,. \tag{B.5}$$

We rearrange and substitute the RHS of the Lagrangian fluid equations to find

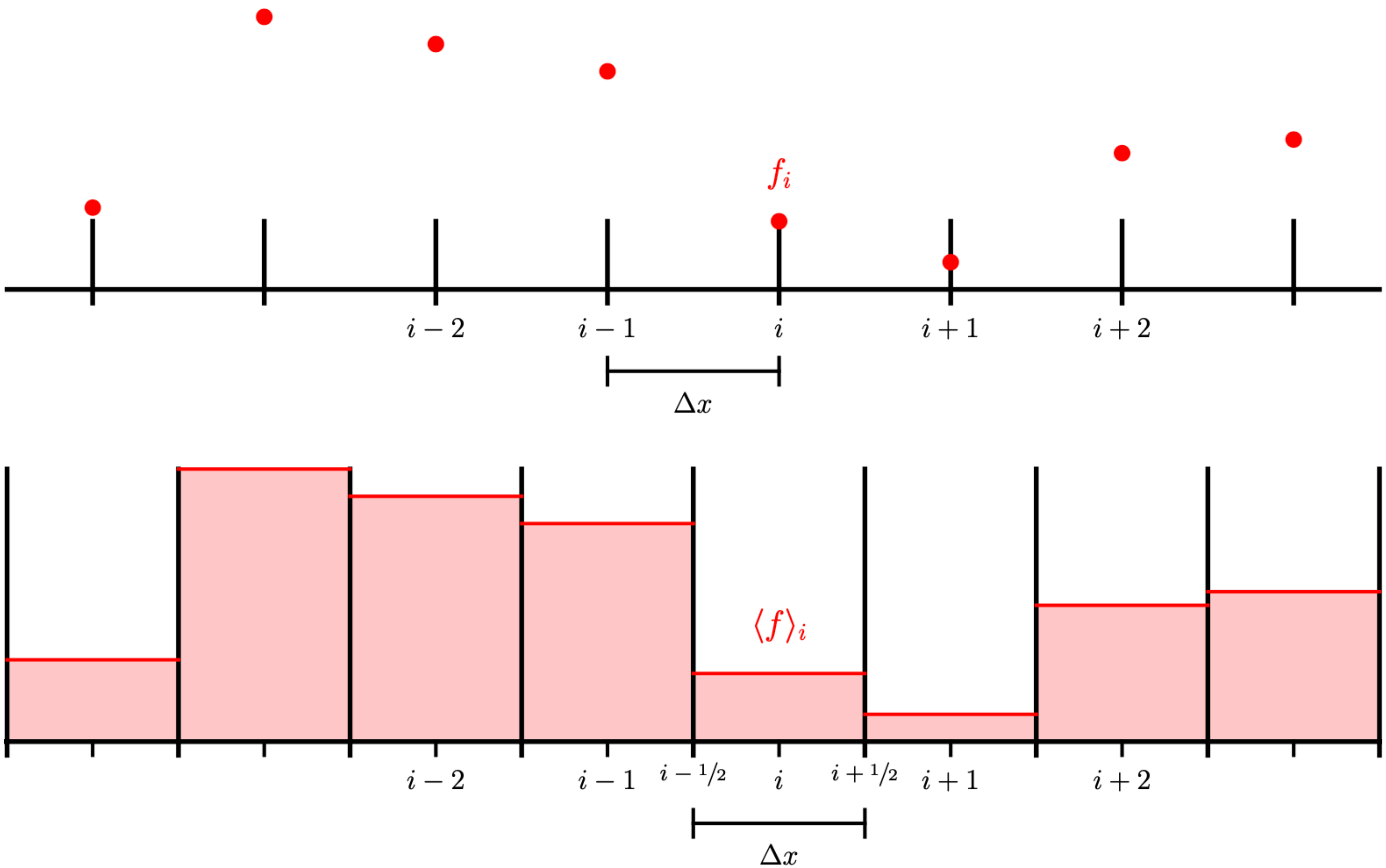
$$\begin{aligned}\frac{\partial E}{\partial t} &= -\boldsymbol{u}\cdot\boldsymbol{\nabla}E - E\boldsymbol{\nabla}\cdot\boldsymbol{u} - \boldsymbol{u}\cdot\boldsymbol{\nabla}P - \rho\boldsymbol{u}\cdot\boldsymbol{\nabla}\Phi + \rho\frac{\mathrm{D}\Phi}{\mathrm{D}t} - P\boldsymbol{\nabla}\cdot\boldsymbol{u} + \Gamma - \Lambda \\ &= -\boldsymbol{u}\cdot\boldsymbol{\nabla}(E+P) - (E+P)\boldsymbol{\nabla}\cdot\boldsymbol{u} + \rho\frac{\partial\Phi}{\partial t} + \Gamma - \Lambda \\ &= -\boldsymbol{\nabla}\cdot[(E+P)\boldsymbol{u}] + \rho\frac{\partial\Phi}{\partial t} + \Gamma - \Lambda\end{aligned} \tag{B.6}$$

and thus

$$\boxed{\frac{\partial E}{\partial t} + \boldsymbol{\nabla}\cdot([E+P]\boldsymbol{u}) = \rho\frac{\partial\Phi}{\partial t} + \Gamma - \Lambda} \tag{B.7}$$

Computational methods

Finite difference vs. finite volume



Zingale 2021

Reading

- Zingale §1-1.2.1, §2, §3.1-2, §7.1