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FOUR-BODY EFFECTS IN GLOBULAR CLUSTER BLACK HOLE COALESCENCE

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ABSTRACT

In the high density cores of globular clusters, multibody interactions are expected to be common, with the result that black holes in binaries are hardened by interactions. It was shown by Sigurdsson & Hernquist (1993) and others that $10\,M_\odot$ black holes interacting exclusively by three-body encounters do not merge in the clusters themselves, because recoil kicks the binaries out of the clusters before the binaries are tight enough to merge. Here we consider a new mechanism, involving four-body encounters. Numerical simulations by a number of authors suggest that roughly 20-50% of binary-binary encounters will eject one star but leave behind a stable hierarchical triple. If the orbital plane of the inner binary is strongly tilted with respect to the orbital plane of the outer object, a secular Kozai resonance, first investigated in the context of asteroids in the Solar System, can increase the eccentricity of the inner body significantly. We show that in a substantial fraction of cases the eccentricity is driven to a high enough value that the inner binary will merge by gravitational radiation, without a strong accompanying kick. Thus the merged object remains in the cluster; depending on the binary fraction of black holes and the inclination distribution of newly-formed hierarchical triples, this mechanism may allow massive black holes to accumulate through successive mergers in the cores of globular clusters. It may also increase the likelihood that stellar-mass black holes in globular clusters will be detectable by their gravitational radiation.

Subject headings: black hole physics — (Galaxy:) globular clusters: general — gravitational waves — stellar dynamics

1. INTRODUCTION

Globular clusters are outstanding testbeds for dynamics. As dense systems with ages many times their core relaxation time, they display such features as core collapse and mass segregation, and they are almost certainly affected strongly by the presence of even a small number of binaries. It has long been speculated that various processes might produce relatively massive black holes in their cores (e.g., Wyller 1970; Bahcall & Ostriker 1975; Frank & Rees 1976; Lightman & Shapiro 1977; Marchant & Shapiro 1980; Quinlan & Shapiro 1987; Portegies Zwart et al. 1999; Ebisuzaki et al. 2001). Recent observations of some dense clusters provide tentative evidence for black holes as massive as $2500\,M_\odot$ at their centers (Gebhardt et al. 2000).

Qualitatively, it seems entirely reasonable that large black holes should grow in the cores of many clusters. Even at birth, black holes are much more massive than the average star in a cluster, and hence they sink rapidly towards the core. When in the core, they tend to exchange into binaries. If the binary is hard (i.e., if its binding energy exceeds the average kinetic energy of a field star), then a subsequent interaction with a field star tends to harden the binary (e.g., Heggie 1975). If this process is repeated often enough, the binary becomes tight enough that it can merge by gravitational radiation, and the black hole becomes larger. If these binaries merge while still in the cluster, sources in globulars could be excellent prospects for detection by the upcoming generation of gravitational wave instruments.

However, it has been shown (e.g., Sigurdsson & Hernquist 1993; Portegies Zwart & McMillan 2000) that if all black holes have initial masses of $10\,M_\odot$, three-body encounters alone do *not* lead to the formation of a large black hole at the center. The reason is that hardening in a binary-single interaction is accompanied by recoil, which kicks the binary out of the cluster before it can merge. Without additional effects, this means that the mergers occur well away from their host globulars. If the initial mass of a black hole is $\gtrsim 50\,M_\odot$, as may result from a high-mass low-metallicity star or rapid merger of main sequence stars, it has enough inertia to remain in the core and grow by coalescence (Miller & Hamilton 2002). But what if only low-mass black holes are produced?

Here we propose a new mechanism for the coalescence of low-mass black holes in globular clusters, involving binary-binary interactions. Studies of such four-body encounters have been comparatively rare, but have shown that in roughly 20-50% of the interactions the final state is an unbound single star plus a stable hierarchical triple system (Mikkola 1984; McMillan, Hut, & Makino 1991; Rasio, McMillan, & Hut 1995). This allows an important new effect: studies of planetary and stellar systems have shown that if there is a large relative inclination between the orbital planes of the inner binary and the outer object of the triple, then over many orbital periods the relative inclination periodically trades off with the eccentricity of the inner binary, sometimes leading to very high eccentricities (Kozai 1962; Harrington 1968, 1974; Lidov & Ziglin 1976). In turn, this can enhance the gravitational radiation rate enormously, leading to merger without a strong kick and allowing even low-mass binary black holes in globulars to be potential gravitational wave sources.

In § 2 we discuss the principles of this resonance, as derived in the case of three objects of arbitrary mass by Lidov & Ziglin (1976). To their treatment we add, in § 3, a simple term that accounts for general relativistic pericenter precession. We show that although, as expected, this precession decreases the maximum attainable eccentricity for a given set of initial conditions, the decrease is typically minor and thus there is significant phase space in which the eccentricity resonance leads to rapid merger. In § 4 we use these results in a simple model for the mergers of black holes, and show that, depending on the fraction of black holes in binaries, this effect can lead to a dramatic increase in the retention of black holes in globulars, and to the growth of $\sim 10^{2-3}\, M_\odot$ black holes in their cores.

2. PRINCIPLES OF THE KOZAI RESONANCE

When looking for changes in the orbital properties of a three-body system that extend over many orbital periods of both the inner binary and the outer tertiary, it is convenient to average the motion over both these periods, a procedure called double averaging. A general analysis of the double-averaged three-body problem has been performed to quadrupolar order for Newtonian gravity by Lidov & Ziglin (1976) in Hill's case, in which the distance of the outer object (of mass m_2) from the inner binary (with component masses m_0 and $m_1 \leq m_0$) is much greater than the semimajor axis of the inner binary. They find that for any set of three masses there is always a relative inclination of orbits such that an inner binary with arbitrarily small initial eccentricity will evolve to $e = 1$. For example, in the restricted three-body problem in which $m_0 \gg m_2 \gg m_1$ (e.g., the Sun, Jupiter, and an asteroid interior to Jupiter's orbit; see Kozai 1962), a relative orbital inclination of 90° will cause the asteroid to evolve to $e = 1$ in a finite time.

However, the growth to such high eccentricity depends on a long series of perturbations from the tertiary that add coherently, and hence requires certain phase relations. An extra source of precession of the pericenter can interfere with this. For example, the orbits of the moons of Uranus are tipped by 97° with respect to Uranus' orbit around the Sun, but their eccentricities stay relatively low due to precession introduced by the quadrupole moment of Uranus. In the case of black holes or other close massive objects, a similar role may be played by the effects of general relativity, which to lowest order includes precession of the pericenter. How does this affect the maximum eccentricity for a given set of initial conditions?

Hill's approximation allows us to treat the system as two nested binaries: the inner pair composed of m_0 and m_1, and a second pair consisting of i) an object of mass $m_0 + m_1$ located at their center of mass and ii) m_2. Defining variables as in Lidov & Ziglin (1976), we let $M_1 = m_0 + m_1$ and $M_2 = m_0 + m_1 + m_2$ be the total masses of the two binaries, and $\mu_1 = m_0 m_1/M_1$, and $\mu_2 = m_2 M_1/M_2$ be their reduced masses. Let the semimajor axes and eccentricities of the two binaries be a_1, e_1 and a_2, e_2, and define i_1 and i_2 to be the inclinations of the binaries relative to the invariant plane of angular momentum of the system. Finally, let $\mu = G m_0 m_1 m_2 / M_1$ and $\epsilon = 1 - e_1^2$.

The double-averaged Hamiltonian $\bar{\mathcal{H}}$ admits several integrals each of which yields a constant of the motion. First, the double-averaging procedure guarantees that a_1 and a_2 are constant. We keep terms in the Hamiltonian up to linear order in a_1/a_2; these quadrupolar terms dominate the evolution of the system for the high relative inclinations of interest here (see Ford, Kozinsky, & Rasio 2000). To this order, e_2 is also constant. The problem has two constants of the motion that are related to angular momentum:

$$\alpha = \epsilon^{1/2}\cos i_1 + \beta \cos i_2 \ , \qquad \beta = \frac{\mu_2\sqrt{M_2}}{\mu_1\sqrt{M_1}}\sqrt{\frac{a_2}{a_1}\,(1-e_2^2)} \ . \tag{1}$$

The constant β (a combination of the constants a_1, a_2, and e_2) represents the total angular momentum of the outer binary while α is the total system angular momentum (with contributions from both the inner and outer binaries). Both α and β are made dimensionless by dividing by $L_1 = \mu_1\sqrt{G M_1 a_1}$, the angular momentum that the inner binary would have if it were on a circular orbit.

The Hamiltonian, $\bar{\mathcal{H}}$, itself is constant. For convenience, we define $\bar{\mathcal{H}} = -k(W + \frac{5}{3})$, with $k = 3\mu a_1^2/\left[8 a_2^3 (1-e_2^2)^{3/2}\right]$ and obtain:

$$W = -2\epsilon + \epsilon\cos^2 I + 5(1-\epsilon)\sin^2\omega\left(\cos^2 I - 1\right) \ , \tag{2}$$

which is equation (30) from Lidov & Ziglin (1976). Here ω is the argument of pericenter of the inner binary and the scaled angular momenta α, β, and $\sqrt{\epsilon}$ form a triangle from which the relative inclination $I = i_1 + i_2$ can be obtained using the law of cosines:

$$\cos I = \frac{\alpha^2 - \beta^2 - \epsilon}{2\beta\sqrt{\epsilon}} \ . \tag{3}$$

The maximum ϵ (and hence minimum e_1) occurs for $\omega = 0$, and the minimum ϵ (and hence maximum e_1) occurs for $\omega = \pi/2$; see Lidov & Ziglin (1976). Therefore, given initial values for ϵ_0 and ω_0, the maximum eccentricity may be derived from conservation of W at $\omega = \pi/2$. The time required to push the system from its minimum to maximum eccentricity, is of order

$$\tau_{\rm evol} \approx f\left(\frac{M_1}{m_2}\frac{b_2^3}{a_1^3}\right)^{1/2}\left(\frac{b_2^3}{G m_2}\right)^{1/2} \tag{4}$$

(e.g., Innanen et al. 1997), where $b_2 = a_2\sqrt{1-e_2^2}$ is the semiminor axis of the tertiary and typically $f \sim$ few for I near 90°, which is the case of interest here.

3. THE KOZAI RESONANCE WITH GR PRECESSION

Post-Newtonian precession may be included in a couple of equivalent ways. One is to modify the Hamiltonian directly, by changing the gravitational potential to simulate some of the effects of general relativity. The modification of the potential is not unique, and depends on which aspect of general relativity is to be reproduced (see Artemova, Bjornsson, & Novikov 1996). For our purposes it is the precession of pericenter that is important (as opposed to, e.g., the location of the innermost stable circular orbit), and hence the correct lowest-order modification is $-GM/r \to (-GM/r)(1 + 3GM/rc^2)$ (Artemova et al. 1996). Averaging the correction term over the orbits of the tertiary and inner binary, we obtain a correction to the double-averaged Hamiltonian of

$$\bar{\mathcal{H}}_{\rm PN} = -\frac{3(G m_0)^2 m_1}{a_1^2 c^2 \epsilon^{1/2}} = -k W_{\rm PN} \ . \tag{5}$$

This result may also be obtained from the first-order general relativistic precession rate of $d\omega = (6\pi G M_1/\left[a_1(1-e_1^2)c^2\right]$ over one binary period (see Misner, Thorne, & Wheeler 1973, p. 1110) using the equation of motion $d\omega/dt =$

$-(2k\sqrt{\epsilon}/L_1)(\partial W/\partial\epsilon)$ derived by Lidov & Ziglin (1976). Substituting and integrating, we find that the first-order post-Newtonian contribution to W is

$$W_{\rm PN} = \frac{8}{\sqrt{\epsilon}}\frac{M_1}{m_2}\left(\frac{b_2}{a_1}\right)^3\frac{GM_1}{a_1c^2} \equiv \theta_{\rm PN}\epsilon^{-1/2} . \qquad (6)$$

in agreement with equation (5). We have also checked our expressions with direct numerical three-body integrations; note that equation (5) corrects a factor of two error in equation (19) of Lin et al. (2000).

Adding the new term $W_{\rm PN}$ to equation (2) and making use of equation (3), we find

$$W = -2\epsilon + \epsilon\cos^2 I + 5(1-\epsilon)\sin^2\omega(\cos^2 I - 1) + \theta_{\rm PN}/\epsilon^{1/2} . \qquad (7)$$

As in the previous section, for a given set of initial conditions, one can therefore solve for the minimum ϵ (maximum e), by setting $\omega = \pi/2$ and using the conservation of W. In general we expect that initially the inner binary will have significant eccentricity caused by perturbations during the four-body encounter, but for simplicity we will assume that the initial eccentricity is small enough that $\epsilon_0 \approx 1$. In the restricted three-body problem in which $m_0 \gg m_2 \gg m_1$ and the initial relative inclination is I_0, the approximate solution for $\epsilon_{\rm min}$ when $5\cos^2 I_0 \ll 3$ (high inclination) and $\theta_{\rm PN} \ll 3$ (weak precession) is

$$\epsilon_{\rm min}^{1/2} \approx \frac{1}{6}\left[\theta_{\rm PN} + \sqrt{\theta_{\rm PN}^2 + 60\cos^2 I_0}\right] . \qquad (8)$$

When $60\cos^2 I_0 \gg \theta_{\rm PN}$ this reduces to the Newtonian solution, in which the maximum eccentricity is $e_{\rm max} = \sqrt{1-(5/3)\cos^2 I_0}$ (Innanen et al. 1997). If instead $I_0 \approx \pi/2$ so that $60\cos^2 I_0 \ll \theta_{\rm PN}$, then $e_{\rm max} \approx 1 - \theta_{\rm PN}^2/9$. More generally, for any set of masses, if $e \to 1$ is allowed in the Newtonian problem then $e_{\rm max} = 1 - \mathcal{O}(\theta_{\rm PN}^2)$ when general relativistic precession is included. Numerically, for $M_1 = M_\odot$ and a_1 =1 AU, $\theta_{\rm PN} = 8\times10^{-8}(M_1/m_2)(b_2/a_1)^3$. Equation (8) shows that in the restricted three-body problem the maximum possible eccentricity (minimum $\epsilon_{\rm min}$) is attained for the initial condition $I_0 = \pi/2$ (initially perpendicular circular orbits). If m_1 has non-negligible mass, so that m_2 dominates the total angular momentum less, then the critical I_0 increases (Lidov & Ziglin 1976). Figure 1 shows the critical inclination in the Newtonian case ($\theta_{\rm PN} = 0$) for several mass ratios and semimajor axes.

We want to know whether this process can cause the inner binary to reach a high enough eccentricity that it merges by gravitational radiation before the next encounter with a star in the globular cluster (which will typically alter the eccentricities and inclinations significantly). Encounters with black holes in globular clusters are usually dominated by gravitational focusing instead of the pure geometrical cross section; this is true within $\sim$ 100AU of a 10 $M_\odot$ black hole, where we have assumed a velocity dispersion of 10 km s^{-1} for the interlopers (see Miller & Hamilton 2002). In this limit the encounter time is

$$\tau_{\rm enc} \approx 6\times10^5 n_6^{-1}(1\ {\rm AU}/a_2)(10\,M_\odot/M_2)\ {\rm yr} , \qquad (9)$$

where the number density of stars in the core of the globular is $10^6 n_6$ pc^{-3}. Note that it is the semimajor axis of the outermost object, m_2, that sets the encounter time scale, because in a stable hierarchical triple a_2 must be a factor of several greater than a_1.

The timescale for merger by gravitational radiation for a high eccentricity orbit is (Peter 1964)

$$\tau_{\rm GR} \approx 5\times10^{17}\left(\frac{M_\odot^3}{M_1^2\mu_1}\right)\left(\frac{a_1}{1\ {\rm AU}}\right)^4\epsilon^{7/2}\ {\rm yr} . \qquad (10)$$

The steep dependence on eccentricity means that shrinkage of the orbit is dominated by the time spent near maximum eccentricity. Assuming that $\tau_{\rm GR} \gg \tau_{\rm evol}$ so that orbital decay occurs over many Kozai oscillation cycles, one finds that the fraction of time spent near $e_{\rm max}$ is of order $\epsilon_{\rm min}^{1/2}$ (Innanen et al. 1997, equation [5]), so that $\tau_{\rm GR} \approx 5\times10^{17}\left(\frac{M_\odot^3}{M_1^2\mu_1}\right)\left(\frac{a_1}{1\ {\rm AU}}\right)^4\epsilon_{\rm min}^3$ yr. The condition for merger before an encounter is then simply $\tau_{\rm GR} < \tau_{\rm enc}$.

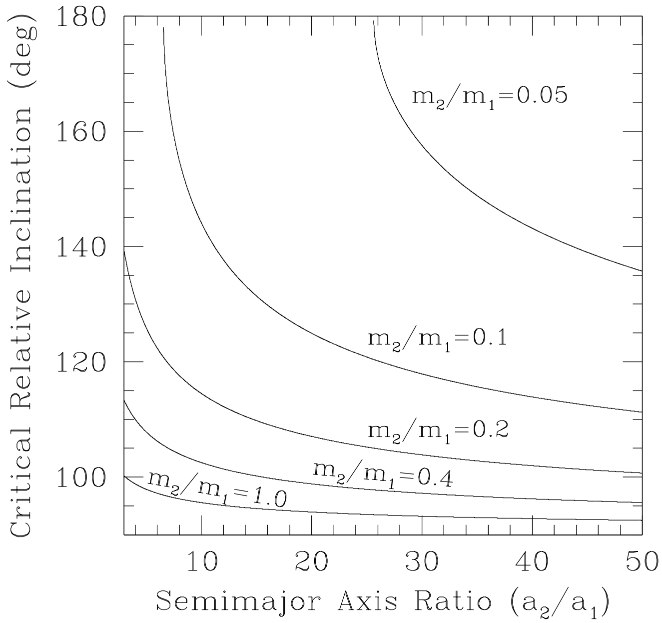

FIG. 1.—Critical relative inclinations for evolution from $e \approx 0$ to $e = 1$ in the Newtonian case; attaining $e = 1$ requires $\alpha = \beta$ in equation (3). Here the inner binary is composed of equal mass stars, $m_0 = m_1$, and the labels on the curves indicate the mass ratio m_2/m_1. As the fraction of the total angular momentum supplied by the tertiary increases (larger β and therefore larger m_2/m_1 or a_2/a_1), the critical inclination trends toward 90°.

Note that in the Newtonian case $\theta_{\rm PN} \equiv 0$, all systems with the same masses, b_2/a_1, and I_0 are dynamically identical, in that the maximum eccentricity does not depend on the individual values of b_2 and a_1. The introduction of post-Newtonian precession breaks this scaling. If b_2/a_1 is fixed, then $\theta_{\rm PN} \propto a_1^{-1}$ and therefore the maximum eccentricity attained is given by $\epsilon_{\rm min} \propto a_1^{-2}$ (cf. equation [8] for the restricted problem). The merger time is $\tau_{\rm GR} \propto a_1^4\epsilon_{\rm min}^3 \propto a_1^{-2}$. That is, a *wider* binary can be pushed to higher eccentricities, and actually merge faster, than a closer binary. Note, however, that the solid angle for this orientation is proportional to $\theta_{\rm PN} \propto a_1^{-1}$, because the optimum angle is usually close to $\pi/2$, so the solid angle is proportional to the cosine of the inclination. Therefore, if binary-binary interactions leave the binary and tertiary inclinations randomly oriented with respect to each other then a smaller fraction of wide binaries will fall into the optimal orientation. Qualitatively this means that as the binary is hardened by various interactions, every time a triple is formed it has a chance to push the eccentricity high enough that the binary merges before the next encounter. The smaller the system, the larger the probability of such an orientation, because both the solid angle and the encounter time are larger.

One way to quantify the probability of merger through the increase of eccentricity is to plot, as a function of the semimajor axis of the inner binary, the range of relative inclinations such that merger occurs before the next encounter of a field black hole with the tertiary (which, being on a wide orbit, will interact before the inner binary will on average). In Figure 2, we assume three $10\,M_\odot$ black holes, with a given a_1 and a_2. From a_2 and an assumed number density of stars in the cluster ($n = 10^6$ pc^{-3}), we compute the average time $\tau_{\rm enc}$ to the next encounter within a distance a_2 of the system. We then determine the range of initial inclinations I such that $\tau_{\rm GR} < \tau_{\rm enc}$, by solving for $\epsilon_{\rm min}$ using equation (7) with the initial conditions $e_1 = e_2 = 0.01$ and $\omega = 0$. Note that for wider tertiary orbits, the total angular momentum of the system is dominated more by the tertiary (larger β), and hence the relative inclination that gives the smallest possible $\epsilon_{\rm min}$ is closer to $90°$ (see Figure 1). If a single Kozai oscillation cycle is longer than $\tau_{\rm enc}$ the system never attains the required high eccentricity. This causes the cutoff in the $a_2 = 10a_1$ and $a_2 = 20a_1$ curves in Figure 2; similar cutoffs exist at $a_1 > 10$ AU for the remaining two curves.

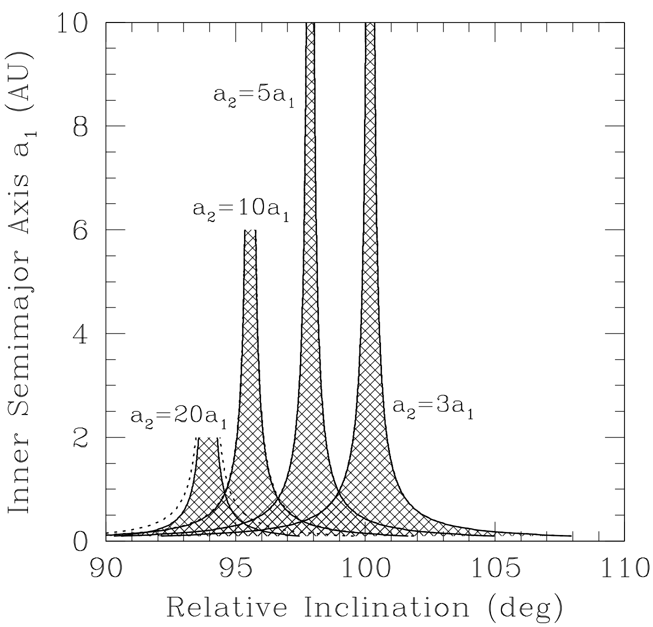

FIG. 2.—Inclination ranges for merger by gravitational radiation. For this graph, we assume that all three black holes have mass $M = 10\,M_\odot$, and we assume a globular core number density $n = 10^6$ pc^{-3} for calculating $\tau_{\rm enc}$. The shaded regions indicate ranges of the relative inclination for which mergers will occur for each of four value of the semimajor axis ratio a_2/a_1: 3, 5, 10, and 20. The peaks, which occur at the locations predicted by the bottom curve in Fig. 1, are truncated where the time to increase the eccentricity of the inner binary is greater than the mean time to an encounter ($\tau_{\rm evol} < \tau_{\rm enc}$). For comparison, the dotted lines are the boundaries of the regions if general relativistic precession is suppressed; only for $a_2 = 20a_1$ is there a noticeable difference.

4. CONCLUSIONS

The level of importance of the Kozai mechanism depends on several factors including: i) details of the interactions between two binaries, ii) details of the interactions between a triple, and either a binary or a single star, and iii) the fraction of black holes in binaries, which in turn relies on the iv) dynamics of the cluster itself. Understanding these interactions statistically will require extensive long-term simulations. However, the Kozai mechanism has the potential to be the dominant process in the interactions of stellar-mass black holes in globulars, if most such black holes are in binaries. When only three-body interactions are considered, very few black holes are retained by the clusters (only 8% in the simulations of Portegies Zwart & McMillan 2000). This occurs because the same processes that harden a binary toward an eventual merger also impart velocity kicks on the binary that ultimately eject it from the globular before it can merge. In contrast, the majority of black holes can be retained if binary-binary interactions dominate.

For example, suppose that a third of those interactions produce stable triples. Subsequent interactions of the tertiary with field stars will change its eccentricity and semimajor axis. If the pericenter distance of the tertiary is less than a few times a_1, then the triple system becomes unstable, normally by ejecting its least massive member. Suppose that there are typically $\sim$2 encounters before the triple is disrupted in this way, and that each encounter of the tertiary that does not create an unstable triple produces a new relative inclination I that is drawn from a uniform distribution in $\cos I$. Suppose also that every time the inner binary interacts strongly its semimajor axis is decreased by ~ 20% (typical for strong interactions of three equal-mass objects; see, e.g., Heggie 1975; Sigurdsson & Phinney 1993). Then, in an $n = 10^6$ pc $^{-3}$ cluster there is a ≈ 50% chance that the inner binary will merge before it hardens to $a_1 \approx 0.2$ AU, at which point the binary recoil velocity $v_{\rm recoil}$ exceeds the $\sim$50 km s^{-1} escape speed typical of the cores of globulars (Webbink 1985). In an $n = 10^5$ pc $^{-3}$ cluster, encounters are less frequent and the fraction rises to ≈ 70%.

Thus, depending on the binary fraction and other properties of black holes in globulars, the majority of black holes could merge before being ejected, and growth of intermediate-mass black holes in globulars may proceed naturally even if no black hole is formed with $M > 10\,M_\odot$. This could influence stellar dynamics in the core, and the gravitational wave signals from globulars, and should be included in future simulations.

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