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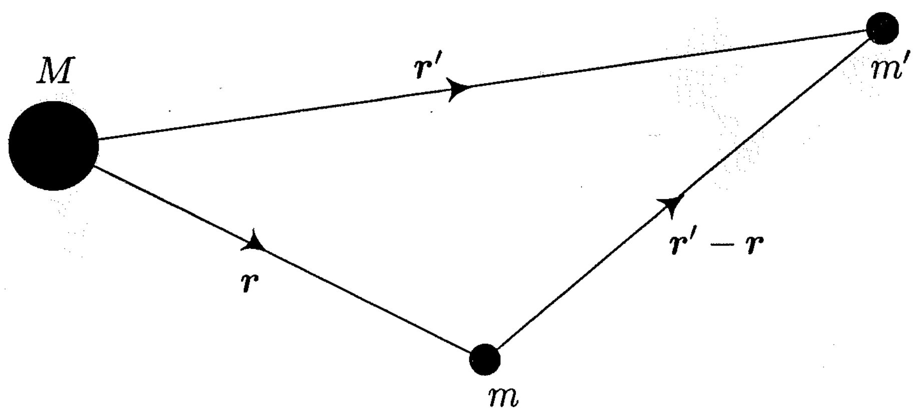

EXPANSION OF THE PLANETARY DISTURBING FUNCTION TO EIGHTH ORDER IN THE INDIVIDUAL ORBITAL ELEMENTS

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prime = outer

Introduction

In the gravitational three body problem two objects with masses m and m' have position vectors $\boldsymbol{r}$ and $\boldsymbol{r}'$ (where $r < r'$) with respect to a central mass, M. Each of the two orbiting masses perturb one another resulting in changes in both orbits. In celestial mechanics the perturbing potential experienced by one of the orbiting masses is called the disturbing function. For example, the disturbing function experienced by the inner mass can be written as

$$\mathcal{R} = \frac{\mu'}{|\boldsymbol{r}' - \boldsymbol{r}|} - \mu' \frac{\boldsymbol{r} \cdot \boldsymbol{r}'}{r'^3} \tag{1}$$

where $\mu' = \mathcal{G}m'$ and $\mathcal{G}$ is the gravitational constant. The corresponding disturbing function for the outer mass is

$$\mathcal{R}' = \frac{\mu}{|\boldsymbol{r} - \boldsymbol{r}'|} - \mu \frac{\boldsymbol{r} \cdot \boldsymbol{r}'}{r^3} \tag{2}$$

where $\mu = \mathcal{G}m$. In equations (1) and (2) the first term on the right hand side is called the direct part and the second the indirect part of the disturbing function.

In order to analyse the resulting perturbations analytically it is necessary to expand the disturbing function in terms of the orbital elements of the two bodies rather than their position vectors. If a, e, I, λ, ϖ and Ω denote the semi-major axis, eccentricity, inclination, longitude of pericentre and longitude of ascending respectively of the inner body, with similar primed quantities for the outer body, then the general form of the expansion is

$$\mathcal{R} = \sum_{j=-\infty}^{\infty} f_j(a, a', e, e', s, s') \cos \varphi_j \tag{3}$$

where $s = \sin \frac{1}{2} I$, $s' = \sin \frac{1}{2} I'$ and φ_j is a permitted linear combination of λ, λ', ϖ, ϖ', Ω and Ω'.

If $\alpha = a/a'$ and ψ denotes the angle between the two position vectors, then we can write

$$\mathcal{R} = \frac{\mu'}{a'} (\mathcal{R}_{\rm D} + \alpha \mathcal{R}_{\rm E}) \tag{4}$$

and

$$\mathcal{R}' = \frac{\mu}{a'} \left(\mathcal{R}_{\rm D} + \frac{1}{\alpha^2} \mathcal{R}_{\rm I} \right) \tag{5}$$

where

$$\mathcal{R}_{\rm D} = \frac{a'}{|\boldsymbol{r} - \boldsymbol{r}'|} \tag{6}$$

and

$$\mathcal{R}_{\rm E} = -\left(\frac{r}{a}\right) \left(\frac{a'}{r'}\right)^2 \cos \psi \tag{7}$$

$$\mathcal{R}_{\rm I} = -\left(\frac{r'}{a'}\right) \left(\frac{a}{r}\right)^2 \cos \psi. \tag{8}$$

In this book we give a literal expansion of $\mathcal{R}_{\rm D}$, $\mathcal{R}_{\rm E}$ and $\mathcal{R}_{\rm I}$ complete to eighth order in the eccentricities and inclinations of the two bodies. The expansion was obtained using the *Mathematica* and *Maple* software packages. Details of the derivation and how the expansion is used in solar system dynamics will be published elsewhere.

Previous high-order expansions such as those by Peirce (*Astron. J.* **1**, 1–8, 31–36, (1849)) and Le Verrier (*Ann. Obs. Paris, Mém.* **1**, 258–331, (1855)) have been in terms of the mutual inclination and mutual node of the two bodies. While this produces a conveniently compact form of the expansion, it makes it more difficult to use, especially when higher powers of the inclinations are involved.

In the expansion each cosine argument has been assigned a unique label. It contains the order of the expansion (always 8 in this case) followed by a letter denoting that the term is associated with the direct (prefix D) or indirect (prefix E for an external perturbation or I for an internal perturbation) part of the disturbing function. The next character denotes the order of the argument. This is defined to be the absolute value of the sum of the coefficients of λ and λ' in the cosine argument. The final number identifies the argument.

For example, to lowest order in the eccentricities and inclinations 8D2.3 (page 108) is

8D2.3		λ'	λ	ϖ'	ϖ	Ω'	Ω
$e\ e'\ s\ s'$		j	$2-j$	-2	0	0	0
0 2 0 0	$\frac{1}{8}\left[2-7j+4j^2-2\alpha D+4j\alpha D+\alpha^2D^2\right]b_{1/2}^{(j-2)}$						

In this expression D denotes the differential operator $\mathrm{d}/\mathrm{d}\alpha$. The quantities $b_i^{(k)}$, where i is a half-integer and k is integer, are Laplace coefficients defined by

$$\frac{1}{2}b_i^{(k)}(\alpha)=\frac{1}{2\pi}\int_0^{2\pi}\frac{\cos k\phi\,\mathrm{d}\phi}{(1-2\alpha\cos\phi+\alpha^2)^i} \tag{9}$$

This can be written in series form as

$$\frac{1}{2}b_i^{(k)}(\alpha)=\frac{i(i+1)\ldots(i+k-1)}{1\cdot2\cdot3\ldots k}\alpha^k\left[1+\frac{i(i+k)}{1(k+1)}\alpha^2+\frac{i(i+1)(i+k)(i+k+1)}{1\cdot2(k+1)(k+2)}\alpha^4+\ldots\right] \tag{10}$$

In the case where $k=0$ the factor outside the brackets is equal to unity.

There are other parts to 8D2.3 but they are associated with higher powers of the eccentricities and inclinations. The expression can be written as

$$\frac{e'^2}{8}\left\{\left(2-7j+4j^2\right)b_{1/2}^{(j-2)}-(2-4j)\alpha\frac{\mathrm{d}b_{1/2}^{(j-2)}}{\mathrm{d}\alpha}+\alpha^2\frac{\mathrm{d}^2b_{1/2}^{(j-2)}}{\mathrm{d}\alpha^2}\right\}\cos\left[j\lambda'+(2-j)\lambda-2\varpi'\right].$$

In those cases where the specific form of the argument is known, the indirect part of the expansion can be searched for contributing terms. For example, if

$$\varphi=3\lambda'-\lambda-2\varpi'$$

with an external perturber, then the appropriate term in the direct part of the expansion is 8D2.3 with $j=3$ and there is a contribution from 8E2.5 (page 357). To lowest order the final form of the term with its associated coefficient is

$$\frac{e'^2}{8}\left\{17b_{1/2}^{(1)}+10\alpha\frac{\mathrm{d}b_{1/2}^{(1)}}{\mathrm{d}\alpha}+\alpha^2\frac{\mathrm{d}^2b_{1/2}^{(1)}}{\mathrm{d}\alpha^2}-27\alpha\right\}\cos\left[3\lambda'-\lambda-2\varpi'\right].$$

It is our intention to make this eighth order expansion available in machine readable form. Research workers interested in obtaining this data are invited to contact either author (`C.D.Murray@qmw.ac.uk` or `D.Harper@qmw.ac.uk`) for further information.

ACKNOWLEDGEMENT

This research was partly supported by the UK Science and Engineering Research Council. One of us (CDM) is grateful to the SERC for the award of an Advanced Fellowship.

Contents

Direct Part, Order 0

8D0.1		λ'	λ	ϖ'	ϖ	Ω'	Ω
e e' s s'		j	$-j$	0	0	0	0

0 0 0 0 $\frac{1}{2} b_{1/2}^{(j)}$

2 0 0 0 $\frac{1}{8}\left[-4j^2 + 2\alpha D + \alpha^2 D^2\right] b_{1/2}^{(j)}$

4 0 0 0 $\frac{1}{128}\left[-9j^2 + 16j^4 - 8j^2\alpha D - 8j^2\alpha^2 D^2 + 4\alpha^3 D^3 + \alpha^4 D^4\right] b_{1/2}^{(j)}$

6 0 0 0 $\frac{1}{4608}\left[-172j^2 + 196j^4 - 64j^6 - 18j^2\alpha D - 33j^2\alpha^2 D^2 + 48j^4\alpha^2 D^2 - 24j^2\alpha^3 D^3 - 12j^2\alpha^4 D^4 + 6\alpha^5 D^5 + \alpha^6 D^6\right] b_{1/2}^{(j)}$

8 0 0 0 $\frac{1}{294912}\left[-7140j^2 + 7801j^4 - 2272j^6 + 256j^8 - 176j^2\alpha D - 1072j^4\alpha D + 256j^6\alpha D - 1192j^2\alpha^2 D^2 + 1072j^4\alpha^2 D^2 - 256j^6\alpha^2 D^2 - 120j^2\alpha^3 D^3 - 78j^2\alpha^4 D^4 + 96j^4\alpha^4 D^4 - 48j^2\alpha^5 D^5 - 16j^2\alpha^6 D^6 + 8\alpha^7 D^7 + \alpha^8 D^8\right] b_{1/2}^{(j)}$

0 2 0 0 $\frac{1}{8}\left[-4j^2 + 2\alpha D + \alpha^2 D^2\right] b_{1/2}^{(j)}$

2 2 0 0 $\frac{1}{32}\left[16j^4 + 4\alpha D - 16j^2\alpha D + 14\alpha^2 D^2 - 8j^2\alpha^2 D^2 + 8\alpha^3 D^3 + \alpha^4 D^4\right] b_{1/2}^{(j)}$

4 2 0 0 $\frac{1}{512}\left[36j^4 - 64j^6 - 34j^2\alpha D + 64j^4\alpha D - 89j^2\alpha^2 D^2 + 48j^4\alpha^2 D^2 + 48\alpha^3 D^3 - 72j^2\alpha^3 D^3 + 52\alpha^4 D^4 - 12j^2\alpha^4 D^4 + 14\alpha^5 D^5 + \alpha^6 D^6\right] b_{1/2}^{(j)}$

6 2 0 0 $\frac{1}{18432}\left[688j^4 - 784j^6 + 256j^8 - 380j^2\alpha D + 464j^4\alpha D - 128j^6\alpha D - 442j^2\alpha^2 D^2 + 616j^4\alpha^2 D^2 - 256j^6\alpha^2 D^2 - 504j^2\alpha^3 D^3 + 384j^4\alpha^3 D^3 - 465j^2\alpha^4 D^4 + 96j^4\alpha^4 D^4 + 180\alpha^5 D^5 - 168j^2\alpha^5 D^5 + 114\alpha^6 D^6 - 16j^2\alpha^6 D^6 + 20\alpha^7 D^7 + \alpha^8 D^8\right] b_{1/2}^{(j)}$

0 4 0 0 $\frac{1}{128}\left[-17j^2 + 16j^4 + 24\alpha D - 24j^2\alpha D + 36\alpha^2 D^2 - 8j^2\alpha^2 D^2 + 12\alpha^3 D^3 + \alpha^4 D^4\right] b_{1/2}^{(j)}$

2 4 0 0 $\frac{1}{512}\left[68j^4 - 64j^6 + 48\alpha D - 178j^2\alpha D + 128j^4\alpha D + 312\alpha^2 D^2 - 305j^2\alpha^2 D^2 + 48j^4\alpha^2 D^2 + 384\alpha^3 D^3 - 120j^2\alpha^3 D^3 + 152\alpha^4 D^4 - 12j^2\alpha^4 D^4 + 22\alpha^5 D^5 + \alpha^6 D^6\right] b_{1/2}^{(j)}$

4 4 0 0 $\frac{1}{8192}\left[153j^4 - 416j^6 + 256j^8 - 408j^2\alpha D + 928j^4\alpha D - 512j^6\alpha D - 2052j^2\alpha^2 D^2 + 1616j^4\alpha^2 D^2 - 256j^6\alpha^2 D^2 + 1440\alpha^3 D^3 - 3152j^2\alpha^3 D^3 + 768j^4\alpha^3 D^3 + 2760\alpha^4 D^4 - 1594j^2\alpha^4 D^4 + 96j^4\alpha^4 D^4 + 1560\alpha^5 D^5 - 288j^2\alpha^5 D^5 + 348\alpha^6 D^6 - 16j^2\alpha^6 D^6 + 32\alpha^7 D^7 + \alpha^8 D^8\right] b_{1/2}^{(j)}$

0 6 0 0 $\frac{1}{4608}\left[-364j^2 + 292j^4 - 64j^6 + 720\alpha D - 834j^2\alpha D + 192j^4\alpha D + 1800\alpha^2 D^2 - 681j^2\alpha^2 D^2 + 48j^4\alpha^2 D^2 + 1200\alpha^3 D^3 - 168j^2\alpha^3 D^3 + 300\alpha^4 D^4 - 12j^2\alpha^4 D^4 + 30\alpha^5 D^5 + \alpha^6 D^6\right] b_{1/2}^{(j)}$

2 6 0 0 $\frac{1}{18432}\left[1456j^4 - 1168j^6 + 256j^8 + 1440\alpha D - 5276j^2\alpha D + 4304j^4\alpha D - 896j^6\alpha D + 13680\alpha^2 D^2 - 14986j^2\alpha^2 D^2 + 4072j^4\alpha^2 D^2 - 256j^6\alpha^2 D^2 + 25920\alpha^3 D^3 - 11736j^2\alpha^3 D^3 + 1152j^4\alpha^3 D^3 + 17400\alpha^4 D^4 - 3465j^2\alpha^4 D^4 + 96j^4\alpha^4 D^4 + 5100\alpha^5 D^5 - 408j^2\alpha^5 D^5 + 702\alpha^6 D^6 - 16j^2\alpha^6 D^6 + 44\alpha^7 D^7 + \alpha^8 D^8\right] b_{1/2}^{(j)}$

0 8 0 0 $\frac{1}{294912}\left[-16260j^2 + 13321j^4 - 3040j^6 + 256j^8 + 40320\alpha D - 50240j^2\alpha D + 14576j^4\alpha D - 1280j^6\alpha D + 141120\alpha^2 D^2 - 63328j^2\alpha^2 D^2 + 7984j^4\alpha^2 D^2 - 256j^6\alpha^2 D^2 + 141120\alpha^3 D^3 - 29928j^2\alpha^3 D^3 + 1536j^4\alpha^3 D^3 + 58800\alpha^4 D^4 - 6078j^2\alpha^4 D^4 + 96j^4\alpha^4 D^4 + 11760\alpha^5 D^5 - 528j^2\alpha^5 D^5 + 1176\alpha^6 D^6 - 16j^2\alpha^6 D^6 + 56\alpha^7 D^7 + \alpha^8 D^8\right] b_{1/2}^{(j)}$

0 0 2 0 $\frac{1}{4}\left[-\alpha\right] b_{3/2}^{(j-1)} + \frac{1}{4}\left[-\alpha\right] b_{3/2}^{(j+1)}$

2 0 2 0 $\frac{1}{16}\left[-2\alpha + 4j^2\alpha - 4\alpha^2 D - \alpha^3 D^2\right] b_{3/2}^{(j-1)} + \frac{1}{16}\left[-2\alpha + 4j^2\alpha - 4\alpha^2 D - \alpha^3 D^2\right] b_{3/2}^{(j+1)}$

$$-64j^2\alpha^3D+16j^3\alpha^3D-10\alpha^4D^2-41j\alpha^4D^2-10\alpha^5D^3-4j\alpha^5D^3$$
$$-\alpha^6D^4\big]\,b_{5/2}^{(j-3)}$$
$$+\tfrac{3}{64}\big[8\alpha^2+28j\alpha^2-2j^2\alpha^2-44j^3\alpha^2+16j^4\alpha^2+24\alpha^3D-18j\alpha^3D$$
$$-64j^2\alpha^3D+16j^3\alpha^3D-10\alpha^4D^2-41j\alpha^4D^2-10\alpha^5D^3-4j\alpha^5D^3$$
$$-\alpha^6D^4\big]\,b_{5/2}^{(j-1)}$$
$$+\tfrac{3}{256}\big[8\alpha^2+28j\alpha^2-2j^2\alpha^2-44j^3\alpha^2+16j^4\alpha^2+24\alpha^3D-18j\alpha^3D$$
$$-64j^2\alpha^3D+16j^3\alpha^3D-10\alpha^4D^2-41j\alpha^4D^2-10\alpha^5D^3-4j\alpha^5D^3$$
$$-\alpha^6D^4\big]\,b_{5/2}^{(j+1)}$$

1 3 0 4

$$\tfrac{3}{256}\big[-24j\alpha^2-42j^2\alpha^2-4j^3\alpha^2+16j^4\alpha^2-40\alpha^3D-94j\alpha^3D-32j^2\alpha^3D$$
$$+16j^3\alpha^3D-50\alpha^4D^2-43j\alpha^4D^2-14\alpha^5D^3-4j\alpha^5D^3-\alpha^6D^4\big]\,b_{5/2}^{(j-3)}$$
$$+\tfrac{3}{64}\big[-24j\alpha^2-42j^2\alpha^2-4j^3\alpha^2+16j^4\alpha^2-40\alpha^3D-94j\alpha^3D-32j^2\alpha^3D$$
$$+16j^3\alpha^3D-50\alpha^4D^2-43j\alpha^4D^2-14\alpha^5D^3-4j\alpha^5D^3-\alpha^6D^4\big]\,b_{5/2}^{(j-1)}$$
$$+\tfrac{3}{256}\big[-24j\alpha^2-42j^2\alpha^2-4j^3\alpha^2+16j^4\alpha^2-40\alpha^3D-94j\alpha^3D-32j^2\alpha^3D$$
$$+16j^3\alpha^3D-50\alpha^4D^2-43j\alpha^4D^2-14\alpha^5D^3-4j\alpha^5D^3-\alpha^6D^4\big]\,b_{5/2}^{(j+1)}$$

1 1 2 4

$$\tfrac{3}{16}\big[2j\alpha^2+4j^2\alpha^2+2\alpha^3D+4j\alpha^3D+\alpha^4D^2\big]\,b_{5/2}^{(j-3)}$$
$$+\tfrac{9}{4}\big[2j\alpha^2+4j^2\alpha^2+2\alpha^3D+4j\alpha^3D+\alpha^4D^2\big]\,b_{5/2}^{(j-1)}$$
$$+\tfrac{3}{16}\big[2j\alpha^2+4j^2\alpha^2+2\alpha^3D+4j\alpha^3D+\alpha^4D^2\big]\,b_{5/2}^{(j+1)}$$
$$+\tfrac{15}{64}\big[2\alpha^3+6j\alpha^3+4j^2\alpha^3+4\alpha^4D+4j\alpha^4D+\alpha^5D^2\big]\,b_{7/2}^{(j-4)}$$
$$+\tfrac{435}{64}\big[2\alpha^3+6j\alpha^3+4j^2\alpha^3+4\alpha^4D+4j\alpha^4D+\alpha^5D^2\big]\,b_{7/2}^{(j-2)}$$
$$+\tfrac{435}{64}\big[2\alpha^3+6j\alpha^3+4j^2\alpha^3+4\alpha^4D+4j\alpha^4D+\alpha^5D^2\big]\,b_{7/2}^{(j)}$$
$$+\tfrac{15}{64}\big[2\alpha^3+6j\alpha^3+4j^2\alpha^3+4\alpha^4D+4j\alpha^4D+\alpha^5D^2\big]\,b_{7/2}^{(j+2)}$$

1 1 0 6

$$\tfrac{5}{64}\big[2\alpha^3+6j\alpha^3+4j^2\alpha^3+4\alpha^4D+4j\alpha^4D+\alpha^5D^2\big]\,b_{7/2}^{(j-4)}$$
$$+\tfrac{45}{64}\big[2\alpha^3+6j\alpha^3+4j^2\alpha^3+4\alpha^4D+4j\alpha^4D+\alpha^5D^2\big]\,b_{7/2}^{(j-2)}$$
$$+\tfrac{45}{64}\big[2\alpha^3+6j\alpha^3+4j^2\alpha^3+4\alpha^4D+4j\alpha^4D+\alpha^5D^2\big]\,b_{7/2}^{(j)}$$
$$+\tfrac{5}{64}\big[2\alpha^3+6j\alpha^3+4j^2\alpha^3+4\alpha^4D+4j\alpha^4D+\alpha^5D^2\big]\,b_{7/2}^{(j+2)}$$

8D2.3	λ'	λ	ϖ'	ϖ	Ω'	Ω
e e' s s'	j	$2-j$	-2	0	0	0

0 2 0 0

$$\tfrac{1}{8}\big[2-7j+4j^2-2\alpha D+4j\alpha D+\alpha^2D^2\big]\,b_{1/2}^{(j-2)}$$

2 2 0 0

$$\tfrac{1}{32}\big[-32+144j-184j^2+92j^3-16j^4+32\alpha D-102j\alpha D+80j^2\alpha D-16j^3\alpha D$$
$$-16\alpha^2D^2+25j\alpha^2D^2+4\alpha^3D^3+4j\alpha^3D^3+\alpha^4D^4\big]\,b_{1/2}^{(j-2)}$$

4 2 0 0

$$\tfrac{1}{512}\big[440-2492j+4962j^2-4785j^3+2428j^4-624j^5+64j^6-440\alpha D+1928j\alpha D$$
$$-2878j^2\alpha D+1908j^3\alpha D-576j^4\alpha D+64j^5\alpha D+220\alpha^2D^2-636j\alpha^2D^2$$
$$+407j^2\alpha^2D^2-40j^3\alpha^2D^2-16j^4\alpha^2D^2-88\alpha^3D^3-12j\alpha^3D^3+120j^2\alpha^3D^3$$
$$-32j^3\alpha^3D^3-10\alpha^4D^4+57j\alpha^4D^4-4j^2\alpha^4D^4+10\alpha^5D^5+4j\alpha^5D^5$$
$$+\alpha^6D^6\big]\,b_{1/2}^{(j-2)}$$

6 2 0 0

$$\tfrac{1}{18432}\big[-3296+24944j-75176j^2+119956j^3-111304j^4+61732j^5-20080j^6$$
$$+3520j^7-256j^8+3296\alpha D-19784j\alpha D+47968j^2\alpha D-60314j^3\alpha D$$
$$+41904j^4\alpha D-16112j^5\alpha D+3200j^6\alpha D-256j^7\alpha D-1648\alpha^2D^2$$
$$+7052j\alpha^2D^2-9400j^2\alpha^2D^2+4103j^3\alpha^2D^2+448j^4\alpha^2D^2-720j^5\alpha^2D^2$$
$$+128j^6\alpha^2D^2+1008\alpha^3D^3-480j\alpha^3D^3-3348j^2\alpha^3D^3+3972j^3\alpha^3D^3$$
$$-1536j^4\alpha^3D^3+192j^5\alpha^3D^3-36\alpha^4D^4-1548j\alpha^4D^4+1575j^2\alpha^4D^4$$
$$-396j^3\alpha^4D^4-312\alpha^5D^5+270j\alpha^5D^5+120j^2\alpha^5D^5-48j^3\alpha^5D^5$$
$$+20\alpha^6D^6+89j\alpha^6D^6-8j^2\alpha^6D^6+16\alpha^7D^7+4j\alpha^7D^7+\alpha^8D^8\big]\,b_{1/2}^{(j-2)}$$

0 4 0 0

$$\tfrac{1}{96}\big[12-14j-40j^2+52j^3-16j^4-12\alpha D-10j\alpha D+48j^2\alpha D-16j^3\alpha D$$

Direct Part, Order 1

8D1.1	λ'	λ	ϖ'	ϖ	Ω'	Ω
e e' s s'	j	$1-j$	0	-1	0	0

1 0 0 0
$$\tfrac{1}{2}\left[-2j-\alpha D\right] b_{1/2}^{(j)}$$

3 0 0 0
$$\tfrac{1}{16}\left[2j-10j^2+8j^3+3\alpha D-7j\alpha D+4j^2\alpha D-2\alpha^2D^2-2j\alpha^2D^2-\alpha^3D^3\right] b_{1/2}^{(j)}$$

5 0 0 0
$$\tfrac{1}{384}\left[-10j-12j^2-26j^3+80j^4-32j^5-5\alpha D+18j\alpha D-37j^2\alpha D+40j^3\alpha D\right.$$
$$-16j^4\alpha D+4\alpha^2D^2-8j\alpha^2D^2-12j^2\alpha^2D^2+16j^3\alpha^2D^2+6\alpha^3D^3$$
$$\left.-14j\alpha^3D^3+8j^2\alpha^3D^3-4\alpha^4D^4-2j\alpha^4D^4-\alpha^5D^5\right] b_{1/2}^{(j)}$$

7 0 0 0
$$\tfrac{1}{18432}\left[-214j-842j^2+494j^3+874j^4+40j^5-480j^6+128j^7+7\alpha D-175j\alpha D\right.$$
$$+277j^2\alpha D-j^3\alpha D-28j^4\alpha D-144j^5\alpha D+64j^6\alpha D-6\alpha^2D^2-54j\alpha^2D^2$$
$$-90j^2\alpha^2D^2+6j^3\alpha^2D^2+240j^4\alpha^2D^2-96j^5\alpha^2D^2-15\alpha^3D^3+48j\alpha^3D^3$$
$$-105j^2\alpha^3D^3+120j^3\alpha^3D^3-48j^4\alpha^3D^3+12\alpha^4D^4-30j\alpha^4D^4-6j^2\alpha^4D^4$$
$$+24j^3\alpha^4D^4+9\alpha^5D^5-21j\alpha^5D^5+12j^2\alpha^5D^5-6\alpha^6D^6-2j\alpha^6D^6$$
$$\left.-\alpha^7D^7\right] b_{1/2}^{(j)}$$

1 2 0 0
$$\tfrac{1}{8}\left[8j^3-2\alpha D-4j\alpha D+4j^2\alpha D-4\alpha^2D^2-2j\alpha^2D^2-\alpha^3D^3\right] b_{1/2}^{(j)}$$

3 2 0 0
$$\tfrac{1}{64}\left[-8j^3+40j^4-32j^5+6\alpha D-10j\alpha D-24j^2\alpha D+44j^3\alpha D-16j^4\alpha D\right.$$
$$-38j\alpha^2D^2+14j^2\alpha^2D^2+16j^3\alpha^2D^2-21\alpha^3D^3-19j\alpha^3D^3+8j^2\alpha^3D^3$$
$$\left.-10\alpha^4D^4-2j\alpha^4D^4-\alpha^5D^5\right] b_{1/2}^{(j)}$$

5 2 0 0
$$\tfrac{1}{1536}\left[40j^3+48j^4+104j^5-320j^6+128j^7-10\alpha D+16j\alpha D-78j^2\alpha D\right.$$
$$-44j^3\alpha D+276j^4\alpha D-224j^5\alpha D+64j^6\alpha D+4\alpha^2D^2+14j\alpha^2D^2$$
$$-248j^2\alpha^2D^2+262j^3\alpha^2D^2+64j^4\alpha^2D^2-96j^5\alpha^2D^2+91\alpha^3D^3$$
$$-198j\alpha^3D^3-37j^2\alpha^3D^3+192j^3\alpha^3D^3-48j^4\alpha^3D^3-28\alpha^4D^4-160j\alpha^4D^4$$
$$+68j^2\alpha^4D^4+24j^3\alpha^4D^4-64\alpha^5D^5-34j\alpha^5D^5+12j^2\alpha^5D^5-16\alpha^6D^6$$
$$\left.-2j\alpha^6D^6-\alpha^7D^7\right] b_{1/2}^{(j)}$$

1 4 0 0
$$\tfrac{1}{128}\left[34j^3-32j^5-24\alpha D-48j\alpha D+41j^2\alpha D+48j^3\alpha D-16j^4\alpha D-96\alpha^2D^2\right.$$
$$-72j\alpha^2D^2+40j^2\alpha^2D^2+16j^3\alpha^2D^2-72\alpha^3D^3-24j\alpha^3D^3+8j^2\alpha^3D^3$$
$$\left.-16\alpha^4D^4-2j\alpha^4D^4-\alpha^5D^5\right] b_{1/2}^{(j)}$$

3 4 0 0
$$\tfrac{1}{1024}\left[-34j^3+170j^4-104j^5-160j^6+128j^7+72\alpha D-120j\alpha D-267j^2\alpha D\right.$$
$$+431j^3\alpha D+124j^4\alpha D-304j^5\alpha D+64j^6\alpha D+48\alpha^2D^2-840j\alpha^2D^2$$
$$+66j^2\alpha^2D^2+714j^3\alpha^2D^2-112j^4\alpha^2D^2-96j^5\alpha^2D^2-624\alpha^3D^3$$
$$-960j\alpha^3D^3+393j^2\alpha^3D^3+264j^3\alpha^3D^3-48j^4\alpha^3D^3-672\alpha^4D^4$$
$$-350j\alpha^4D^4+142j^2\alpha^4D^4+24j^3\alpha^4D^4-217\alpha^5D^5-47j\alpha^5D^5+12j^2\alpha^5D^5$$
$$\left.-26\alpha^6D^6-2j\alpha^6D^6-\alpha^7D^7\right] b_{1/2}^{(j)}$$

1 6 0 0
$$\tfrac{1}{4608}\left[728j^3-584j^5+128j^7-720\alpha D-1440j\alpha D+1198j^2\alpha D+1668j^3\alpha D\right.$$
$$-484j^4\alpha D-384j^5\alpha D+64j^6\alpha D-4320\alpha^2D^2-3600j\alpha^2D^2+2196j^2\alpha^2D^2$$
$$+1362j^3\alpha^2D^2-288j^4\alpha^2D^2-96j^5\alpha^2D^2-5400\alpha^3D^3-2400j\alpha^3D^3$$
$$+1185j^2\alpha^3D^3+336j^3\alpha^3D^3-48j^4\alpha^3D^3-2400\alpha^4D^4-600j\alpha^4D^4$$
$$+216j^2\alpha^4D^4+24j^3\alpha^4D^4-450\alpha^5D^5-60j\alpha^5D^5+12j^2\alpha^5D^5-36\alpha^6D^6$$
$$\left.-2j\alpha^6D^6-\alpha^7D^7\right] b_{1/2}^{(j)}$$

1 0 2 0
$$\tfrac{1}{4}\left[\alpha+2j\alpha+\alpha^2D\right] b_{3/2}^{(j-1)}+\tfrac{1}{4}\left[\alpha+2j\alpha+\alpha^2D\right] b_{3/2}^{(j+1)}$$

3 0 2 0
$$\tfrac{1}{32}\left[-3\alpha+5j\alpha+6j^2\alpha-8j^3\alpha+\alpha^2D+11j\alpha^2D-4j^2\alpha^2D+5\alpha^3D^2\right.$$
$$\left.+2j\alpha^3D^2+\alpha^4D^3\right] b_{3/2}^{(j-1)}$$
$$+\tfrac{1}{32}\left[-3\alpha+5j\alpha+6j^2\alpha-8j^3\alpha+\alpha^2D+11j\alpha^2D-4j^2\alpha^2D+5\alpha^3D^2\right.$$
$$\left.+2j\alpha^3D^2+\alpha^4D^3\right] b_{3/2}^{(j+1)}$$

5 0 2 0
$$\tfrac{1}{768}\left[5\alpha-8j\alpha+49j^2\alpha-14j^3\alpha-64j^4\alpha+32j^5\alpha-3\alpha^2D-2j\alpha^2D\right.$$
$$+61j^2\alpha^2D-72j^3\alpha^2D+16j^4\alpha^2D-22\alpha^3D^2+50j\alpha^3D^2-12j^2\alpha^3D^2$$

Indirect Part (External Perturber), Order 1

8E1.1

λ'	λ	ϖ'	ϖ	Ω'	Ω
1	−2	0	1	0	0

$$-\tfrac{1}{2}e + \tfrac{3}{8}e^3 - \tfrac{5}{96}e^5 + \tfrac{1}{72}e^7 + \tfrac{1}{4}ee'^2 - \tfrac{3}{16}e^3e'^2 + \tfrac{5}{192}e^5e'^2 + \tfrac{1}{128}ee'^4 - \tfrac{3}{512}e^3e'^4 + \tfrac{29}{2304}ee'^6$$
$$+\tfrac{1}{2}es^2 - \tfrac{3}{8}e^3s^2 + \tfrac{5}{96}e^5s^2 - \tfrac{1}{4}ee'^2s^2 + \tfrac{3}{16}e^3e'^2s^2 - \tfrac{1}{128}ee'^4s^2 + \tfrac{1}{2}es'^2 - \tfrac{3}{8}e^3s'^2 + \tfrac{5}{96}e^5s'^2$$
$$-\tfrac{1}{4}ee'^2s'^2 + \tfrac{3}{16}e^3e'^2s'^2 - \tfrac{1}{128}ee'^4s'^2 - \tfrac{1}{2}es^2s'^2 + \tfrac{3}{8}e^3s^2s'^2 + \tfrac{1}{4}ee'^2s^2s'^2$$

8E1.2

λ'	λ	ϖ'	ϖ	Ω'	Ω
1	0	0	−1	0	0

$$\tfrac{3}{2}e - \tfrac{3}{4}ee'^2 - \tfrac{3}{128}ee'^4 - \tfrac{29}{768}ee'^6 - \tfrac{3}{2}es^2 + \tfrac{3}{4}ee'^2s^2 + \tfrac{3}{128}ee'^4s^2 - \tfrac{3}{2}es'^2 + \tfrac{3}{4}ee'^2s'^2 + \tfrac{3}{128}ee'^4s'^2$$
$$+\tfrac{3}{2}es^2s'^2 - \tfrac{3}{4}ee'^2s^2s'^2$$

8E1.3

λ'	λ	ϖ'	ϖ	Ω'	Ω
2	−1	−1	0	0	0

$$-2e' + e^2e' + \tfrac{1}{32}e^4e' + \tfrac{29}{576}e^6e' + \tfrac{3}{2}e'^3 - \tfrac{3}{4}e^2e'^3 - \tfrac{3}{128}e^4e'^3 - \tfrac{5}{24}e'^5 + \tfrac{5}{48}e^2e'^5 + \tfrac{1}{18}e'^7 + 2e's^2$$
$$-e^2e's^2 - \tfrac{1}{32}e^4e's^2 - \tfrac{3}{2}e'^3s^2 + \tfrac{3}{4}e^2e'^3s^2 + \tfrac{5}{24}e'^5s^2 + 2e's'^2 - e^2e's'^2 - \tfrac{1}{32}e^4e's'^2$$
$$-\tfrac{3}{2}e'^3s'^2 + \tfrac{3}{4}e^2e'^3s'^2 + \tfrac{5}{24}e'^5s'^2 - 2e's^2s'^2 + e^2e's^2s'^2 + \tfrac{3}{2}e'^3s^2s'^2$$

8E1.4

λ'	λ	ϖ'	ϖ	Ω'	Ω
2	−3	−1	2	0	0

$$-\tfrac{3}{4}e^2e' + \tfrac{3}{4}e^4e' - \tfrac{111}{512}e^6e' + \tfrac{9}{16}e^2e'^3 - \tfrac{9}{16}e^4e'^3 - \tfrac{5}{64}e^2e'^5 + \tfrac{3}{4}e^2e's^2 - \tfrac{3}{4}e^4e's^2 - \tfrac{9}{16}e^2e'^3s^2$$
$$+\tfrac{3}{4}e^2e's'^2 - \tfrac{3}{4}e^4e's'^2 - \tfrac{9}{16}e^2e'^3s'^2 - \tfrac{3}{4}e^2e's^2s'^2$$

8E1.5

λ'	λ	ϖ'	ϖ	Ω'	Ω
1	0	−2	1	0	0

$$\tfrac{3}{16}ee'^2 + \tfrac{1}{16}ee'^4 + \tfrac{75}{2048}ee'^6 - \tfrac{3}{16}ee'^2s^2 - \tfrac{1}{16}ee'^4s^2 - \tfrac{3}{16}ee'^2s'^2 - \tfrac{1}{16}ee'^4s'^2 + \tfrac{3}{16}ee'^2s^2s'^2$$

8E1.6

λ'	λ	ϖ'	ϖ	Ω'	Ω
3	−2	−2	1	0	0

$$-\tfrac{27}{16}ee'^2 + \tfrac{81}{64}e^3e'^2 - \tfrac{45}{256}e^5e'^2 + \tfrac{27}{16}ee'^4 - \tfrac{81}{64}e^3e'^4 - \tfrac{999}{2048}ee'^6 + \tfrac{27}{16}ee'^2s^2 - \tfrac{81}{64}e^3e'^2s^2 - \tfrac{27}{16}ee'^4s^2$$
$$+\tfrac{27}{16}ee'^2s'^2 - \tfrac{81}{64}e^3e'^2s'^2 - \tfrac{27}{16}ee'^4s'^2 - \tfrac{27}{16}ee'^2s^2s'^2$$

8E1.7

λ'	λ	ϖ'	ϖ	Ω'	Ω
1	0	0	1	0	−2

$$\tfrac{3}{2}es^2 - \tfrac{3}{4}ee'^2s^2 - \tfrac{3}{128}ee'^4s^2 - \tfrac{3}{2}es^2s'^2 + \tfrac{3}{4}ee'^2s^2s'^2$$

8E1.8

λ'	λ	ϖ'	ϖ	Ω'	Ω
1	−2	0	1	−1	1

$$-ess' + \tfrac{3}{4}e^3ss' - \tfrac{5}{48}e^5ss' + \tfrac{1}{2}ee'^2ss' - \tfrac{3}{8}e^3e'^2ss' + \tfrac{1}{64}ee'^4ss' + \tfrac{1}{2}es^3s' - \tfrac{3}{8}e^3s^3s' - \tfrac{1}{4}ee'^2s^3s'$$
$$+\tfrac{1}{8}es^5s' + \tfrac{1}{2}ess'^3 - \tfrac{3}{8}e^3ss'^3 - \tfrac{1}{4}ee'^2ss'^3 - \tfrac{1}{4}es^3s'^3 + \tfrac{1}{8}ess'^5$$

8E1.9

λ'	λ	ϖ'	ϖ	Ω'	Ω
1	0	0	−1	−1	1

$$3ess' - \tfrac{3}{2}ee'^2ss' - \tfrac{3}{64}ee'^4ss' - \tfrac{3}{2}es^3s' + \tfrac{3}{4}ee'^2s^3s' - \tfrac{3}{8}es^5s' - \tfrac{3}{2}ess'^3 + \tfrac{3}{4}ee'^2ss'^3 + \tfrac{3}{4}es^3s'^3$$
$$-\tfrac{3}{8}ess'^5$$

Indirect Part (Internal Perturber), Order 1

8I1.1

λ'	λ	ϖ'	ϖ	Ω'	Ω
1	−2	0	1	0	0

$$-2e + \tfrac{3}{2}e^3 - \tfrac{5}{24}e^5 + \tfrac{1}{18}e^7 + ee'^2 - \tfrac{3}{4}e^3e'^2 + \tfrac{5}{48}e^5e'^2 + \tfrac{1}{32}ee'^4 - \tfrac{3}{128}e^3e'^4 + \tfrac{29}{576}ee'^6 + 2es^2 - \tfrac{3}{2}e^3s^2 + \tfrac{5}{24}e^5s^2 - ee'^2s^2 + \tfrac{3}{4}e^3e'^2s^2 - \tfrac{1}{32}ee'^4s^2 + 2es'^2 - \tfrac{3}{2}e^3s'^2 + \tfrac{5}{24}e^5s'^2 - ee'^2s'^2 + \tfrac{3}{4}e^3e'^2s'^2 - \tfrac{1}{32}ee'^4s'^2 - 2es^2s'^2 + \tfrac{3}{2}e^3s^2s'^2 + ee'^2s^2s'^2$$

8I1.2

λ'	λ	ϖ'	ϖ	Ω'	Ω
0	1	−1	0	0	0

$$\tfrac{3}{2}e' - \tfrac{3}{4}e^2e' - \tfrac{3}{128}e^4e' - \tfrac{29}{768}e^6e' - \tfrac{3}{2}e's^2 + \tfrac{3}{4}e^2e's^2 + \tfrac{3}{128}e^4e's^2 - \tfrac{3}{2}e's'^2 + \tfrac{3}{4}e^2e's'^2 + \tfrac{3}{128}e^4e's'^2 + \tfrac{3}{2}e's^2s'^2 - \tfrac{3}{4}e^2e's^2s'^2$$

8I1.3

λ'	λ	ϖ'	ϖ	Ω'	Ω
2	−1	−1	0	0	0

$$-\tfrac{1}{2}e' + \tfrac{1}{4}e^2e' + \tfrac{1}{128}e^4e' + \tfrac{29}{2304}e^6e' + \tfrac{3}{8}e'^3 - \tfrac{3}{16}e^2e'^3 - \tfrac{3}{512}e^4e'^3 - \tfrac{5}{96}e'^5 + \tfrac{5}{192}e^2e'^5 + \tfrac{1}{72}e'^7 + \tfrac{1}{2}e's^2 - \tfrac{1}{4}e^2e's^2 - \tfrac{1}{128}e^4e's^2 - \tfrac{3}{8}e'^3s^2 + \tfrac{3}{16}e^2e'^3s^2 + \tfrac{5}{96}e'^5s^2 + \tfrac{1}{2}e's'^2 - \tfrac{1}{4}e^2e's'^2 - \tfrac{1}{128}e^4e's'^2 - \tfrac{3}{8}e'^3s'^2 + \tfrac{3}{16}e^2e'^3s'^2 + \tfrac{5}{96}e'^5s'^2 - \tfrac{1}{2}e's^2s'^2 + \tfrac{1}{4}e^2e's^2s'^2 + \tfrac{3}{8}e'^3s^2s'^2$$

8I1.4

λ'	λ	ϖ'	ϖ	Ω'	Ω
0	1	1	−2	0	0

$$\tfrac{3}{16}e^2e' + \tfrac{1}{16}e^4e' + \tfrac{75}{2048}e^6e' - \tfrac{3}{16}e^2e's^2 - \tfrac{1}{16}e^4e's^2 - \tfrac{3}{16}e^2e's'^2 - \tfrac{1}{16}e^4e's'^2 + \tfrac{3}{16}e^2e's^2s'^2$$

8I1.5

λ'	λ	ϖ'	ϖ	Ω'	Ω
2	−3	−1	2	0	0

$$-\tfrac{27}{16}e^2e' + \tfrac{27}{16}e^4e' - \tfrac{999}{2048}e^6e' + \tfrac{81}{64}e^2e'^3 - \tfrac{81}{64}e^4e'^3 - \tfrac{45}{256}e^2e'^5 + \tfrac{27}{16}e^2e's^2 - \tfrac{27}{16}e^4e's^2 - \tfrac{81}{64}e^2e'^3s^2 + \tfrac{27}{16}e^2e's'^2 - \tfrac{27}{16}e^4e's'^2 - \tfrac{81}{64}e^2e'^3s'^2 - \tfrac{27}{16}e^2e's^2s'^2$$

8I1.6

λ'	λ	ϖ'	ϖ	Ω'	Ω
3	−2	−2	1	0	0

$$-\tfrac{3}{4}ee'^2 + \tfrac{9}{16}e^3e'^2 - \tfrac{5}{64}e^5e'^2 + \tfrac{3}{4}ee'^4 - \tfrac{9}{16}e^3e'^4 - \tfrac{111}{512}ee'^6 + \tfrac{3}{4}ee'^2s^2 - \tfrac{9}{16}e^3e'^2s^2 - \tfrac{3}{4}ee'^4s^2 + \tfrac{3}{4}ee'^2s'^2 - \tfrac{9}{16}e^3e'^2s'^2 - \tfrac{3}{4}ee'^4s'^2 - \tfrac{3}{4}ee'^2s^2s'^2$$

8I1.7

λ'	λ	ϖ'	ϖ	Ω'	Ω
0	1	1	0	0	−2

$$\tfrac{3}{2}e's^2 - \tfrac{3}{4}e^2e's^2 - \tfrac{3}{128}e^4e's^2 - \tfrac{3}{2}e's^2s'^2 + \tfrac{3}{4}e^2e's^2s'^2$$

8I1.8

λ'	λ	ϖ'	ϖ	Ω'	Ω
1	−2	0	1	−1	1

$$-4ess' + 3e^3ss' - \tfrac{5}{12}e^5ss' + 2ee'^2ss' - \tfrac{3}{2}e^3e'^2ss' + \tfrac{1}{16}ee'^4ss' + 2es^3s' - \tfrac{3}{2}e^3s^3s' - ee'^2s^3s' + \tfrac{1}{2}es^5s' + 2ess'^3 - \tfrac{3}{2}e^3ss'^3 - ee'^2ss'^3 - es^3s'^3 + \tfrac{1}{2}ess'^5$$

8I1.9

λ'	λ	ϖ'	ϖ	Ω'	Ω
0	1	−1	0	1	−1

$$3e'ss' - \tfrac{3}{2}e^2e'ss' - \tfrac{3}{64}e^4e'ss' - \tfrac{3}{2}e's^3s' + \tfrac{3}{4}e^2e's^3s' - \tfrac{3}{8}e's^5s' - \tfrac{3}{2}e'ss'^3 + \tfrac{3}{4}e^2e'ss'^3 + \tfrac{3}{4}e's^3s'^3 - \tfrac{3}{8}e'ss'^5$$

$+\frac{15}{32}\left[-2\alpha^3+9j\alpha^3-4j^2\alpha^3+4\alpha^4D-4j\alpha^4D-\alpha^5D^2\right]b_{7/2}^{(j+1)}$

2 0 3 3 $\frac{15}{32}\left[-8\alpha^2+13j\alpha^2-4j^2\alpha^2+6\alpha^3D-4j\alpha^3D-\alpha^4D^2\right]b_{5/2}^{(j)}$
$+\frac{15}{16}\left[-2\alpha^3+9j\alpha^3-4j^2\alpha^3+4\alpha^4D-4j\alpha^4D-\alpha^5D^2\right]b_{7/2}^{(j-1)}$
$+\frac{45}{32}\left[-2\alpha^3+9j\alpha^3-4j^2\alpha^3+4\alpha^4D-4j\alpha^4D-\alpha^5D^2\right]b_{7/2}^{(j+1)}$

8D4.39

	λ'	λ	ϖ'	ϖ	Ω'	Ω
$e\ e'\ s\ s'$	j	$4-j$	-1	-1	1	-3

1 1 3 1 $\frac{3}{8}\left[4\alpha^2+6j\alpha^2-4j^2\alpha^2+2\alpha^3D-4j\alpha^3D-\alpha^4D^2\right]b_{5/2}^{(j-1)}$

3 1 3 1 $\frac{3}{64}\left[-184\alpha^2-156j\alpha^2+350j^2\alpha^2-140j^3\alpha^2+16j^4\alpha^2-44\alpha^3D+274j\alpha^3D\right.$
$-144j^2\alpha^3D+16j^3\alpha^3D+64\alpha^4D^2-49j\alpha^4D^2-6\alpha^5D^3-4j\alpha^5D^3$
$\left.-\alpha^6D^4\right]b_{5/2}^{(j-1)}$

1 3 3 1 $\frac{3}{64}\left[48\alpha^2+60j\alpha^2-34j^2\alpha^2-36j^3\alpha^2+16j^4\alpha^2+72\alpha^3D-26j\alpha^3D\right.$
$-48j^2\alpha^3D+16j^3\alpha^3D-6\alpha^4D^2-35j\alpha^4D^2-10\alpha^5D^3-4j\alpha^5D^3$
$\left.-\alpha^6D^4\right]b_{5/2}^{(j-1)}$

1 1 5 1 $\frac{3}{16}\left[-4\alpha^2-6j\alpha^2+4j^2\alpha^2-2\alpha^3D+4j\alpha^3D+\alpha^4D^2\right]b_{5/2}^{(j-1)}$
$+\frac{45}{32}\left[-6\alpha^3-2j\alpha^3+4j^2\alpha^3+4j\alpha^4D+\alpha^5D^2\right]b_{7/2}^{(j-2)}$
$+\frac{15}{16}\left[-6\alpha^3-2j\alpha^3+4j^2\alpha^3+4j\alpha^4D+\alpha^5D^2\right]b_{7/2}^{(j)}$

1 1 3 3 $\frac{15}{16}\left[-4\alpha^2-6j\alpha^2+4j^2\alpha^2-2\alpha^3D+4j\alpha^3D+\alpha^4D^2\right]b_{5/2}^{(j-1)}$
$+\frac{15}{8}\left[-6\alpha^3-2j\alpha^3+4j^2\alpha^3+4j\alpha^4D+\alpha^5D^2\right]b_{7/2}^{(j-2)}$
$+\frac{45}{16}\left[-6\alpha^3-2j\alpha^3+4j^2\alpha^3+4j\alpha^4D+\alpha^5D^2\right]b_{7/2}^{(j)}$

8D4.40

	λ'	λ	ϖ'	ϖ	Ω'	Ω
$e\ e'\ s\ s'$	j	$4-j$	-1	1	-1	-3

1 1 3 1 $\frac{3}{8}\left[12\alpha^2+22j\alpha^2-4j^2\alpha^2+14\alpha^3D+\alpha^4D^2\right]b_{5/2}^{(j-3)}$

3 1 3 1 $\frac{3}{64}\left[-872\alpha^2-1180j\alpha^2+1010j^2\alpha^2-228j^3\alpha^2+16j^4\alpha^2-900\alpha^3D+586j\alpha^3D\right.$
$\left.-76j^2\alpha^3D+40\alpha^4D^2+55j\alpha^4D^2-8j^2\alpha^4D^2+22\alpha^5D^3+\alpha^6D^4\right]b_{5/2}^{(j-3)}$

1 3 3 1 $\frac{3}{64}\left[144\alpha^2+228j\alpha^2-18j^2\alpha^2-100j^3\alpha^2+16j^4\alpha^2+376\alpha^3D+186j\alpha^3D\right.$
$\left.-84j^2\alpha^3D+182\alpha^4D^2+23j\alpha^4D^2-8j^2\alpha^4D^2+26\alpha^5D^3+\alpha^6D^4\right]b_{5/2}^{(j-3)}$

1 1 5 1 $\frac{3}{16}\left[-12\alpha^2-22j\alpha^2+4j^2\alpha^2-14\alpha^3D-\alpha^4D^2\right]b_{5/2}^{(j-3)}$
$+\frac{45}{32}\left[-26\alpha^3-22j\alpha^3+4j^2\alpha^3-16\alpha^4D-\alpha^5D^2\right]b_{7/2}^{(j-4)}$
$+\frac{15}{16}\left[-26\alpha^3-22j\alpha^3+4j^2\alpha^3-16\alpha^4D-\alpha^5D^2\right]b_{7/2}^{(j-2)}$

1 1 3 3 $\frac{9}{16}\left[-12\alpha^2-22j\alpha^2+4j^2\alpha^2-14\alpha^3D-\alpha^4D^2\right]b_{5/2}^{(j-3)}$
$+\frac{15}{16}\left[-26\alpha^3-22j\alpha^3+4j^2\alpha^3-16\alpha^4D-\alpha^5D^2\right]b_{7/2}^{(j-4)}$
$+\frac{15}{4}\left[-26\alpha^3-22j\alpha^3+4j^2\alpha^3-16\alpha^4D-\alpha^5D^2\right]b_{7/2}^{(j-2)}$

8D4.41

	λ'	λ	ϖ'	ϖ	Ω'	Ω
$e\ e'\ s\ s'$	j	$4-j$	1	-1	-1	-3

1 1 3 1 $\frac{3}{8}\left[-4\alpha^2+10j\alpha^2-4j^2\alpha^2-2\alpha^3D+\alpha^4D^2\right]b_{5/2}^{(j-1)}$

3 1 3 1 $\frac{3}{64}\left[184\alpha^2-580j\alpha^2+498j^2\alpha^2-156j^3\alpha^2+16j^4\alpha^2+44\alpha^3D+6j\alpha^3D\right.$
$\left.-12j^2\alpha^3D-64\alpha^4D^2+41j\alpha^4D^2-8j^2\alpha^4D^2+6\alpha^5D^3+\alpha^6D^4\right]b_{5/2}^{(j-1)}$

1 3 3 1 $\frac{3}{64}\left[-48\alpha^2+108j\alpha^2-50j^2\alpha^2-28j^3\alpha^2+16j^4\alpha^2-72\alpha^3D+86j\alpha^3D\right.$
$\left.-20j^2\alpha^3D+6\alpha^4D^2+9j\alpha^4D^2-8j^2\alpha^4D^2+10\alpha^5D^3+\alpha^6D^4\right]b_{5/2}^{(j-1)}$

1 1 5 1 $\frac{3}{16}\left[4\alpha^2-10j\alpha^2+4j^2\alpha^2+2\alpha^3D-\alpha^4D^2\right]b_{5/2}^{(j-1)}$

	λ'	λ	ϖ'	ϖ	Ω'	Ω
8I8.37	6	2	−5	−1	−2	0
$-\frac{27}{40}ee'^5s'^2$						

	λ'	λ	ϖ'	ϖ	Ω'	Ω
8I8.38	7	1	−6	0	−2	0
$-\frac{16807}{46080}e'^6s'^2$						