ORBITAL DYNAMICS AND THE STRUCTURE OF FAINT DUSTY RINGS

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Douglas Peary Hamilton
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Douglas Peary Hamilton, Ph.D.
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The orbital perturbations that act on objects circling a planet vary in strength depending on the sizes of both the particle and its orbit. We examine three cases that are difficult to treat with the standard tools of celestial mechanics: *i*) large distant satellites, *ii*) small objects on distant orbits, and *iii*) tiny particles orbiting near a planet.

The dominant perturbation in the first case is the tidal component of solar gravity. Taking as our example an asteroid on a circular orbit about the Sun, we numerically determine the size and three-dimensional shape of the surface beyond which circum-asteroidal debris is unlikely to be present. We present scaling laws that allow this result to be applied to objects with different masses, semimajor axes, and eccentricities. Small objects on distant orbits are highly perturbed by radiation pressure, which rapidly causes many of them to escape or to impact the asteroidal surface. We determine that, for the asteroid Gaspra (radius $\approx 10\,\mathrm{km}$), debris smaller than centimeter-sized will disappear from distant orbits in just a few years. We generalize our results for application to arbitrary asteroids.

Micron-sized grains, the principal constituents of the many diffuse rings circling within a few planetary radii of the giant planets, are dominantly perturbed by electromagnetic and radiation forces. We derive orbit-averaged equations that govern the evolution of such grains subject to these perturbations; our expressions are valid at all non-resonant locations. Resonant locations are treated by expanding the electromagnetic perturbation analogously to the derivation of the disturbing function of celestial mechanics. We compare our electromagnetic expansion to previous gravitational expansions; similarities lead to the discovery of simple orbital symmetries that constrain the possible consequences of any perturbation.

We use the above expressions to explore the dynamics of the micron-sized grains that make up Saturn's E ring and find that a coupling between planetary oblateness, electromagnetism, and radiation pressure generates highly-eccentric orbits. The distribution of material along an ensemble of these elliptical orbits agrees well with the E ring's observed radial and vertical structure. As a consequence of their highly-elliptical orbits, dust grains strike embedded satellites and nearby rings at large velocities. We argue that these energetic collisions sustain the E ring at its current optical depth against the erosive effects of grain-grain collisions.

Biographical Sketch

Douglas Peary Hamilton was born on January 1, 1966 in Seattle, Washington, although shortly thereafter his family moved to Fairbanks, Alaska. One of his earliest memories is watching the first manned moon landing on July 20, 1969 on a battered old black and white TV. Another strong memory is of waking up in the middle of the night to see a particularly spectacular Aurora Borealis. "What makes the aurora?" was added to the other great unknown questions of the time: "Why do jets leave white trails in the sky?" and "Why do my parents sleep in longer on my birthday than on any other day of the year?"

Doug's family moved to California in 1973 or so and then back to Alaska in 1981, but not before he picked up an avid love of bicycling and a desire to ride "across" things, like the Golden Gate Bridge and the Santa Cruz Mountains. This interest culminated in a six-week, 4200-mile, bicycle ride across the United States in 1982.

Back in Alaska, he discovered one of the advantages of living in a state with more moose than people by posting the top score in a state-wide high school math contest for three consecutive years. This and other successes convinced him to major in mathematics and, after graduating from high school in 1984, he chose to attend Stanford, one of the great southern universities.

While taking the requisite mathematics classes, Doug also found time to explore the sciences, consequently delaying his declaration of the math major. In the autumn quarter of his junior year, the university sent him a threatening letter demanding that he declare a major within the month and, in a moment of panic, he chose physics instead of math.

But the decision was a good one, and he satisfied all requirements, receiving a B.S. in Physics with Distinction and Honors in 1988. Astronomy also figured significantly in Doug's Stanford experience; he taught an observational astronomy course for three of his four college years and submitted an undergraduate honors project entitled "Theoretical optical line intensities from intracluster plasmas."

Accepting a Sage fellowship to attend another great southern university – Cornell – Doug flung himself in new directions, sampling a number of diverse research fields including geophysics, magnetospheric physics, ion beams, and chemistry. He dropped all of these in favor of orbital mechanics in the summer after his second year, well before receiving a threatening note from the university.

While at Cornell, Doug published several single- and multiple-author research articles in *Icarus*, a journal edited by his advisor, Joe Burns. A few of his more recent papers have found their way into *Nature* and (fingers crossed) into *Science*. He cheerfully admits to pirating material from several of these works to form this thesis. Doug has given invited colloquia at the University of Iowa in Iowa City and at Queen Mary College in London. He has observed the planets, their satellites, and their rings from the 200-inch telescope atop Mt. Palomar.

Upon completion of his Ph.D., Doug accepted a NSF-NATO fellowship for study at the Max Planck Institut für Kernphysik in Heidelberg, Germany. In Heidelberg, he plans to work with Eberhard Grün analyzing data from the Galileo and Ulysses spacecraft while ordering beers in increasingly fluent German.

To my parents, Lisa Wichmann Hamilton and Thomas Dudley Hamilton.

cknowledgments

Facing the blank pages that are to become this acknowledgement section, I worry about forgetting to mention someone important, someone who later reads these pages and thinks “hmmpph, didn't mention me.” What a faux pas it would be to neglect to thank ones advisor, or parents, or the numerous colleagues and friends who have made life in Ithaca just a little bit easier! Yet five years is a long time, and memories are too often imperfect. Therefore, I'd like to apologize in advance for the omissions that I am bound to make – don't take it personally; let's talk about it over a beer.

First and foremost, I'd like to acknowledge my thesis advisor, Joe Burns. Besides teaching me the things that all students need to know (where rings come from, how satellites move, how often asteroids knock together etc.), Joe has given me invaluable advice in many areas including scientific writing, NASA politics, public speaking, scientific ethics, and, perhaps most importantly, career opportunities for a newly-graduated planetary scientist. Over the years, I've gained a great admiration for Joe's clear and concise writing style, elements of which I still strive to emulate. The touch of Joe's accomplished editorial pen, which improved virtually every page of this thesis, is a blessing to anyone toying with the idea of reading beyond these few pages.

I have spent many pleasurable hours in the company of Phil Nicholson, a meticulous scientist whose careful attention to detail truly sets him apart. To Phil, I owe my introduction to observational astronomy: two unforgettable observing runs at the 200 inch telescope atop Mt. Palomar. Although the subsequent data analysis and interpretation that Phil and I have worked on together does not appear in this thesis, it has been invaluable in broadening my experience as a graduate student. Besides being an excellent observer, Phil speaks knowledgeably on an incredibly wide range of topics; it would be impossible to list all of the ideas and insights that I have gleaned from him over the course of innumerable conversations.

Besides Joe and Phil, I wish to thank two additional scientists, Dick Lovelace and Dave Hammer, who rounded out my special committee. Dave Hammer served as my minor advisor, representing the interests of the Electrical Engineering Department; I also had the good fortune to have taken a very lucidly taught course in plasma physics from him. Dick Lovelace graciously agreed to be the chairman of my committee and serve as the link between my original department (Applied and Engineering Physics), and my adopted department (Astronomy and Space Sciences). Each of these committee members was faced with the daunting task of reading this thesis in its entirety; their insightful comments improved the quality of the final product.

I have benefited immensely from following in the footsteps of some of Joe's distinguished former students. Bob Kolvoord, who was just finishing his degree as I began mine, had good advice to share on how to survive the rigors of graduate student life and Les Schaffer's periodic returns to Ithaca were, and still are, punctuated with his unquenchable enthusiasm. Although Mark Showalter left Ithaca long before I arrived, his monumental works modeling the faint rings of Jupiter and Saturn were secure platforms from which to launch my own investigations of orbital dynamics. I am also grateful to Mihaly Horanyi, a colleague now working in Boulder, Colorado. When I became interested in numerically following the orbital motion of particles in faint dusty rings, Mihaly selflessly assisted me by sharing a similar program that he had already written.

The smiling, sunny disposition of Cheryl Hall, who assists Joe in editorial matters, brightened many a gloomy Ithaca day. I spent enjoyable lunch hours in the company of Cheryl and her sister Tami, and from them I learned the true meaning of "proposal season." Additional thanks are due to Barbara Boetcher who hand-drafted several of the figures for this thesis and to the staff of the first floor – Su Clark, "Wild" Bill Hoffman, Mina Gentile, Sandy Mack, and Elisabeth Bilson – who assisted in the day-to-day affairs that we all too often take for granted: timely paychecks, mail and telephone services, large xeroxing jobs, etc. The graduate students made life easier in many ways, in particular, by making the office a friendly and relaxed place in which to work.

I am also indebted to the people at the Alaska Student Loan Office, to whom I owe more than I can ever repay.

Several "off-duty" activities helped to make the past five years in Ithaca especially enjoyable. I have been fortunate enough to find many hiking and camping companions, and go exploring with some regularity. I visited the Grand Canyon with Peter Powers, climbed in the San Francisco peaks in Arizona with Pascal Lee, in the high peaks region of the Adirondacks with Peter and Mike Opie, in the White Mts. of New Hampshire with Vicky Chang, Tom Megeath, and Laura Jalso, and up California's Mt. Whitney with Peter, Ignacio Mosqueira, Tom Schossau, Jeff Wagoner, and Claudia Lopez. From Jeff, I learned rudimentary ice climbing skills which we put to the test on Wyoming's Grand Teton and Montana's Granite Peak. We tested our newly acquired snowboarding skills high on the flanks of Utah's King mountain, and camped on the desolate plains of Froze-to-Death Plateau.

Summer in Ithaca is a glorious time – at least compared to winter – and exploring Tompkins County and the surrounding area by bicycle proved to be par-

ticularly enjoyable. Peter Powers, Marian Silberstein, and many others helped to make such rides especially memorable. Besides granola, water is the one thing found in Ithaca in abundance, and I've enjoyed the excellent sailing, windsurfing, and kayaking that the area has to offer. Vicky Chang was particularly enthusiastic, teaching me to sail catamarans on Cayuga lake and leading a unique kayaking trip through the flooded TOPS parking lot. In winter, a motley crew of downhill skiers gathered once a week for a $7.50 night-skiing fest at Greek Peak; Shuichi Kawasaki, Vicky, Marian, and Shelley Drazen, may your quest for powder bring happy returns.

Cool Ithaca evenings were spiced up by the bimonthly meetings of the Jalepeño Pizza Club. Convening at the Chariot, club members savored the bite of the sacred pepper. Each of the club's distinguished members, Eric Avera, Elaine Bukherovich, Paul Kimoto, Tom Megeath, Jayaram Chengalur, Chris Koresko, and Matt Ashby, lived up to the clubs hallowed motto: "I've had worse." Noble and inspired poetry was also written:

Roiling smoke on the horizon,
The spoor of antlered beasts,
Unmistakable autumn signs,
That herald Pepper feasts.

Avera (private communication 1992). Ignacio, Brett Gladman, Jeff Moersch and many Jalapeño club members also accompanied me on semi-regular runs up to Coyote Loco for those tasty margaritas.

For the past three years, I've enjoyed playing badminton with the Taiwanese Badminton Club. Liang-Yuh Chen, Chen-Ho Chien, Tan-Chen Lee, Chih-Wen Wu, Cheng-Pei Lei, To-Chia Wu, Li-Min Sun, Mingchein Kuo, Chien-Hui Ma, Dingyah Yang, Jeng Fan and many many others taught me a little of Taiwan and helped to improve my game. Shay Shay, my friends.

Several individual people helped keep life in Ithaca interesting and exciting. Brian Jakubowski, one of the wild-and-craziest dudes on the planet, accompanied me on two glorious European adventures and together we donned scuba gear and descended into the frigid and murky depths of lake Cayuga. Brian, I'll never forget our arrival in Glasgow early one December morning – Here we are now, entertain us! Vicky and I shared many memorable adventures in the Ithaca area and during difficult thesis times, she stepped in with emergency lasagnas and those amazing chocolate-chocolate brownies. Niels Otani and Lauren Kiefer's no-nonsense approach to bridge made evenings of card-playing especially enjoyable.

The point has often been made, at least in learned circles, that a group of four or more men living together constitutes a fraternity (Megeath, Ph.D. thesis; Chengalur, Ph.D. thesis). Accordingly, I'd like to thank the many "brothers" of 305 Wyckoff, past and present, for the good times that we've had over the years: Peter Powers, Tom Megeath, Alex and Hanne Schwartz, Brian Jakubowski, Shuichi Kawasaki, Jayaram Chengalur, Matt Ashby, John Miles, Ignacio Mosqueira,

Lauren Doyle, and Jeff Moersch.

Finally, and most importantly, I appreciate the support of my family which was not diminished by the vast distance between Ithaca and everywhere else.

Although I leave Ithaca and friends behind, I bring with me many happy memories and take solace in the fact that even after my departure, I can still be found at the Chapter house.

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Chapter 1

Introduction

1.1 Modern Celestial Mechanics

The cyclic phases of the moon, the timing and duration of lunar and solar eclipses, and the motions of the Sun, Moon, and planets are problems that have captivated the imaginations of all who have witnessed such celestial events; accordingly predictions thereof have challenged the minds of many of the world's greatest thinkers. From the late sixteenth century, when Copernicus' heliocentric view of the solar system first vied with Ptolemy's idea of a geocentric universe, through the ensuing years during which Kepler, Galileo, Newton and Einstein made their pivotal contributions, and up to the present era of spacecraft reconnaissance, our understanding of celestial mechanics has steadily improved. Over the past few decades, celestial mechanics has undergone a transformation from a largely theoretical pursuit into a practical discipline. This occurred as planetary and satellite fly-bys, each requiring accurate descriptions of spacecraft trajectories and planetary positions, became commonplace. The spectacular images relayed back to Earth from the Voyager spacecraft, for example, would not have been possible without the detailed trajectory information necessary for planet and satellite rendezvouses, and precision camera-pointing. An additional reason to understand orbital motion is to minimize the danger of debilitating collisions with tiny, unseen, and rapidly-moving bits of space debris. The thrust of this thesis – orbital dynamics and the structure of faint dusty rings – is strongly motivated by such concerns for spacecraft safety.

1.2 Why Study Dust?

Dust is ubiquitous throughout the solar system, being found in orbit around Earth (McDonnell *et al.* 1992), Mars (Dubinin *et al.* 1990), and the giant planets (Burns *et al.* 1984, Smith *et al.* 1989, Esposito *et al.* 1991), jettisoned from comets to form elegant tails (Grün and Jessberger 1990) and from the jovian system in apparently periodic streams (Grün *et al.* 1993), and strewn throughout the inner solar system's zodiacal cloud, where it concentrates in bands near the most prominent asteroid families (Dermott *et al.* 1985) and perhaps at certain resonant locations (Jackson and Zook 1989). Because small particles are especially sensitive to non-gravitational forces, they can be driven to unusual places. For example, micron-sized particles make up the wedge-shaped diffuse E ring of Saturn (Showalter *et al.* 1991), while the complex and beautifully intricate spokes of Saturn's B ring are hypothesized to arise from tiny grains electrostatically levitated off larger ring members (Goertz and Morfill 1983, Grün *et al.* 1983, Tagger *et al.* 1991). Resonant electromagnetic forces acting on small charged dust particles may provide the explanation for the abrupt transition between Jupiter's faint ring and its vertically-extended ethereal halo (Burns *et al.* 1985). During Voyager's Neptune fly-by, the spacecraft's plasma wave and planetary radio astronomy instruments discovered a tenuous cloud of dust in yet another unlikely locale, over Neptune's northern polar region (Gurnett *et al.* 1991, Warwick *et al.* 1989).

The facts that small particles are both difficult to detect, but also present in vast quantities throughout the solar system, greatly enhance the potential for a catastrophic spacecraft-projectile encounter. The most dangerous locations are in the vicinity of larger parent objects: amid the ring systems and satellite retinues of the giant planets, inside cometary halos and tails and, increasingly, within a few planetary radii of Earth as man-made orbital debris – paint chips, fuel droplets, pieces of hardware, old spacecraft, and the collisional products thereof – accumulates (Kessler 1985). While the last locale is undoubtably the most threatening to Earth-orbiting satellites, Shuttle missions, and the proposed Space Station Freedom, the first menaces the orbiters Galileo and Cassini, which will spend long periods of time in the environs of Jupiter and Saturn, respectively. Because of large relative velocities, and hence energetic collisions, objects only a centimeter across can annihilate an entire spacecraft while millimeter-sized particles are capable of inflicting considerable damage, perhaps destroying individual instruments. The latter fact was dramatically underscored by the crippling of the European Giotto mission during its traverse of comet Halley's halo in 1986. Somewhat smaller particles, in the submicron to tenths of millimeters range, can scour optical surfaces, interfere with electrical systems, and, over time, degrade various sensitive components of a spacecraft. Because of the great expense of planetary missions, the prevalence of orbital debris, and the distinct threat that such debris represents, considerable planning has been done to insure safe orbital tours for both Galileo and Cassini (see Section 1.4).

Despite the fact that orbiting debris often stars as the black-robed villain of celestial mechanics, ever plotting to intercept unwary space-faring vessels, its other role – that of an instructor – should not be forgotten for much can be learned from studying the distribution and orbital motion of these tiny motes. Paint chips and antiquated satellites tell us little of recent terrestrial history that we do not already know, but samples of cometary and asteroidal particles collected in space (*e.g.,* from the Long Duration Exposure Facility, McDonnell *et al.* 1992) and on the wings of specially outfitted airplanes (Brownlee 1985) hint at the origin of "shooting stars" occasionally seen flashing across the night sky. Rapidly-moving dust particles recently detected near Jupiter by the Ulysses spacecraft seem to be interstellar in origin, bearing clues to events that occurred far beyond the limits of our robotic exploration (Grün *et al.* 1993). The organization of dust into faint ethereal rings highlights various dynamical processes which can be used to better understand more complex rings dominated by large closely-packed members, and additional dust within these dense rings provides further tracers of ongoing processes. Thus the smallest particles carry information pertinent to some of the most profound questions of celestial mechanics - How do rings form? What processes govern their structure? And, ultimately, how did the ring-like primordial planetary nebula originate, condense, and evolve into the solar system we know today?

To address these questions, an intimate understanding of the relevant forces acting on dust grains and the consequent orbital evolution that they induce is essential. Accordingly, a major goal of this thesis is to develop a set of tools capable of describing orbital motions and then to apply these tools in simple models of existing phenomena. The knowledge gleaned from such an exercise is of both practical and philosophical use: practical since, by understanding the motion of these particles, we can minimize the threat to our spacecraft, and philosophical in that once we better understand whence these tiny messagers originate, perhaps we will be able to better decipher the information that they carry.

1.3 lassical elestial ec anics

Much of the substance of this thesis involves the application of perturbation theories to determine the evolution of orbiting particles imposed by particular perturbing accelerations. These theories require accurate descriptions of the accelerations as well as nearly-correct baseline solutions from which to compute deviations; such schemes were first employed during the eighteenth century development of the disturbing function of celestial mechanics. Since many of our results rely heavily on orbital perturbation theories and, in the case of Chapter 7, closely parallel the derivation of the disturbing function, we briefly summarize relevant results and place them in historical context.

The first accurate description of planetary orbital motions was found empirically nearly four centuries ago by Johannes Kepler, who made extensive use of Tycho Brahe's meticulous naked-eye observations of Mars. Because of the high ellipticity of Mars' path around the Sun, Kepler was forced to discard the notion of perfectly circular orbits and instead formulated the following three laws of planetary motion:

1. Planets move along elliptical orbits with the Sun at one focus.
2. The radius vector to the planet traces out equal areas in equal time.
3. The square of a planet's orbital period is proportional to the cube of the semimajor axis of its elliptical orbit about the Sun.

Later, Sir Isaac Newton showed that these rules followed naturally from the mutual gravitational attractions of two spherical bodies. But although elliptical motion is an exact solution to the two-body problem, it only approximates the actual motion of a planet around the Sun or that of a satellite about a planet. Deviations from purely elliptical motion occur because of the gravitational attractions of additional objects, the non-spherical shapes of these bodies, and even the minuscule corrections of Einstein's general theory of relativity.

In the usual case, perturbations are dominated by the direct gravitational attraction of the primaries and the orbits are nearly elliptical; thus Keplerian motion can be used as a baseline solution and the actual path followed can be determined from perturbative theories. Much of the early work in celestial mechanics focused on efforts to describe and approximate the gravitational effects of one planet on another. High-order expansions of the disturbing function in terms of the elliptical elements of planetary orbits were first worked out by Peirce (1849) and Le Verrier (1855) (*cf.* Brouwer and Clemence 1961): today's version uses computer algebra to derive extremely accurate and computationally extensive expansions (Murray and Harper 1993). In another problem of interest, the perturbations arising from the gravitational attraction of an arbitrarily-shaped planet can also be modeled by a disturbing function, and this allows the motions of close planetary satellites to be predicted very accurately (Kaula 1966). With these techniques, one can, in principle, understand the motions of most planets, comets, asteroids, and natural and artificial satellites found in our solar system.

1.4 Brief Summary of apters

This thesis addresses two types of problems that fall outside the scope of the above-mentioned classical tools of celestial mechanics: *i*) those where large perturbing accelerations make expansions inappropriate, and *ii*) those involving motions of micron-sized dust particles that are strongly influenced by non-gravitational accelerations.

The first subject is discussed in Chapters 2–4, which investigate the region where material may stably orbit an asteroid. This study was motivated by concerns for the safe passage of Galileo, which made historic first fly-bys of the asteroids 951 Gaspra (October 29, 1991) and 243 Ida (August 28, 1993); the results of our study were used by the Galileo team to decide how close, and from what direction, to approach these primordial objects. For distant circum-asteroidal orbits, the solar tidal force's pull on a particle is nearly as strong as the asteroid's gravitational grip; hence the Sun and the asteroid vie for domination of drifting debris. Numerical investigations are needed to follow the orbital evolution of such particles although some analytical constraints do exist. In Chapter 2, we consider an asteroid on a circular orbit around the Sun, in Chapter 3 we extend this analysis to arbitrarily elliptical heliocentric orbits, and in Chapter 4 we add the effects of solar radiation pressure which, due to the asteroid's weak gravitational field, is a relatively strong perturbation for potentially destructive particles in the millimeter and centimeter size range. Because objects smaller than these are rapidly driven from circum-asteroidal orbits by radiation forces, the near-asteroidal environment is predicted to be relatively free of orbiting debris and hence benign to passing spacecraft. Galileo's unscathed fly-by of both Gaspra and Ida, and the negative results of its onboard dust detector substantiate these claims (Grün *et al.* 1992, E. Grün 1993, private communication). Scaling relations are derived that allow the results of Chapters 2–4 to be applied to asteroids of different masses, eccentricities, and distances from the Sun.

Chapters 5–7 focus on the dynamics of micron-sized particles in circumplanetary orbits, an interesting and challenging problem because of the unusual array of physical processes that influence the motions of these tiny motes. The strongest perturbations are radiation forces, which arise from the transfer of momentum due to the absorption and re-emission of solar photons, and electromagnetic forces which occur in the spinning magnetospheres of the giant planets. Because of many uncertainties, particularly in the nature of the plasma surrounding the giant planets, researchers have usually sought to isolate and model a single perturbation in order to understand its influence on an orbit. For example, Burns *et al.* (1979) Mignard (1982, 1984), and Mignard and Hénon (1984) analytically describe the influence of radiation pressure and other effects associated with the transfer of momentum from solar photons and the solar wind. The equilibrium electrical potential of an isolated dust grain immersed in a plasma has been studied by Whipple (1981), Meyer-Vernet (1982), Whipple *et al.* (1985) and others. Various resonances associated with electromagnetic forces have been identified, among them "Lorentz resonances" with spatially-periodic magnetic fields (Burns *et al.* 1985, Schaffer and Burns 1987), "shadow resonances" (Horanyi and Burns 1991), and "resonant charge variations" (Burns and Schaffer 1989, Northrop *et al.* 1989). The dynamics of grains moving through the convected solar wind field about a planet have been addressed by Horanyi *et al.* (1990, 1991). Despite the important role that small particles may play in various features of the solar system, a comprehensive treatment of the orbital histories of circumplanetary dust is not yet available (*cf.* Schaffer 1989).

A first attempt to comprehensively and simultaneously treat the largest perturbative accelerations acting on circumplanetary micron-sized dust, electromagnetism and radiation pressure, is presented in Chapter 5. Although these non-gravitational accelerations are large, the planet's attraction usually dominates and perturbative schemes are appropriate. We employ the method of orbit-averaging which has the advantage of suppressing all but the secular terms (those that are independent of orbital longitudes); in most cases, the secular terms constitute the perturbation's dominant long-term effects. The resulting set of equations determine the orbital evolution of small grains throughout the inner magnetosphere excepting at certain resonant locations.

In Chapter 6, we use our knowledge of the orbital motions of micron-sized dust to understand the peculiar three-dimensional structure of the saturnian E ring. We find that E-ring grains orbit Saturn in unusually elliptic orbits which imply a previously unsuspected method for the generation and sustenance of faint rings. We argue that collisions of E-ring particles with the satellites immersed in the ring are sufficiently energetic to generate new ring material and that this process sustains the ring. Besides being consistent with the main properties of the E ring – the radial location of its peak brightness, the numerical value of that brightness, the ring's radial extent and its vertical structure – we find that our model agrees with a number of independent observations – the coloration and surface properties of the embedded satellites, the presence of large amounts of OH in the inner magnetosphere, and the high dust content of neighboring rings. Our results may be useful in planning for, and can be tested by, both the 1995-6 edge-on appearance of Saturn's rings and the Cassini mission to Saturn, which will make multiple passes through the E ring.

Motivated by the strong evidence for electromagnetic resonances causing the transition between the main jovian ring and its inner halo, we return, in Chapter 7, to systematically expand the electromagnetic perturbation at resonance locations in a manner similar to that employed in the derivation of the disturbing function of celestial mechanics. Besides providing a methodology for treating the motions of dust everywhere in the inner magnetosphere, we investigate similarities and differences in the properties displayed by electromagnetic and gravitational resonances. We separate these properties into three groups: *i*) those shared by all orbital perturbations, *ii*) those that are common just to mean-motion resonances, and *iii*) those that are unique to individual resonances. Properties in group *i*) are shown to follow from simple physical symmetries which apply not only to the perturbations considered here, but to all quantities that are expressed in terms of orbital elements, while those in group *ii*) arise from shared integrals of the motion. As is often the case in research, study of a new phenomenon (here Lorentz resonances) gives unexpected insights into a well-researched related area (gravitational resonances).

Finally, in Chapter 8 we conclude by discussing directions for further study. A particularly promising line of research that we are currently pursuing is a re-examination of the dynamics in the jovian ring system. Many of the ideas discussed in Chapters 5–7 seem to be simultaneously at work in these diffuse rings, and the upcoming impact of comet Shoemaker-Levy 9 into Jupiter in July 1994 makes our study especially timely. The cometary impact may cause the main and gossamer rings to brighten, perhaps yielding clues to processes relevant to their formation. The more distant distributions of dust in the uranian and neptunian magnetospheres are also intriguing and can be studied with the methods of Chapters 5–7. As each of these new areas are investigated, the need for improved theories will undoubtably arise, thereby driving the understanding of the orbital motions of circumplanetary dust yet another step forward.

r 2

Or i l ili y Z s s r i s i Z r E ri i y[1]

2.1 Intro uction

While two questions – "How much material is likely to be in orbit around an asteroid?" and "Exactly where will that material be?" – are interesting to planetary scientists and celestial mechanicians, they are critically important to those spacecraft mission planners who must decide how closely to approach such objects. It is well known that, in the absence of perturbations, orbiting particles can move on Keplerian paths at all distances from an isolated asteroid. In reality, however, gravitational perturbations from the Sun and, to a lesser extent, the planets will limit the zone in which particles can stably orbit.

Since the problem of N gravitationally attracting bodies is well known to be analytically unsolvable for $N > 2$, numerical methods must be employed to obtain quantitative estimates of the motion of a test particle in the vicinity of an asteroid that itself circles the Sun. We neglect planetary perturbations since these are at least a thousand times weaker than solar effects (*cf.* Chauvineau and Mignard 1990b) and treat a three-body problem consisting of the Sun, an asteroid, and an orbiting particle. The three-body problem has been numerically integrated many times previously (consult Szebehely 1967 for historical references while for more recent work see Zhang and Innanen 1988, Murison 1989b, Chauvineau and Mignard 1990a,b) but the space of possible parameters is so large that the three-body problem's complete solution is, fundamentally, not understood. Fortunately the problem that we wish to solve is more restricted, although still analytically intractable.

We treat the case of hierarchical masses since the asteroid's mass is insignificant relative to the solar mass, yet is very large in comparison to particles likely

[1] This chapter is based on the paper: Hamilton, D.P., and J.A. Burns (1991), Orbital stability zones about asteroids *Icarus* **92**, 118–131 [copyright 1991 by Academic Press, Inc.].

Table 2.1 Parameters of Amphitrite and Gaspra

Object	A (AU)	E	R_A (km)	μ	ρ (g/cm^3)	r_H (R_A)
Amphitrite	2.55	0.00	100	5×10^{-12}	2.38	452
Gaspra	2.20	0.17	10	5×10^{-15}	2.38	390

to be orbiting it. Hierarchical masses provide a limiting case of both Hill's problem and the restricted three-body problem (Hénon and Petit 1986). We further narrow the space of parameters by giving the asteroid a circular orbit around the Sun, by choosing to study only those orbits that are weakly bound to the asteroid, and by starting test particles out on initially circular orbits. The second choice is made in order to explore the transition region between bound and unbound orbits and hence to delineate the zone in which the material could be stably trapped.

In the numerical examples to follow, we model the asteroid 29 Amphitrite, a previously planned target of Galileo (see also Zhang and Innanen 1988), as having a circular orbit of radius $A = 2.55\,\text{AU}$, and an asteroid/Sun mass ratio $\mu = 5.0 \times 10^{-12}$. In reality, Amphitrite's orbit is moderately eccentric ($E = 0.07$). For an assumed asteroid radius of $R_A = 100$ km, the chosen μ corresponds to a reasonable density of $\rho = 2.38$ g/cm^3. These parameters, as well as ones appropriate for Gaspra, Galileo's actual target, are listed in Table 2.1. The final column in the table lists the radius of the Hill sphere which we define at the end of the next section. Our investigation confirms and extends the study of Zhang and Innanen (1988) by using heuristic models to understand the nature of the observed orbits, by considering motion out of the orbital plane, by illustrating the shape of the volume filled by particles on stable orbits, by showing how results can be scaled to other asteroids, and by placing the problem in the context of modern ideas on chaos (*cf.* Chauvineau and Mignard 1990a,b; Murison 1989b).

2.2 Equation of otion

We use two *non-inertial* coordinate systems (Fig. 2.1), each with its origin on the asteroid which itself orbits the Sun: *non-rotating* coordinates that keep their axes fixed with respect to the distant stars, and *rotating* coordinates that maintain their axes fixed relative to the Sun. In each, the asteroid's orbit lies in the xy plane. Because the orbits we consider are only weakly bound to the asteroid, solar perturbation forces are relatively large and, accordingly, most paths are more easily understood when viewed in a reference frame rotating with the asteroid's mean motion $\Omega\hat{\mathbf{z}}$ around the Sun ($(xyz)_{rot}$ in Fig. 2.1). The mean motion is

Figure 2.1 Two non-inertial coordinate systems are shown as they follow the asteroid on its circular orbit of radius A about the Sun. The xyz system stays fixed in its angular orientation while the $(xyz)_{rot}$ system rotates uniformly so that the Sun always is at $x_{rot} = -A$. In the non-rotating system the Sun is initially at $x = -A$ and it moves with angular speed Ω around the asteroid in the plane $z = 0$. In most integrations the particle starts along the Sun-asteroid line at $(x = d, y = 0, z = 0)$ with a velocity in the non-rotating frame that would put it on a circular orbit if the Sun were not present.

a vector that points normal to the orbit ($\hat{\mathbf{z}}$ is the unit vector in the positive z direction) and has magnitude

$$\Omega = \sqrt{\frac{GM_\odot}{A^3}}, \tag{2.1}$$

where G is the gravitational constant, A is the Sun-asteroid distance, and $M_\odot$ is the mass of the Sun. The acceleration of a particle orbiting the asteroid is then approximately given by Hill's equation (Szebehely 1967):

$$\frac{d^2\mathbf{r}}{dt^2} = -\frac{GM_A}{r^2}\hat{\mathbf{r}} + \frac{GM_\odot}{A^3}(3\mathbf{x}_{rot} - \mathbf{z}) - 2\mathbf{\Omega} \times \mathbf{v}_{rot}, \tag{2.2}$$

where $\mathbf{r}$ is the vector pointing from the asteroid to the particle, $\hat{\mathbf{r}}$ is the corresponding unit vector, $\mathbf{v}_{rot}$ is the particle's velocity measured in the rotating frame, and M_A is the mass of the asteroid. The terms on the right side of Eq. (2.2) are due to the asteroid's direct gravity, the combination of solar tidal and centripetal effects, and the Coriolis effect, respectively. Henceforth, the full second term will be referred to as the "tidal" term. In the derivation of Eq. (2.2), we have neglected quantities that are second order in r/A. These terms, if included, would break the symmetry of the tidal term around the $(yz)_{rot}$ plane. We have observed consequences of this broken symmetry in a few numerically integrated escape orbits, but do not judge it to be significant in estimating the trapped region or in describing most orbits.

In order for the reader to gain insight into the trajectories to be shown later, we now discuss some of the properties of the accelerations in Eq. (2.2). In these descriptions we will call an orbit *prograde* if the particle's angular velocity around the asteroid is in the same sense as the asteroid's angular velocity around the Sun; for a *retrograde* orbit, the particle's angular velocity is in the opposite sense. Figure 2.2 is a sketch showing how the direction and magnitude of the various accelerations change along a hypothetical orbit that is coplanar and oval-shaped in the rotating frame. Notice that the accelerations all act in different directions: the direct term always points toward the asteroid, the tidal term invariably aligns parallel or antiparallel to the solar direction, and the Coriolis term is always perpendicular to the orbit. Furthermore, the direct acceleration is inward and thus acts to bind particles to the asteroid, while the tidal acceleration, which has a component that points outward, acts to expel them from the system. Because the Coriolis acceleration depends on the sign of the velocity, it points outward for prograde orbits but inward for retrograde ones; thus the Coriolis acceleration tends to stabilize the latter but disrupt the former.

Finally note that along the orbit the tidal acceleration increases with growing separation distance, while all other accelerations decrease. Comparing the directions of the accelerations in the prograde and retrograde cases, we can already see that retrograde orbits should be stable out to greater distances than prograde

Figure 2.2 Sketches of the accelerations (magnitudes and directions) that are experienced by a particle at various places along a coplanar oval orbit whose long axis is aligned with the solar direction; the asteroid is at the origin. The direct acceleration is caused by the asteroid's gravitational attraction of the particle. The "tidal" acceleration is due to the local imbalance between the Sun's attraction and that needed to cause the asteroid's circular path (see Eq. 2.2). The sign of the Coriolis acceleration depends on whether the particle moves in the same (prograde) or opposite (retrograde) angular sense as the asteroid in its orbit about the Sun.

ones since in the former situation the Coriolis acceleration is inward while in the latter it is outward. Numerical experiments support this statement as does nature's laboratory: the outermost moons of Jupiter and Saturn are on retrograde orbits.

In order to quantify the radial dependences of the accelerations, they are plotted in Fig. 2.3 as functions of distance from the asteroid for the special case of a circular coplanar orbit. Recall that we've used Amphitrite as our model asteroid (Table 2.1). To adjust the axes of this and all of the following plots to your favorite asteroid, simply multiply distances measured in R_A by the factor $(\rho/2.38\,\mathrm{g\,cm^{-3}})^{1/3}(A/2.55\,\mathrm{AU})$, where ρ is the new asteroid's density and A is its semimajor axis. The justification for this scaling will be presented in Section 2.3.3; we also note here that differences in asteroid orbital eccentricities cannot be accommodated although we will have more to say about this in Chapter 3.

All of the curves plotted in Fig. 2.3 are normalized by the local direct acceleration of the asteroid's gravity. Since the strength of the tidal acceleration depends on azimuthal position (see Fig. 2.2), it varies along even a circular orbit and thus here we plot its maximum value. The total accelerations for prograde [curve P] and retrograde [curve R] orbits, as plotted in Fig. 2.3, were obtained by taking the various terms and simply adding them; even though this addition ignores the vector character of these accelerations, we believe that it is instructive.

In the limit of small separations (*i.e.,* on the left side of Fig. 2.3), the perturbation accelerations [curves C(P), C(R), and T] tend to zero, and thus both prograde and retrograde orbits approach the two-body solutions: circles and ellipses about the asteroid. Accordingly, the curves of Fig. 2.3 are most applicable in this inner region, since only there do circular orbits actually exist. Nevertheless, the curves provide useful guides for estimating magnitudes in more complicated situations. Of course, care must be exercised in their application, especially when estimating the magnitude of the Coriolis acceleration which, due to its velocity dependence, will vary substantially with the actual path taken.

Both the restricted three-body problem and Hill's problem admit an integral of the motion that can be derived by integrating, over time, the scalar product of Eq. (2.2) with the velocity $\mathbf{v}_{rot}$. Chauvineau and Mignard's (1990a) expression for this Jacobi integral can be generalized to three dimensions as

$$v_{rot}^2 - \Omega^2(3x^2 - z^2) - \frac{2GM_A}{r} = -C, \tag{2.3}$$

where C is the Jacobi constant and $r = (x^2 + y^2 + z^2)^{1/2}$. The three terms on the right of Eq. (2.3) are the kinetic, "tidal," and direct terms respectively. This Jacobi constant is conserved in a rotating frame centered on the asteroid and is related to the more usually defined Jacobi constant (see Szebehely 1967) which is conserved in a rotating frame centered on the Sun. Subsequently we will give the quantity "$-C/2$" the name "energy" to distinguish it from the heliocentric energy

Figure 2.3 The various accelerations acting on a particle as it moves along a circular coplanar orbit about the asteroid are plotted versus separation from the asteroid; all accelerations are normalized to G, the local gravitational attraction of the asteroid, which decreases as the inverse square of the separation. The various perturbations, shown dotted, are all zero for orbits atop the asteroid (*i.e.,* at zero separation); T is the maximum "tidal" term, C(P) is the prograde Coriolis acceleration and C(R) is the retrograde Coriolis acceleration. P and R are the total perturbations that act on prograde and retrograde particles, respectively, ignoring the vector nature of the actual forces.

(energy of a body orbiting the Sun) and the two-body energy (energy of a body orbiting the asteroid if the Sun were not present). Since a particle's velocity must always remain real, and since C is fixed uniquely by initial conditions (position and speed), Eq. (2.3) restricts the motion of any particle to lie within those regions of space where the following inequality is satisfied:

$$\Omega^2(3x^2 - z^2) + \frac{2GM_A}{r} \geq C. \tag{2.4}$$

The lines along which the velocity is zero (*i.e.,* those places where the left-hand side of Eq. 2.4 equals C) are called zero-velocity or Hill curves. An escape criterion that can been invoked is that whenever, for given initial conditions, a particle lies within a zero-velocity surface that is closed about the asteroid, the particle cannot escape that region. Of course the converse does not hold: there is no guarantee that, just because the Hill curve is open, the particle will necessarily escape in a finite time. Diagrams of zero-velocity curves can be found in many basic celestial mechanics texts (*e.g.,* Danby 1988); particularly nice three-dimensional views are given in Lundberg *et al.* (1985). The distance to the positions along the x_{rot}-axis at which the zero-velocity surface first opens can be computed to be $r_H = (\mu/3)^{1/3}A$ for Hill's problem (Danby 1988). These points are two of the three co-linear Lagrange points (the other is on the far side of the Sun) and their distance from the asteroid defines the radius of the Hill sphere (see Table 2.1). The co-linear Lagrange points are unstable equilibrium points; a particle placed with zero velocity in one of these positions will remain there forever, but particles starting arbitrarily close will depart the neighborhood.

2.3 General Remarks on t e Solution

2.3.1 Integrations

Our numerical integrations call upon an efficient integrator that utilizes both the Bulirsch-Stoer and Runge-Kutta methods (Press *et al.* 1987). The routine takes advantage of the speed of the Bulirsch-Stoer technique, falling back on the Runge-Kutta scheme during close approaches between the two bodies (*cf.* Murison 1989a).

In our integrations the particle was generally started along the Sun-asteroid line, on the far side of the minor planet (Fig. 2.1). It was usually given a velocity that would place it on a circular orbit around the asteroid if perturbations from the Sun were absent. In many simulations the plane of the particle's orbit was given an initial inclination i with respect to the plane of the asteroid's orbit. The inclination is positive to the heliocentric north, and reaches 180° for a purely retrograde orbit. With these initial conditions, the ones used most frequently, the only degrees of freedom are the initial separation distance and the initial inclination. We also explored other initial conditions for the particle (*i.e.,* a spectrum of starting longitudes, different launch speeds, and arbitrarily-directed initial velocity vectors) to assess the generality of our results.

2.3.2 Nature of Orbits

Since the relative strengths of the various perturbations change with separation (Fig. 2.3), orbits may have quite different characteristics depending on their distances from the asteroid (Chauvineau and Mignard 1990a). Within a few asteroidal radii, orbits are simple Keplerian ellipses since the asteroid's gravity dominates all perturbations (see Fig. 2.3 and the earlier discussion). Farther out, perturbations become large enough to induce orbital planes and pericenters to precess noticeably, although the orbits retain their basic Keplerian nature. As the distance is increased still further we come to a region in which quasiperiodic stable orbits are intermingled with chaotic paths. An orbit is quasiperiodic if it contains only a finite number of incommensurate frequencies. In many of our experiments, the period corresponding to the particle's dominant frequency is seen to be commensurate with the asteroid's orbital period; such a commensurate "locking" between the forcing frequency and the natural response of a system is a common feature of nonlinear systems (Guckenheimer and Holmes 1983).

This quasiperiodic/chaotic zone gradually gives way to the realm of escape orbits which we define as those trajectories that depart the vicinity of the asteroid, but the division between these regions is not clearly defined; in fact, in the circular restricted three-body problem the boundary between these regions is self-similar in a fractal-like manner (Murison 1989b). In an area where escape orbits predominate, isolated "islands" of stable quasiperiodic orbits can occur (Chauvineau and Mignard 1990a). And, likewise, in regions where mostly quasiperiodic orbits exist, a few escape orbits can be found. Although the regions are not entirely disconnected, we observe that beyond a certain "stability boundary," the number of stable orbits drops very sharply. Our goal in this chapter is to understand the shape of this boundary that separates orbits bound to the asteroid from those that escape its influence. Since chaotic orbits are prevalent in the transition zone, our results for the size of the stability zone are probably conservative: longer integrations would have shown additional escapes (Wisdom 1982). But, to a first approximation, we can determine the locus of points forming the stability boundary by looking at the outermost regions where the majority of orbits are stable. Chaos necessarily permeates these outer regions, since a particle's fate certainly depends sensitively on initial conditions (Murison 1989b).

Since the results of Chauvineau and Mignard (1990a), which follow on the pioneering study of Hénon (1970), are so relevant to our findings, they will be summarized here. These authors use the surface-of-section technique to study the stability of motions in Hill's problem. They find that, for prograde orbits that have a Jacobi constant much greater than the critical value at which the Hill curves no longer enclose the asteroid, the motions are regular: trajectories are nearly periodic, and stable. Employing non-dimensional units, in which the gravitational constant, the asteroid's mean motion, and its Hill radius are set to unity ($G = 1, \Omega = 1$, and $r_H = 1$; these choices set $M_A = 3$), the critical Jacobi constant occurs at $C = 9$. At values somewhat above 9 (from 9.2 to 9.3604 to be precise), the topological structure of the mapping is such that new periodic orbits are introduced as C is lowered; more and more of these periodic islands appear as $C = 9.2$ is approached and the regularity of the mapping is lost. At 9.2 and below, chaotic trajectories appear in parts of the mapping. These ergodic regions tend to fill up more and more of the phase space until, with C near 9, little of the surface of section is populated with periodic islands; instead virtually all is a sea of chaos. Note that up to this point, since all the zero-velocity curves corresponding to $C > 9$ encircle the asteroid, the motions are bounded with the particles remaining about the asteroid, albeit moving along chaotic paths. However, once the Jacobi constant falls below 9, suddenly the ergodic region becomes connected with external parts of the phase space. That is, however, not to say that all particles will necessarily escape in a finite time, merely that it is energetically possible for particles with $C < 9$ to find their way through the ergodic region and escape. Some regular direct orbits do exist for $8.88 < C < 9.00$, although they cover little of the available phase space. For retrograde orbits Chauvineau and Mignard (1990a) find quite different results. With $C >> 9$ the mapping is usually regular and, as in the prograde case, chaos appears when C is a bit larger than 9. The striking difference is that many regular retrograde orbits are seen to persist for values of C well below the critical value, unlike the prograde situation. For completeness, we note that there are also a small number of pathological orbits that oscillate between the direct and retrograde states.

To help the reader connect the results of Chauvineau and Mignard (1990a) to the trajectories that we will be plotting later, we now show that a one-to-one correspondence exists between our usual initial conditions and the Jacobi constant. Recall that we start a particle at $(d, 0, 0)$, with a velocity that is inclined at an angle i from the xy plane and whose speed in the non-rotating frame is $(GM_A/d)^{1/2}$. From Eq. (2.3) the Jacobi constant for this initial condition is:

$$C = -\left[\left(\frac{GM_A}{d}\right)^{1/2}\cos i - \Omega d\right]^2 - \frac{GM_A}{d}\sin^2 i + 3d^2\Omega^2 + \frac{2GM_A}{d}. \qquad (2.5)$$

Figure 2.4 is a plot of C versus the starting distance d for various inclinations i; the plotted Jacobi constant is given in the non-dimensional units used by Chauvineau and Mignard (1990a).

Figure 2.4 The Jacobi constant, in non-dimensional units ($G = 1, \Omega = 1, r_H = 1$), is plotted for the family of orbits studied in this chapter. These orbits are initially circular, are started from the positive x-axis, and are inclined by an angle i with respect to the xy plane. The critical Jacobi constant ($C = 9$) is also plotted. If the Jacobi constant of a particular orbit lies above the critical line, that particle is bound to the asteroid for all time. If, however, it lies below the critical line, the particle is energetically able to escape, although it is not required to do so.

2.3.3 Scaling to Other Asteroids

Even though most of our simulations considered a specific case ($\mu = 5 \times 10^{-12}$ and $A = 2.55\,\mathrm{AU}$), we can apply our results to other asteroids with different semi-major axes and mass ratios. Consider a system of N gravitationally interacting bodies viewed from an inertial frame. All forces in the system are gravitational, so the strength of each interaction varies as the inverse square of distance. In particular, if all distances are multiplied by a factor ζ, the forces retain their directions and are reduced by ζ^2. One can then rescale time so that the resulting system of differential equations is identical to the original set: therefore, as long as the initial velocities are also appropriately modified, identical orbital paths will result. So, for example, if the asteroid's distance from the Sun is doubled, particle orbits around the asteroid will have the same shape as in the original case if starting distances from the asteroid are doubled and velocities are reduced by a factor of $2^{1/2}$. Thus, the orbits scale with the asteroid's semimajor axis A.

Employing similar ideas to a change in the asteroid's mass, we find that the orbit scales with $\mu^{1/3}$ for the case of the three-body Hill problem with the asteroid-particle distance much less than the distances to the Sun. This approximation is well satisfied for the motion of bound satellites of asteroids. For the distant satellites of the jovian planets, however, higher-order terms in the mass ratio μ are important and the scaling law is less valid. When combined, the distance and mass scaling laws imply the powerful assertion that for *each orbit* existing around one asteroid, a corresponding orbit, differing *only* in absolute size, exists around a second asteroid provided that the two asteroids have the same orbital eccentricity. The ratio of the sizes of the two orbits is equal to the ratio of the radii of their respective Hill spheres: $r_H = (\mu/3)^{1/3}A$. If the sizes are measured in asteroid radii, as in our plots, they scale as $\rho^{1/3}A$. In particular the orbital stability zone, which is the union of all stable orbits, scales as this ratio. The sizes of the Hill spheres of Amphitrite and Gaspra are listed in Table 2.1.

At any rate, it is clear that Hill sphere scaling differs from $\mu^{2/5}A$, the size of the sphere of influence, that has been used by some mission planners to estimate the region within which material could be stably trapped. We recall that the sphere of influence is defined as that surface along which it is equally valid to consider the motion of the particle relative to the Sun with the asteroid as a perturber as it is to consider the motion of the particle relative to the asteroid with the Sun as a perturber (Roy 1978). That is to say, the sphere of influence is the locus of points where the ratios of the perturbing forces to the direct forces in the two cases are equal. This sphere lies within the Hill sphere for $\mu < 0.004$ but the difference only becomes significant (Chebotarev 1964) when μ is very small, as in the case under consideration here. Amphitrite's sphere of influence has a radius of $115R_A$.

As an example of scaling, we consider orbits about Galileo's target asteroid 951 Gaspra. To apply our Amphitrite plots given below to an asteroid with

Gaspra's parameters, but zero orbital eccentricity, distances measured in asteroid radii should simply be multiplied by the ratio of the semimajor axes, namely 2.20/2.55=0.86 (Table 2.1).

2.4 nalytic Escape riteria

Many estimates of analytical escape criteria for circular orbits have been made; most follow either from considering the Jacobi constant that will open the zero-velocity curves or from equating forces in a rotating frame (see Fig. 2.3). Szebehely (1978) has used the former method to predict that circular orbits will escape when they are beyond $r_H/3$. Markellos and Roy (1981) refined Szebehely's treatment by including all of the terms in the Jacobi equation (Eq. 2.5 with $i = 0^o$ and $i = 180^o$) to derive critical distances of $\sim 0.49 r_H$ for prograde circular orbits and $\sim 0.28 r_H$ for retrograde circular orbits (see Fig. 2.4). These distances are lower limits for escape; particles starting on circular orbits within these distances are constrained by closed zero-velocity surfaces that encircle the asteroid. Our numerical results for initially circular orbits are $\sim 0.49 r_H$ for prograde orbits and $\sim r_H$ for retrograde ones (see Section 2.6.1). The agreement of the prograde results is impressive, while that of the retrograde results is appalling. But there is a simple explanation: the method outlined above ignores the influence of the Coriolis acceleration on the particle since the scalar product of the Coriolis term in Eq. (2.2) with $\mathbf{v}_{rot}$ is zero. The effect of this omission is abundantly clear in the results of Markellos and Roy which predict that retrograde orbits are less stable than prograde ones, even though the directions of prograde and retrograde Coriolis accelerations imply the converse (see Fig. 2.2). In fact, we find that prograde orbits slip away as soon as escape is energetically possible, pushed outward by the omitted Coriolis acceleration, while retrograde orbits linger, held in by this acceleration.

Equating forces in a rotating reference frame was originally applied by King (1962) who showed that direct gravity balances the "tidal" force along the x-axis at a distance r_H. Innanen (1979) added the effects of the Coriolis force to obtain limiting radii for prograde and retrograde orbits of $0.69 r_H$ and $1.44 r_H$, respectively. This work contains a subtle error which involves the translation of the particle's velocity into the rotating frame; after correction of this mistake, we find that the limiting radii calculated via Innanen's method should be $0.80 r_H$ and $2.60 r_H$, respectively (these distances are the points where the normalized force curves P and R attain a value of zero in Fig. 2.3). This method shows that retrograde orbits are stable out to much greater distances than prograde ones, but gives poor agreement with numerical results (see the discussion of Fig. 2.3 for an explanation of why this method gives poor predictions).

Various arguments (see, *e.g.,* Keenan and Innanen 1975) have been given for the reason why retrograde orbits are so much more stable than prograde ones, but one that we find especially appealing relies on the nature of epicycles, the paths of particles on elliptical orbits as seen from a coordinate system that moves at the mean orbital rate; epicyclic motions are retrograde and, for small eccentricities, take place along a 2:1 ellipse aligned with the long axis in the direction of the orbital motion. That is to say, if a particle were at a great separation from the asteroid such that it felt virtually no attraction to the asteroid but it had an elliptical path of the same semimajor axis as the asteroid's, it would be observed in the rotating system to travel along a retrograde path (see Chauvineau and Mignard 1990a). In a very real sense the retrograde motion is preferred whereas prograde motion must be forced.

2.5 In ivi ual Examples

2.5.1 Coplanar Trapped Orbits

Our numerical experiments for the Amphitrite case show that all trajectories that start as circular prograde orbits within $\sim 224R_A$ ($C = 9.0000$) are bound, while most of those outside this range escape from the asteroid. Since we are concerned with the outer limit where material can still be retained by the asteroid, we show an orbit (Fig. 2.5) that is close to the stability limit, namely one that was initially circular at $221R_A$ ($C = 9.0505$). The displayed orbit, is quasiperiodic with two dominant frequencies: one is the inverse of the synodic period and the other is about eight times slower. The regular appearance of this orbit in the rotating frame is due to the fact that the two dominant frequencies are close to a ratio of integers. Relevant timescales are the asteroid's orbital period (4.08 Earth years), and the sidereal period of an unperturbed satellite at $221R_A$ (0.80 years). The unit of time in this and the following plots is taken to be an asteroid year (the period of the asteroid's orbit around the Sun).

We can qualitatively understand the orbital evolution of Fig. 2.5 by considering the acceleration (Eq. 2.1) along an initially circular orbit. At first, the path is elongated into an elliptical shape by the action of the tidal term since the Coriolis term does not change a circular orbit (an orbit that is circular in the sidereal frame will also be circular in the synodic frame; the Coriolis acceleration in this simple case merely accounts for the difference in orbital velocity measured in the two frames). As the orbit elongates and is flattened further, the Coriolis acceleration becomes increasingly asymmetrical (see Fig. 2.2); the strengthened Coriolis acceleration near pericenter enhances radial accelerations there whereas the corresponding acceleration is diminished near apocenter (Fig. 2.3). In fact, the direction of the Coriolis acceleration near apocenter can switch sign if the eccentricity is high enough (remember that it is the velocity in the rotating frame that appears in Eq. 2.1); although such a reversal does not occur in any of the planar orbits displayed in this chapter, we have noticed it in other integrations.

Figure 2.5 The path of a particle started on a prograde coplanar circular orbit at $221R_A$ ($C = 9.0505$) as seen in the rotating coordinate system. The asteroid's position is given by an x; the particle's initial location by the small triangle with one point showing the direction of the initial velocity; and the particle's location at the end of the integration by the solid square. The Sun lies out the negative x_{rot}-axis throughout the integration. The heavy line shows the zero-velocity curve specified by the initial conditions (see Eq. 2.4) and the stars show the positions of the nearby Lagrange points (and accordingly the size of the Hill sphere).

Now the fact that the Coriolis acceleration near apocenter is less than that necessary to maintain a circular orbit allows local gravity to more effectively compete with the tidal force. This competition is most apparent in highly-eccentric orbits where the apocenter end of the ellipse appears to be flattened (Fig. 2.5). The asymmetry of the Coriolis acceleration acts to circularize the orbit, and eventually it dominates the elongating effect of the tidal force. In the example under discussion, this occurs after the third synodic period. The elongation slows, stops, and reverses itself. The orbit then becomes more circular until the tidal force again dominates the Coriolis force and the process repeats. The period of this cycle is eight times the synodic period as was mentioned above.

The entire orbital path of the prograde satellite shown in Fig. 2.5 lies well within the zero-velocity curve, defined by Eq. (2.4), that corresponds to the initial conditions. This occurs because a significant fraction of the "energy" in the Jacobi integral remains in kinetic "energy". It is apparent from the zero-velocity curve that the specified starting conditions have too little initial "energy" to allow escape.

The dynamical history of an orbiting particle can be described in terms of its initial position and velocity or, equally well, in terms of its four osculating orbital elements for a two-dimensional problem (Danby 1988). The osculating orbital elements for a bound orbit are defined to be those that describe the ellipse that the particle would follow if all perturbations were turned off. These elements, which we define in the non-rotating frame, change with time as perturbations cause the particle to deviate from true elliptical motion. Of the orbital elements, the orbital semimajor axis a is the most significant when addressing escape since the size of the orbit, $2a$, formally becomes infinite and then attains negative values as the particle goes through the escape process. The time histories of the osculating orbital elements that describe the path about the asteroid shown in Fig. 2.5 are displayed in Fig. 2.6. Here the periodic nature of the solution is clearly visible. We note that the semimajor axis vs. time curve has local extrema near the points where the orbit crosses the x_{rot} and y_{rot} axes. This feature arises because orbital energy is directly related to the semimajor axis (Burns 1976) and because the work done by the tidal force changes sign in each quadrant of the $(xy)_{rot}$ plane. In general, the work done by the tidal force will change sign four times in a single orbit, although this need not occur at the points where the orbit crosses the axes.

In the next example (Fig. 2.7), the particle starts along a retrograde circular orbit twice as large as the first example; it begins at $x_{rot} = 445R_A, y_{rot} = 0$ ($C = 1.5518$), very close to the transition between bound and unbound retrograde orbits. The unperturbed sidereal orbital period is about 2.3 Earth years or nearly 4/7 of an asteroid year. As in the prograde case, quasiperiodic retrograde orbits are also common; this one has two major frequencies that are not quite a ratio of integers as can be seen in Fig. 2.8 which presents the histories of the osculating

Figure 2.6 The time history of the orbit shown in Fig. 2.5. Plotted are the particle's osculating orbital semimajor axis a, orbital eccentricity e, and orbital radius r as functions of time in asteroid years. The orbit is most perturbed when it is farthest from the asteroid. It is bound and almost periodic.

Figure 2.7 The path of a particle started on a retrograde coplanar circular orbit at $445R_A$ ($C = 1.5518$) as observed in the rotating coordinate system. See Fig. 2.5's caption for a description of the symbols. The orbit is bound and has a very regular appearance.

orbital elements a and e. For this retrograde orbit, the zero-velocity curves do not constrain the motion (Chauvineau and Mignard 1990a) since, as a result of the small C (due to the large apparent velocity of a retrograde orbit as measured in the rotating frame), the curves do not enclose the asteroid. Nevertheless, the particle is obviously bound; indeed we note that it is strongly influenced by the asteroid since its orbital shape is not the 2:1 ellipse that would be characteristic of heliocentric epicyclic motion.

To analyze the particle's motion, consider the perturbing effects of the tidal and Coriolis terms on a circular orbit (see Fig. 2.9 which shows the first three loops about the asteroid of Fig. 2.7). Initially the tidal term dominates, since the Coriolis acceleration does not change the shape of a circular orbit. This pushes the particle in the x_{rot} direction (arc AB) which displaces the orbit as a whole to the right (positive x_{rot}). When the particle moves to the left side of the asteroid, it is much closer to the asteroid due to this displacement (arc BC). Thus at point C the tidal term, being proportional to x_{rot} (see Fig. 2.2) , is smaller than it was at A. Hence the total contribution of the tidal force along BC is smaller than the integrated effect along AB, resulting in a net displacement of the orbit to the right. In addition, the Coriolis acceleration, which is stronger over arc BC than over arc AB due to a larger velocity, dominates the weakening tidal force. The particle then swings around the asteroid (arc CD), mostly under the influence of the asteroid's gravity, and out to large r where Monsieur Coriolis starts to tug it to the left (arc DB). The tidal force switches sign again, and pulls the particle outward along arc BE to the point E, where it has roughly the negative of its initial velocity and position: the cycle repeats.

2.5.2 Coplanar Escape Orbits

Figures 2.10 and 2.11 show planar escape orbits that have initial conditions that are close to the bound orbits of Figs. 2.5 and 2.7; thus all of these orbits lie near the stability boundary. In those cases where escape is marginal (such as all those discussed here), the direction of escape is always near the Sun-asteroid line because the outwardly directed tidal term is maximum there (Fig. 2.2). This result, which remains valid even for inclined orbits, can also be understood readily from the zero-velocity surface which opens first along the Sun-asteroid line (see Fig. 2.10). Of course, with large enough initial "energy" (or, equivalently, small enough C for the zero-velocity curves to be wide open), objects can escape in any direction, but in all of the cases that concern us, objects depart from the asteroid with little extra energy because the particle is initially bound (*i.e.,* its energy in the two-body system composed of the asteroid and the particle is initially negative) and the perturbation forces can modify this energy only slowly. In fact, the Coriolis acceleration, being perpendicular to the orbital velocity, can do no work and thus does not alter the orbital energy at all.

Figure 2.8 The time history of some osculating orbital elements for the retrograde orbit displayed in Fig. 2.7.

Figure 2.9 The first few loops of the orbit shown in Fig. 2.7. The letters on the path are used in the text to describe various arcs along which particular accelerations dominate the motion.

Figure 2.10 The trajectory of a coplanar prograde particle that escapes after starting on a circular orbit at $227.25 R_A$ (C=8.9423). The symbols are defined in Fig. 2.5's caption. Note that, in contrast to Fig. 2.5, the initial conditions here are such that the zero-velocity curve is open to heliocentric space and the particle, after bouncing chaotically around within the zero-velocity bottle, eventually slips out the neck to move along an elliptic heliocentric orbit having properties described in the text.

Figure 2.11 The trajectory of a coplanar retrograde particle that escapes after starting on a circular orbit at $450R_A$ (C=1.5421). See Fig. 2.5's caption for a description of the symbols used. Note that on the last loop the path extends well beyond the radius of the Hill sphere and that the particle transfers to a prograde orbit before escaping. In this case, the transfer to a prograde orbit occurs in both the rotating and non-rotating frames. The character of the escape path is discussed in the text.

Figure 2.10 shows a chaotic prograde orbit started at $227.25 R_A$ ($C = 8.9423$) that escapes inward toward the Sun. The fact that the zero-velocity surface accurately delimits the accessible region of space is apparent. Note that since, by definition, speeds must be zero on zero-velocity surfaces, particles approach the surface perpendicular to it so as to form orbital cusps.

Because the asteroid's orbit is circular, one can very simply calculate the parameters of the solar orbit that is attained by particles escaping from it. Since the particle departs the asteroid with a very low velocity relative to the rotating frame, we can ignore this velocity as well as later influences of the asteroid (since it is so small and so distant) when estimating the particle's heliocentric energy which determines directly the orbital semimajor axis of the particle in its new path around the Sun (Burns 1976). The particle's velocity in the rotating frame is lowest near the inner Lagrange point (see Fig. 2.10), so at this point its angular velocity about the Sun closely matches that of the asteroid. Making the simplification that the particle starts from the inner Lagrange point with zero velocity in the rotating frame, one can calculate the specific (*i.e.,* per unit mass) heliocentric kinetic energy of the particle $\Omega^2(A - r_H)^2/2$, and its specific potential energy, $-GM_\odot/(A - r_H)$. Equating the sum of these two energies to the total specific heliocentric energy, $-GM_\odot/2A_g$, we find that the semimajor axis of the particle's new orbit about the Sun is $A_g = (A - 4r_H)$. Since the particle's initial velocity in the non-rotating frame is perpendicular to the solar direction and the particle initially falls toward the Sun, the Lagrange point must be aphelion of the new solar orbit. Solving the equation for aphelion $A_g(1 + E_g) = A - r_H$ yields (to first order) an eccentricity of $E_g = 3r_H/A$. Since the escaped particle's semimajor axis is smaller than the asteroid's, the particle's orbital period is shorter, so its path trails off to the upper left as viewed in the frame rotating with the asteroid's mean motion (Fig. 2.10). Alternatively the direction of departure can be understood in the rotating frame by considering the effects of the Coriolis acceleration.

Figure 2.11 shows a retrograde orbit starting at $450 R_A$ ($C = 1.5421$) that becomes prograde just prior to escape. Arguments similar to those for the prograde orbit can be used to find $A_g = A + 4r_H$ and $E_g = 3r_H/A$; thus the particle's escape path trails off to the lower right. Again, although it is not as clear as in the prograde case, the point of lowest relative velocity occurs near a Lagrange point.

2.5.3 Inclined Orbits

Figures 2.12 and 2.13, which are plotted in non-rotating coordinates, show orbits with initial inclinations of 70°. Fig. 2.12, where the trajectory is seen as projected onto the xz plane, displays an orbit that starts out roughly circular at a distance of $230 R_A$ ($C = 7.2747$) but changes to an oval shape that becomes narrower and

Figure 2.12 The trajectory of a particle started on a circular orbit at $230R_A$ with an inclination of 70^{o} as viewed in a projection onto the xz plane of the non-rotating system (C=7.2747). The symbols are defined in Fig. 2.5's caption. This particle eventually escapes.

Figure 2.13 An xz projection of a 50 year integration of a particle started on a circular orbit at $250R_A$ with an initial inclination of 70° ($C = 6.9306$). This particle, like many others on three-dimensional orbits with inclinations satisfying $60^\circ < i < 120^\circ$, is seen to reach roughly the same z value regardless of x.

narrower until, on the last loop, the direction of rotation actually reverses! When viewed in three dimensions, the ellipse is tilted out of the asteroid's orbital plane by approximately 45° and the direction of its major axis is such that the latter's projection onto the orbital plane lies along the initial Sun-asteroid line. The ellipse is not as narrow as it appears in this projection since it also extends in the $\hat{\mathbf{y}}$ direction. To lessen confusion in the diagram, we have elected not to show the further evolution of the orbit but will describe it. The highly-eccentric orbit is seen to broaden slowly until it is approximately circular. At this point, the cycle begins to repeat with the circular orbit slowly becoming more eccentric, but after a second close approach to the asteroid, the particle escapes. In many orbits (*e.g.*, Fig. 2.13) this cycle continues without an escape. Each time the approximately circular orbit begins to increase its eccentricity, the major axis of the new ellipse is found to be tilted at $\sim 45°$ from the xy plane and to lie along the re-oriented Sun-asteroid line. The axis can be tilted either toward or away from the Sun, and can lie either primarily above the xy plane or primarily below it due to the symmetry of the tidal term. Fig. 2.13 shows an orbit started $x = 250R_A$ ($C = 6.9306$) that was followed for ten circuits of the asteroid around the Sun. Notice that the maximum z values attained by the orbit are approximately independent of x. This characteristic, which was observed on many orbits started near the critical distance with $60° < i < 120°$, has an important influence on the shape of the stability zone as described below.

In the depicted case, tidal perturbations alone must be responsible for the motion since the results are plotted in non-rotating coordinates, where no Coriolis term appears. The form of the tidal acceleration in the non-rotating frame is $\Omega^2(3\mathbf{x}_{rot} - \mathbf{r})$, which differs from the second term of Eq. (2.2) since that term included the centrifugal acceleration of the rotating frame. In the following, we lump the radial part of the tidal term in with the asteroid's gravity, and consider only the effects of the $\mathbf{x}_{rot}$ term. Consider a particle that would be on an elliptical orbit primarily in the xz plane in the absence of perturbations (Fig. 2.14), and ignore for the moment the fact that the Sun is not always along the x-axis. We see that, starting from $x = 0$, the tidal perturbation pushes the particle to larger values of x than would be experienced in a two-body problem. Because of this added acceleration, the particle drops along an orbital path that brings it closer to the asteroid than its unperturbed counterpart. Throughout the region of close approach, the tidal force is negligible so that we can approximate the motion there by the solution to the two-body problem. Hence, after one revolution, the particle emerges on a more highly-eccentric ellipse, and the cycle repeats. The outcome of the narrowing ellipse is either an impact with the asteroid or a reversal of the direction of rotation (see Fig. 2.14). If the latter occurs, the tidal acceleration operates in the opposite way to broaden the orbit out to a circle where the whole process begins anew. Because of passage through many of these very narrow ellipses, the probability for a particle on an orbit of this

Figure 2.14 Effect of tidal forces on an inclined elliptical orbit. Notice that the actual orbital path for a single revolution around the asteroid is displaced to the right from where an unperturbed elliptical path would lie. This, of course, is due to the tidal acceleration. The orbit shown is part of that in Fig. 2.12.

type to impact the asteroid is very large. We note that the reverse of such an impact orbit offers a mechanism by which material, blasted from the surface of the asteroid by a collision, could be put into distant orbits.

The essence of this argument is unchanged when we take into account that the Sun is not always along the x-axis as measured in the non-rotating frame. Therefore, in general, the tidal acceleration contains both x and y components that vary in time. Because the particle's orbital motion remains primarily in the xz plane, the direction of the tidal acceleration varies roughly sinusoidally as this plane moves with the asteroid's angular frequency around the Sun. Thus generally the x component of the tidal acceleration dominates the y component for the simple reason that the orbit never samples large y values. The argument can be generalized for orbits whose motions are primarily in the $x'z$ plane where x' is some linear combination of x and y. Orbits with inclinations in the range $60^o < i < 120^o$ have their motions primarily in some $x'z$ plane, and thus exhibit this type of dynamical motion.

2.6 lobal Structure

2.6.1 Escape as a Function of Inclination

To explore the effects of orbital inclination on the stability of particles, we studied weakly bound orbits that began at various inclinations but otherwise chose the same initial conditions for purposes of comparison. We define the critical distance as the initial displacement within which most orbits remain bound, and outside of which most escape. We find that the critical distance displays a strong dependence on initial inclination. Naturally, because of the problem's fractal-like nature (Murison 1989b), occasional orbits within the critical distance escape, while some others outside this distance are bound; in this sense the critical "distance" represents a very complex structure that cannot be truly represented by a single line. The number of these exceptions, however, decreases rapidly as one moves away from the transition region.

Figure 2.15 shows the results of almost seven hundred different integrations in which the initial distance and initial inclination were varied in increments of $10R_A$ and 10^o respectively. The diagram distinguishes between orbits that escape, those that remain captured, and those that crash into the asteroid. Note that the collision orbits occur predominantly for inclinations around 90^o where orbits undergo the hazardous "narrowing ellipse" motion described above. It is apparent that there is a fairly crisp "boundary" between the bound and escape orbits; this boundary is the critical distance. Most of the graph's features can be interpreted as due to the Coriolis acceleration. Taking a circular orbit for illustration, consider the radial part of the Coriolis term (*i.e.,* toward or away from the asteroid), which is proportional to $\cos i$ and which therefore attains its maximum

Figure 2.15 The critical distance, which divides stable from unstable orbits, as a function of initial inclination. All particles are injected on initially unperturbed circular orbits along the Sun-asteroid line. A large solid dot signifies an orbit that remains near the asteroid for at least 5 asteroid years, a small dot is an orbit that escapes in less than this amount of time, and an open circle with a dot inside is an orbit that strikes the asteroid. Note that orbits with $i > 90^o$, particularly those that approach purely retrograde orbits, are stable out to much greater distances than coplanar prograde paths (see text for discussion).

inward and outward strengths at $i = 180^{\circ}$ and $i = 0^{\circ}$, respectively. This predicts the upward trend of the critical distance with inclination in Fig. 2.15.

We find a local minimum in the critical distance near $i = 90^{\circ}$ confirming previous results of Keenan (1981). This feature and the rough symmetry for $\pm\ 30^{\circ}$ around $i = 90^{\circ}$ can be explained by abrupt inclination shifts that we have observed in orbits with initial inclinations in the range $60^{\circ} < i < 120^{\circ}$. We have found that many escape orbits with inclinations i in this range switch to orbits with an inclination $\sim 180^{\circ} - i$ via the narrowing ellipse process outlined in Section 2.5.3, and thus escape for both i and $180^{\circ} - i$ orbits can occur at the smaller inclination where the Coriolis binding acceleration is weaker. Together, these two effects predict the overall shape of Fig. 2.15. Non-radial Coriolis accelerations, which are maximum near $i = 90^{\circ}$, may also influence the structure and exact location of the minimum.

2.6.2 The "Stability Boundary"

Figure 2.16 illustrates the shape of the boundary within which stable orbits lie. The surface represents the maximum z value attained by a particle as a function of x_{rot} and y_{rot}, not for a single orbit, but for the union of nearly 1000 stable orbits lying within the critical distance in Fig. 2.15. The rare stable orbits found in regions where unstable orbits predominate were not included (see prior discussion of the fractal-like nature of the stability boundary and Fig. 2.15). The output of our integration routines is a series of points in the rotating system (x_{rot}, y_{rot}, z) through which a given orbit passes. We divided the $(xy)_{rot}$ plane up into a 20×20 grid of $60\,\mathrm{km} \times 60\,\mathrm{km}$ squares and recorded the maximum z value occurring above each square from the union of all of the points in each of the stable orbits. The data were then interpolated out to an 80×80 grid to optimize the viewing.

We also exploited two symmetries to quadruple the effective number of input orbits to Fig. 2.16. It can be shown that the transformation of initial conditions $z \to -z, v_z \to -v_z$, results in an orbit that is the reflection of the original orbit through the xy plane (see Eq. 2.2). This follows most simply from considerations of the symmetry of the gravitational forces in an inertial frame centered on the Sun. Thus each of our orbits has a mirror image through the $(xy)_{rot}$ plane and we can incorporate this image by taking not the maximum z, but the maximum $|z|$ attained. This effectively doubles the number of input orbits. Furthermore, the transformation $(\mathbf{r} \to -\mathbf{r}, \mathbf{v}_{rot} \to -\mathbf{v}_{rot})$ also yields identically shaped orbits in Hill's problem, so we can again double the number of input orbits. All told, there are $4 \times 239 \approx 1,000$ separate initial conditions incorporated in Fig. 2.16, each pertaining to an orbit that is stable for at least 5 asteroid years.

Fig. 2.16 shows that the stability surface is roughly flat on top with very steep sides. The plateau region is at an average height of about $285 R_A$ above

Figure 2.16 Plot of the upper half of the stability surface viewed from pitch=60°, yaw=10°, and roll=0° as suggested by the reference cube. Note that the scale is distorted due to the viewing angle. The flattened surface is at an approximate altitude of $z = 285R_A$, and the surface drops off precipitously to the roughly circular base region ($r \sim 480R_A$). To determine this surface we took the exterior envelope of the orbits of about 1000 particles that were started near the critical distance but remained captured for 5 asteroid years. Thus, if pathological cases are ignored, particles found within the surface are generally bound to the asteroid while those outside are not. See the text's discussion for more details about how this figure was constructed. This figure clearly illustrates that stable orbits are more closely confined in the polar region.

the xy plane with the highest orbit rising to $307 R_A$ above the plane; its base is roughly circular with a radius of about $480 R_A$. The flattened polar region arises from the fact that maximum z values attained by orbits with $60^o < i < 120^o$ are roughly independent of x and y (see Fig. 2.12). The plotted surface is not based on enough different orbits to validate comments on the second order structure of the surface; in addition, we remind the reader that this surface pertains to particular initial conditions, and thus the detailed shape may change somewhat with different modes of injection.

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3.1 nalytic Treat ent

3.1.1 Equation of Motion

The study of orbital stability in Chapter 2 assumes an asteroid on a circular orbit and although an exact scaling law can connect results for asteroids with different masses and distances from the Sun, no such scaling to asteroids with other orbital eccentricities is expected to be possible. Since many asteroids and comets are on significantly elliptic orbits, this chapter will explore the consequences of non-zero orbital eccentricity on the stability of circum-asteroidal orbits.

An asteroid on an elliptic orbit moves around the Sun at a non-uniform angular rate which, written as a vector, is:

$$\mathbf{\Omega} = \frac{d\nu}{dt}\hat{\mathbf{z}} = \left(\frac{GM_\odot}{R^3}\right)^{\frac{1}{2}}(1 + E\cos\nu)^{\frac{1}{2}}\hat{\mathbf{z}}, \tag{3.1}$$

where R is the instantaneous distance from the Sun given by

$$R = \frac{A(1-E^2)}{1+E\cos\nu}; \tag{3.2}$$

and A, E, and ν are the asteroid's semimajor axis, eccentricity and true anomaly, respectively (see Fig. 3.1). Equation (3.1) reduces to Eq. (2.1) in the limit $E \to 0$. The true anomaly ν, which gives the angular location of the particle relative to pericenter, is a periodic function of time; thus Ω and R also vary periodically. To study orbits in the vicinity of the asteroid, it is desirable to work in a coordinate

[1]This chapter is based on the paper: Hamilton, D.P., and J.A. Burns (1992), Orbital stability zones about asteroids II. The destabilizing effects of eccentric orbits and of solar radiation, *Icarus* **96**, 43–64 [copyright 1992 by Academic Press, Inc.]

Figure 3.1 An eccentric orbit showing the definitions of some of the variables used in the text. The Sun lies at one focus of the ellipse and the asteroid's true anomaly ν is the angle between the asteroid and pericenter as seen from the Sun. The instantaneous Sun-asteroid distance R is minimum at pericenter ($\nu = 0$) where it attains the value $A(1 - E)$.

system centered on the asteroid and rotating with it at the instantaneous angular velocity $\mathbf{\Omega}$ around the Sun. We generalize Eq. (2.2) and find, to first-order in r/R, the equation of motion for a particle in such a frame:

$$\frac{d^2\mathbf{r}}{dt^2} = -\frac{GM_A}{r^2}\hat{\mathbf{r}} + \frac{GM_\odot}{R^3}[(3\mathbf{x} - \mathbf{z}) + E\cos\nu(\mathbf{x} + \mathbf{y}) + 2E\sin\nu(x\hat{\mathbf{y}} - y\hat{\mathbf{x}})] - 2\mathbf{\Omega}\times\mathbf{v}_{rot}. \tag{3.3}$$

Recall that $\mathbf{r} = r\hat{\mathbf{r}} = \mathbf{x} + \mathbf{y} + \mathbf{z}$ is the vector pointing from the asteroid to the particle, M_A is the mass of the asteroid, and $\mathbf{v}_{rot}$ is the particle's velocity measured in the rotating frame. We omit "rot" subscripts from position coordinates in this chapter and the next since in these chapters we work exclusively in the rotating frame. The velocity in the rotating frame is related to that in the non-rotating frame by

$$\mathbf{v}_{rot} = \mathbf{v} - (\mathbf{\Omega} \times \mathbf{r}) \tag{3.4}$$

where $\mathbf{v}$ is the test particle's velocity relative to non-rotating coordinates; its magnitude for a circular orbit is simply $(GM_A/r)^{1/2}$. Taking $E = 0$ in Eq. (3.3), we recover Hill's equation (Eq. 2.2). The three terms in Eq. (3.3) without explicit eccentricity dependence are the asteroid's gravitational attraction, the "tidal acceleration" and the Coriolis acceleration; these terms are discussed in greater detail in Section 2.2. The new terms are only present for non-zero eccentricity and so will be dubbed the "eccentric" terms in the discussion below. The term with $E\cos\nu$ dependence is a correction to the centrifugal acceleration which arises from the difference in the asteroid's actual angular velocity from the angular velocity it would have if it were on a circular orbit at the same distance. Near pericenter, the asteroid's angular velocity exceeds that which it would have on a circular orbit (Eq. 3.1) and hence there is an enhanced centrifugal acceleration away from the asteroid. Similarly, near the asteroid's apocenter, the angular velocity is significantly lower than it would be on a corresponding circular orbit; consequently the "eccentric centrifugal acceleration" is inwardly directed.

The term proportional to $E\sin\nu$ arises from the non-uniform rate of rotation of the reference frame; it vanishes at pericenter and apocenter where the angular acceleration (the time derivative of Eq. 3.1) is zero. This acceleration always lies in the xy plane and is tangent to a circle surrounding the asteroid. In contrast to the other accelerations discussed above, this acceleration can have a substantial component directed parallel or antiparallel to the particle's velocity; "energy" is added to the orbit in the former case and removed from it in the latter. Since the term has a $\sin\nu$ dependence, it causes "energy" to be added to prograde orbits as the asteroid moves from apocenter to pericenter and removed during the return to apocenter. Retrograde orbits lose "energy" as the asteroid drops toward pericenter but regain it over the second half of the cycle. For many orbits, there is little net change in the "energy" over the asteroid's complete orbital period. Nevertheless, acting over long times, we expect this acceleration to be destabilizing since it produces behavior analogous to a random walk in orbital "energy". Those orbits whose orbital "energy" is increased may eventually be driven to escape.

3.1.2 Hill Sphere at Pericenter Scaling

In this section, our goal is to find a simple analytic way to extend results obtained for an asteroid with a given semimajor axis, eccentricity, and mass to a second asteroid with different values of these quantities. In Hill's problem when the asteroid's eccentricity was zero, we found that such an extension was possible and that distances scale like the radius of the asteroid's Hill sphere $r_H = (\mu/3)^{1/3}A$, where $\mu \equiv M_A/M_\odot$ is the asteroid-Sun mass ratio. Thus, for example, if an interesting orbit were discovered to exist around one asteroid with zero eccentricity, an orbit with the *same shape* exists around all other asteroids which move on circular paths. This follows from the fact that Hill's problem in dimensionless form is parameter free.

These ideas extend readily to the case when the asteroid has non-zero eccentricity. To non-dimensionalize Eq. (3.3), we choose to measure distances in units of the asteroid's Hill radius and angular velocities in units of the asteroid's mean motion $n_\odot \equiv (GM_\odot/A^3)^{1/2}$. With these choices and the definitions given in Eqs. (3.1) and (3.2), we can rewrite Eq. (3.3) as follows:

$$\frac{d^2\mathbf{r}}{d\tau^2} = -\frac{3}{r^2}\hat{\mathbf{r}} + \frac{1+E\cos\nu}{1-E^2}^{\,3}[(3\mathbf{x}-\mathbf{z}) + E\cos\nu(\mathbf{x}+\mathbf{y}) + 2E\sin\nu(x\hat{\mathbf{y}} - y\hat{\mathbf{x}})]$$
$$-2\frac{(1+E\cos\nu)^2}{(1-E^2)^{1.5}}(\hat{\mathbf{z}}\times\mathbf{v}) - 2\frac{(1+E\cos\nu)^4}{(1-E^2)^3}(\mathbf{x}+\mathbf{y}), \qquad (3.5)$$

where $\tau = n_\odot t$ is the dimensionless time and $\mathbf{v}$ is the particle's dimensionless velocity measured in the non-rotating frame. Since the only parameter in Eq. (3.5) is E (ν is a function of time), it follows that with a given eccentricity, the equations of motion are identical for asteroids of different sizes and distances from the Sun; changing these quantities only affect how we define the dimensionless units. In short, since distances are measured in Hill radii, our results scale with that distance. The more interesting question, however, is the following: How can we scale results from one asteroid to another when the two have different orbital eccentricities?

Clearly an exact scaling of results is impossible given the ν dependence of Eq. (3.5); accordingly we attempt to find an approximation valid for the orbits that we are most interested in, namely those that narrowly avoid escaping from the asteroid. Physical intuition and Eq. (3.5) show that the perturbation accelerations felt by an orbiting particle are maximum when the asteroid is near the pericenter of its orbit. In general, therefore, weakly bound particles have their closest brush with escape during the asteroid's pericenter passage and, given slightly more "energy", many of these particles would be expected to escape during this time. If we are only interested in determining what will happen to the system in the short term (a few orbits of the asteroid around the Sun), and are only worried about marginal escapes, which occur near pericenter, then in some sense we can ignore what happens over the rest of the orbit. Taking $\nu = 0$ in Eq. (3.5), we claim that, apart from small differences in the centrifugal and Coriolis terms due to the faster angular velocity at pericenter, the result is just the equation of motion for orbits around an asteroid with $E' = 0$ and $A' = A(1-E)$. In other words, for the purposes of studying marginal escapes on short timescales, an asteroid moving through its pericenter can be reasonably well approximated by a second asteroid moving on a circular orbit at the pericenter distance of the first. A similar tack is taken by Lecar *et al.* (1992) in quite a different context.

In order to justify this claim, we must show that the perturbation accelerations arising from the asteroid's faster angular velocity at pericenter are small compared to the perturbations due to the asteroid's closer distance to the Sun. Consider first the "tidal" acceleration arising from solar tidal and centrifugal effects which is given by the second term on the right side of Eq. (3.5). Evaluated at pericenter ($\nu = 0$) this term becomes:

$$\mathbf{a}_{Tidal} = \frac{[3\mathbf{x} - \mathbf{z} + E(\mathbf{x} + \mathbf{y})]}{(1-E)^3}. \tag{3.6}$$

Expanding Eq. (3.6) in a Taylor series in E, we find the first-order term is given by:

$$E[9\mathbf{x} - 3\mathbf{z}] + E[\mathbf{x} + \mathbf{y}], \tag{3.7}$$

where the first term in brackets arises from the asteroid's closer distance to the Sun and the second comes from the increased angular velocity. The distance terms are significantly larger, especially for particles along the x-axis where escape invariably occurs. This remains true for higher-order terms in the Taylor expansion although the magnitude of the difference decreases somewhat. Treating the Coriolis acceleration in the same manner, we find that at pericenter it can be written in the form

$$\mathbf{a}_{Coriolis} = -2\frac{[1+E]^{\frac{1}{2}}}{[1-E]^{\frac{3}{2}}}\hat{\mathbf{z}} \times \mathbf{v} - 2\frac{[1+E]}{[1-E]^3}(\mathbf{x} + \mathbf{y}); \tag{3.8}$$

here the terms in the denominators arise from the asteroid's closer distance to the Sun while those in the numerators are due to the variation of the asteroid's velocity along its elliptic path. As in Eq. (3.7), we find the change in distance is the dominant effect, accounting for $\gtrsim 75\%$ of the variance in the Coriolis acceleration for all values of the eccentricity.

Since the terms arising from the asteroid's increased angular velocity at pericenter are small compared to the terms owing to its location closer to the Sun, we can – as a first approximation – ignore the velocity terms. A particle's equation of motion around an asteroid near pericenter is then identical to the equation of motion of a particle around a second asteroid on a circular orbit at the pericenter distance of the first. Furthermore, since the stability of weakly bound orbits is put to the greatest test during the asteroid's pericenter passage, the most important factor determining escape is clearly how closely the asteroid approaches the Sun. The synthesis of these results suggests that the size of the asteroid's stability zone is simply proportional to the asteroid's pericenter distance. Combined with the results for an asteroid on a circular orbit (Section 2.3.3), we have that *the size of an asteroid's stability zone is roughly proportional to the size of the Hill sphere calculated at the asteroid's pericenter* ($\sim (\mu/3)^{1/3} A(1-E)$).

This is a very strong assertion. It states that, if we can ascertain the size of the stability zone for one asteroid, we can estimate it for other asteroids with different masses, semimajor axes and eccentricities. As noted in Section 2.3.3, scaling to an asteroid with a different semimajor axis is mathematically exact and scaling to an asteroid with a different mass only errs to the order of the asteroid-Sun mass ratio which is entirely negligible. Thus any given orbit around one asteroid has a counterpart around another asteroid with an *identical shape* if the eccentricities of the two asteroids are the same. Since the stability surface is composed of multiple orbits all of which scale in this way, it does too. We have now shown that for orbits of short duration around asteroids with different eccentricities, most (perhaps 70 - 80%) of the effects of eccentricity on the size of the stability surface can be accounted for by scaling the surface as the Hill sphere calculated at the asteroid's pericenter. In the sections to follow, we use our $E = 0$ results (Fig. 2.15) to make predictions for asteroids with non-zero eccentricity and then compare these predictions with actual numerical integrations. We also discuss the validity of the approximations made for three special cases: prograde, retrograde and $i = 90^{\circ}$ orbits.

3.1.3 The Jacobi Integral

First, however, we digress slightly and consider the Jacobi integral which, after all, is one of the most powerful results available for the circular restricted problem of three bodies. In the circular case, the Jacobi integral allows the derivation of zero-velocity curves (ZVCs) which place simple, but often useful, restrictions on the portion of space accessible to particles starting with given initial conditions. In Section 2.2, we applied these surfaces to an asteroid on a circular orbit; here we examine the difficulties inherent in extending this analysis to asteroids on eccentric orbits.

Attempting to obtain the Jacobi integral in the standard way, we first take the scalar product of Eq. (3.3) with $\mathbf{v}_{rot}$ to obtain

$$\mathbf{v}_{rot} \cdot \dot{\mathbf{v}}_{rot} + \frac{GM_A}{r^2}\dot{r} = \frac{GM_\odot}{R^3}[(3x\dot{x} - z\dot{z}) + E\cos\nu(x\dot{x} + y\dot{y}) + 2E\sin\nu(x\dot{y} - y\dot{x})], \quad (3.9)$$

where the Coriolis term has vanished since it is perpendicular to $\mathbf{v}_{rot}$. The next step is to integrate Eq. (3.9) over time. The terms on the left are directly integrable, but those on the right, especially the last one, are more stubborn. These right-hand terms are implicit functions of time through both the particle's coordinates and the asteroid's true anomaly, and hence they cannot be integrated for an unknown orbit. Thus we find a Catch-22: although a Jacobi integral exists for the case where the primaries orbit along ellipses, it is not known how to express the integral in a useful manner (Szebehely and Giacaglia 1964). That is to say, to obtain useful information from the Jacobi integral, the trajectory of the particle must be known but knowledge of the particle's trajectory makes the information contained in the integral redundant!

Once again, because we are mainly interested in orbits during the asteroid's pericenter passage, we look for a result that can be applied in that region. Taking $\nu = 0$ in Eq. (3.9) eliminates the final term and allows the time integration to be performed. Carrying out the integration and switching to slightly different dimensionless units ($GM_\odot/r_p^3 \equiv 1$, $(\mu/3)^{1/3}r_p \equiv 1$, where $r_p = A(1-E)$ is the pericenter distance), we obtain:

$$C = \frac{6}{r} + 3x^2 - z^2 + E(x^2 + y^2) - v_{rot}^2. \quad (3.10)$$

This equation with $v_{rot} = 0$ determines the shape of the ZVCs instantaneously at the asteroid's pericenter. The application of Eq. (3.10) is approximate, and even then strictly limited to a small time Δt near a single passage of an asteroid through pericenter; similar conclusions are reached through more rigorous derivations (Szebehely and Giacaglia 1964, Ovenden and Roy 1961). If one attempts to apply Eq. (3.10) to two successive pericenter passages, unmodeled effects such as the final term in Eq. (3.9), acting in the interim might alter C, the Jacobi "constant." Fortunately, such modifications are usually small for short time periods, and we can normally apply Eq. (3.10) to orbits followed for a few pericenter passages of the asteroid.

Comparing Eq. (3.10) to the equivalent expression for a circular orbit we find that the two differ only by the excess centrifugal potential $E(x^2 + y^2)$. As an illustration of the slight difference, we calculate the locations where the zero-velocity surfaces surrounding the asteroid first open up. These positions occur at saddle points of Eq. (3.10) (with $v_{rot} = 0$) which are also equilibrium points of Eq. (3.3) (with $\nu = 0$). Setting the partial derivatives of Eq. (3.10) equal to zero, we find that the openings of the ZVCs occur at the points ($x = \pm x_{crit}, y = 0, z = 0$), where x_{crit} and the corresponding Jacobi "constant" are given by:

$$x_{crit} = {\frac{3}{3+E}}^{\frac{1}{3}}; C_{crit} = \frac{9}{x_{crit}}. \qquad (3.11)$$

As the eccentricity is increased in Eq. (3.11), the opening of the zero-velocity surfaces occurs closer to the asteroid; this can be qualitatively understood by noting that the equilibrium points occur nearer the asteroid as a result of the additional outwardly directed centrifugal acceleration at pericenter. For $E = 0$, we recover the more familiar results $x_{crit} = 1$ and $C_{crit} = 9$ (see Section 2.3.2 and Chauvineau and Mignard 1990a); while, conversely, taking the extreme case $E = 1$, we obtain $x_{crit} \approx 0.91$ and $C_{crit} \approx 9.9$, differences of only $\sim 10\%$. We conclude, as above, that the influence of the additional centrifugal acceleration is minimal.

3.2 Integrations

3.2.1 General

Now that some intuition has been developed about the effect of the asteroid's orbital eccentricity, we will present the results of our numerical integrations. For comparison purposes, we take the particle to have the same initial conditions used in Chapter 2 (initially in the asteroid's orbital plane on an initially circular orbit) and use an asteroid like Amphitrite (Table 2.1), but with different orbital eccentricities. The addition of orbital eccentricity, however, complicates matters by requiring the specification of two extra items, namely the eccentricity of the orbit and the asteroid's position along its orbit at the time the particle is launched. The second of these complications has lesser significance since we follow the test particle's motion during the time it takes the asteroid to complete five orbits around the Sun ($\approx$ 20 years); thus usually the influence of different starting positions should be minimal. For simplicity, therefore, we choose to start the asteroid at the apocenter of its heliocentric orbit in all of the following integrations. This choice should provide a stringent test of our neglect of the "eccentric" terms in the above discussion since these terms are allowed to act for some time before escape, which generally occurs during the pericenter passage, is possible. Even with this reduction of the problem, a thorough exploration of the three-dimensional phase space (asteroid's eccentricity, particle's inclination, particle's starting distance) would require approximately (10 eccentricities) x (20 inclinations) x (25 starting distances) = 5000 initial conditions. To reduce this to a more manageable number we will take four two-dimensional slices through this phase space, three at constant inclinations representing the three important classes of orbits (prograde, retrograde, and highly-inclined), and one at the measured eccentricity of the asteroid Gaspra (Table 2.1).

3.2.2 Prograde Orbits

Prograde orbits provide the best test of the ideas presented above since, at least in the circular case, particles on such orbits usually escape very quickly whenever their ZVCs are open (see Fig 2.10). We might be tempted, therefore, to predict that escapes will occur when the ZVC evaluated at the asteroid's pericenter is open, but before we can confidently make such a prediction, an additional factor must be considered. Imagine that escape is energetically possible as the asteroid nears pericenter, but the particle is located at a disadvantageous spot for escape to occur, say 90° away from the Sun-asteroid line. Then to provide a fair chance for escape, we must either require that the asteroid remain near pericenter long enough for the particle to complete a reasonable fraction of one orbit around the asteroid or, equivalently, we must integrate through multiple pericenter passages so that many opportunities to escape arise, some of which will find the particle in a favorable position. The prograde orbits with the longest periods are those near the limits of stability; these have synodic periods that are about 1/4 of the asteroid's period if the minor planet is on a circular orbit. For an eccentric asteroid orbit with the same semimajor axis, the stability zone is smaller and the particles orbit even faster. Thus we expect that five pericenter passages of the asteroid about the Sun should usually allow the particle ample opportunity to escape.

Figure 3.2 shows the results of nearly two hundred orbital integrations carried out for initially circular prograde orbits at a variety of distances from asteroids with differing eccentricities. We treat the full range of possible eccentricities; the low-to-moderate values are generally applicable to asteroids, while the larger are more appropriate for comets. The boundary line extends the critical distance found for $i = 0$ orbits in Fig. 2.15 to asteroids with non-zero orbital eccentricity using the scaling result of Section 3.1.2. The division plots as a straight line in the (e, R_A) coordinates used in Fig. 3.2 because the critical distance, like the size of the stability zone, is proportional to the asteroid's pericenter distance $A(1-E)$. For these prograde orbits, the line also selects the initial condition corresponding to the critical pericenter ZVC (ignoring the small eccentricity dependence discussed in Section 3.1.3). Thus only particles with initial conditions above the line have ZVCs that are instantaneously open near pericenter. It is apparent that no orbits below the line escape; note, however, that this trapping is not necessarily required by the argument of closed ZVCs because accelerations that were ignored in developing these ZVCs can cause orbits to cross them. Nevertheless, as we argued above, these accelerations should be small, so the fact that no escapes are seen to occur from below the boundary is encouraging. Furthermore, there is only a single bound orbit that lies significantly above the division. This lone particle was never in the right place to get a boost from M. Coriolis at pericenter; it would almost certainly escape with increased integration time.

The distribution of orbits that strike the asteroid in Fig. 3.2 displays an in-

Figure 3.2 The orbital fate of nearly 200 particles on *prograde* orbits around an asteroid at 2.55 AU. Each particle was given the velocity that would put it on an initially circular path around the minor planet. A solid circle signifies a particle that remains in the asteroid's vicinity for at least twenty years, a small dot corresponds to a grain that escapes into heliocentric space, while an open circle with a dot inside represents a particle that strikes the asteroid's surface. The diagonal line is the predicted division between bound and escape orbits; its derivation is based on scaling the Hill sphere at pericenter as developed in the text.

teresting regularity. All of these crash orbits are found above the division line at which particles become unbound. The lack of crash orbits below the line is consistent with the character of bound prograde and retrograde orbits which are usually very regular in appearance and rarely display chaotic behavior (*cf.* Chauvineau and Mignard 1990a). However, as we will see presently, the separation of bound and crash orbits observed here for prograde orbits is not a result that can be extended to three-dimensional paths.

3.2.3 Inclined Orbits

Bound orbits with inclinations in the range $60^o < i < 120^o$ have many similar characteristics (Section 2.6.1); accordingly we choose $i = 90^o$ orbits as typical examples of this class. The largest of these orbits is comparable to the largest of the prograde orbits, so the maximum period for bound, inclined orbits is also about 1/4 of an asteroid period. By the argument advanced above, five pericenter passages of the asteroid about the Sun should be enough to allow most particles that are destined to escape to be dislodged. But we have found that the opening of the ZVCs is not a good indicator of escape for orbits with $i \gtrsim 30^o$ since the Coriolis acceleration for these orbits does not have the large radially outward component characteristic of that for prograde orbits. We therefore discontinue our use of critical ZVCs as an escape criterion, instead focusing on Hill sphere scaling as described in Section 3.1.2 to connect our results for an asteroid on a circular orbit to those with non-zero eccentricity.

The line in Fig. 3.3 shows the application of this scaling. It does remarkably well, although not nearly as well as in the prograde case. The reason for this is clear. A prograde orbit will almost always escape if the corresponding ZVC is open, and will rarely escape if the ZVC is closed; this idea is reflected in the sharpness of the empirical boundary seen in Fig. 3.2 (recall, however, that for eccentric asteroids the ZVC is just an approximation). Inclined orbits, on the other hand, are not so strictly constrained. Many remain at least temporarily in the asteroid's vicinity even if their ZVCs are wide open; hence the division line between bound and unbound inclined orbits is "fuzzier" than the division in the prograde case. Several bound orbits are located in the region dominated by escape orbits and a few escape orbits are even found below the line in the region where this criterion asserts that orbits should be bound. Notice also that crash orbits are inextricably interwoven with both bound and escape paths. This result is consistent with a similar one for the circular case where many inclined crash orbits are found in the vicinity of the critical distance (Fig. 2.15). The ubiquity of crash orbits under these circumstances is a direct consequence of the dynamics of such orbits discussed in Section 2.5.3 in some detail.

Figure 3.3 Same as Fig. 3.2 for initially circular orbits with inclination $i = 90^{\circ}$. As in Fig. 3.2, the approximate theoretical division separating bound and escape orbits matches the data quite impressively; the decrease of stability with increasing eccentricity is very evident.

3.2.4 Retrograde Orbits

The situation for retrograde paths about elliptically orbiting asteroids is not as good as for the two cases discussed above for several reasons. First, since bound retrograde orbits are relatively large, their periods are about four times the period of the biggest prograde orbits; this implies that integrations of five asteroid years may not be sufficiently long to explore the full dynamical range. In addition, since these orbits are about twice the size of the ones considered previously, the asteroid's gravity is much weaker and the perturbations are significantly larger (see Fig. 2.3). Consequently the unmodeled parts of these forces are more important for retrograde orbits than for either prograde or inclined ones. As an example, the Coriolis acceleration pulls more strongly inward at the asteroid's pericenter for retrograde orbits than simple scaling would suggest and this augments the stability of these orbits around asteroids on eccentric paths. Finally the point at which the ZVCs first open for retrograde orbits is only about 25% of the distance to where escapes first occur assuming an asteroid on a circular orbit. The constraint provided by the retrograde ZVCs, therefore, is almost useless (*cf.* Section 2.5.1 and Chauvineau and Mignard 1990a).

Figure 3.4 shows our results for planar retrograde orbits. The scaling law that worked so well for the prograde and inclined orbits clearly fails here: many bound orbits are found above the line where the theory predicts only escape orbits. The behavior is not even linear; notice the abrupt drop in stability that occurs for an asteroid eccentricity of 0.7. This steep fall-off suggests that longer integrations would lead to additional escapes, at least near this edge. Furthermore, the finger of escape orbits extending into the bound orbits at a distance of about 300 asteroid radii also hints that the bound orbits above the finger will escape given a few more pericenter passages. But increasing the integration time will not solve all of the problems encountered here. We recall the results of Zhang and Innanen (1988) who, after tracking orbits for 1000 years, found that the critical distance for initially circular retrograde orbits around asteroids with eccentricities of 0.0 and 0.07 were 445 and $358R_A$, respectively. The $E = 0$ result agrees with our finding for a 20 year integration; thus, scaling to the pericenter of an $E = 0.07$ orbit (see Fig. 3.4), we would predict a critical distance of $410R_A$, or about 15% larger than the numerical result. Evidently the analysis of these retrograde orbits is hampered by both insufficient integration times and inadequate approximations.

3.2.5 Gaspra

As a final test and an independent verification of the ideas addressed above, and motivated by the destination of a certain spacecraft, we carried out a more thorough investigation of the stability of orbits about an idealization of the asteroid 951 Gaspra. We use values for Amphitrite (Table 2.1) and Gaspra's true eccentricity $E = 0.17$ to facilitate direct comparisons with our previous figures. Because

Figure 3.4 Same as Fig. 3.2 for initially circular *retrograde* orbits. Note the sparsity of orbits that strike the asteroid. For retrograde orbits, the calculated bound-escape division disagrees with the data for reasons that are discussed in the text (compare Figs. 3.2 and 3.3).

our integrations are for an object at 2.55 AU, the results need to be scaled for application to the true Gaspra which orbits at 2.20 AU. Distances measured in R_A, as the ordinates are in the following three figures, must therefore be reduced by the ratio of the semimajor axes of the two asteroids.

Figure 3.5 shows the fate of particles as a function of their starting distance and initial inclination for an asteroid with Gaspra's eccentricity of 0.17 (*cf.* Fig. 2.15 which has $E = 0$). We estimate the critical distance by taking, for each inclination column, the outermost bound orbit such that there are no escape orbits below it; this procedure eliminates freak orbits such as the one at $(i = 70^o, d = 470R_A)$. The results, critical distance as a function of inclination, are plotted in Fig. 3.6 along with similar results for an asteroid with $E = 0$ (from Fig. 2.15). The dotted line in Fig. 3.6 is the expected result for $E = 0.17$ which has been scaled from the $E = 0$ data; comparing the predictions to the actual integrations, we see that prograde and inclined orbits actually escape at distances slightly less than predicted, but well within expected errors arising from the neglected effects. In those regions of Fig. 3.5 where there are many crash orbits, the division between bound and escape orbits is poorly constrained; this leads to a "choppiness" in the critical distance which is observed in the $E = 0.17$ data near $i = 90^o$ in Fig. 3.6. Unlike prograde and inclined orbits, retrograde ones exhibit little loss of stability; once again suspicion falls on insufficient integration times.

To describe the volume in which bound material might be present about asteroids on circular heliocentric orbits, we used the "stability surface" (Fig. 2.16) Note that its typical radius up to latitudes of 35° is nearly constant and is significantly larger than its vertical dimension which is approximately constant for latitudes greater than 35° (*i.e.,* its shape is like a sphere with the poles sliced off). Because polar orbits are less stable than retrograde ones for asteroids on elliptic orbits as well as those on circular paths (Fig. 3.6), we anticipate a similar morphology for the stability surface in the current case. Fig. 3.7 plots the largest out-of-plane distance (z coordinate) from the union of all orbits with a given starting inclination that lie within the critical distance; for comparison, we also plot results for a circular asteroid orbit. We see that the maximum height to which material around Gaspra can rise is only about 75% the value it would have above an asteroid on a circular orbit. The dotted line in Fig. 3.7, the prediction of direct Hill-sphere-scaling of results for $E = 0$, suggests that the value should be 83%. Clearly the correlation between the dotted line and the $E = 0.17$ data is worse in Fig. 3.7 than it is in Fig. 3.6; this difference reflects changes in the orbital evolution of the inclined orbits under accelerations ignored in our analysis.

These numerical experiments indicate that bound debris should not present beyond about $200R_A$ above Gaspra's orbital pole. We remind the reader that our study has dealt only with the question of which orbits are stable and which are unstable. To actually estimate the probability that a spacecraft might strike something would require a knowledge of the population and loss mechanisms

Figure 3.5 The fate of about 650 particles started at different inclinations for an asteroid on an orbit with semimajor axis $A = 2.55\,\mathrm{AU}$ and an eccentricity $E = 0.17$; solid circles, open circles, and small dots correspond to bound orbits, crash orbits, and escape orbits, respectively. Note the prevalence of impacts for orbits with inclinations near 90 deg (*cf.* Fig. 2.15 and nearby text). We can scale this plot for application to Gaspra ($A = 2.20\,\mathrm{AU}$ and $E = 0.17$): since the eccentricities of the two asteroids are identical, and differences in their masses are accounted for by measuring distances in R_A, the ordinate need only be multiplied by the ratio of the two semimajor axes, namely $2.20/2.55 \approx 0.86$.

Figure 3.6 Maximum starting distance for those initially circular orbits that remained bound to the asteroid (for about 20 years) as a function of the orbiting particle's initial inclination. Data are plotted for two values of the asteroid's orbital eccentricity, $E = 0$ and $E = 0.17$; in both cases $A = 2.55$. The dotted line is the prediction for $E = 0.17$ derived from scaling the $E = 0$ result with the Hill sphere at pericenter. In this case the two semimajor axes are identical, so scaling is accomplished by simply multiplying the $E = 0$ results by $1 - 0.17 = 0.83$. The plot clearly shows the erosion of the zone of stability caused by increasing the asteroid's orbital eccentricity.

Figure 3.7 Maximum height above the asteroid's orbital plane attained by the particles from Fig. 3.6; as in that figure, the dotted line is the prediction for the lower set of data obtained by scaling from the upper set. The data displayed here show that as the asteroid's eccentricity is increased, orbits that rise to large heights above the orbital plane disappear faster than our simple scaling would suggest.

for circum-asteroidal orbits. Most discussions of debris sources (Weidenschilling *et al.* 1989, Burns and Hamilton 1991) favor the likelihood that circum-asteroidal debris, if any exists at all, will be produced much closer to the minor planet than the distant orbits considered here. Thus our criterion is likely to be quite conservative; that is, a spacecraft should be able to safely pass much closer to the asteroid than the $200R_A$ quoted above. In the next chapter, we investigate the effects of another perturbing acceleration that clears the circum-asteroidal environment – solar radiation pressure.

r 4

R i i P r r i s Dis r i s[1]

4.1 Intro uction

In Chapters 2 and 3 we discussed distant circum-asteroidal orbits that are strongly perturbed by the solar tidal force. Because the direct gravitational acceleration toward the asteroid is so weak in an absolute sense, radiative processes impart non-trivial perturbations for particles smaller than a few centimeters across. The spatial distribution of millimeter and centimeter-sized objects around an asteroid is of considerable practical interest since impacts with such objects are lethal to a swiftly-passing spacecraft. Accordingly, in this chapter we focus on the orbital dynamics of radiatively perturbed particles and put limits on the extent of circum-asteroidal debris in this size range.

The perturbations we consider arise from the absorption and subsequent re-emission of solar photons and corpuscular radiation. Of the many forces (radiation pressure, Poynting-Robertson drag, Yarkovsky effect, etc. - see the review by Burns *et al.* 1979) that arise from this process, radiation pressure is by far the strongest. Radiation pressure arises primarily from the absorption of the momentum of solar photons and consequently is directed radially outward from the Sun. The force's strength is proportional to the solar flux density which has the same inverse square radial dependence as the Sun's gravity; hence radiation pressure is usually written as a dimensionless quantity β times solar gravity. For spherical particles that obey geometrical optics,

$$\beta = 5.7 \times 10^{-5} \frac{Q_{pr}}{\rho_g r_g}, \tag{4.1}$$

[1]This chapter is based on the paper: Hamilton, D.P., and J.A. Burns (1992), Orbital stability zones about asteroids II. The destabilizing effects of eccentric orbits and of solar radiation, *Icarus* **96**, 43–64 [copyright 1992 by Academic Press, Inc.]

where r_g and ρ_g are the particle's radius and density in cgs units and Q_{pr} is a constant whose value depends on the optical properties of the grain (Burns *et al.* 1979). This result applies to particles larger than about a half-micron, the wavelength of a photon at the peak of the solar spectrum. When a particle's characteristic size is similar to the wavelength of incident light, Mie scattering occurs, Q_{pr} is no longer constant, and β becomes a complex function of particle size. In contradiction to Eq. (4.1), which predicts that the strength of radiation pressure will increase for smaller particles, it actually decreases (Burns *et al.* 1979) because most solar photons are in the visible and such photons interact only weakly with very small grains. In the rest of this work, we will confine ourselves to large grains that obey Eq. (4.1).

4.2 Heliocentric vs. ircu planetary Or its

The acceleration of an isolated particle on a heliocentric orbit is determined by the sum of the inward force of solar gravity and the outward force of radiation pressure, which can be combined into a single $1/r^2$ force with magnitude $(1 - \beta)$ times solar gravity. The grain's orbital dynamics is then identical to the gravitational two-body problem with a reduced solar mass; if a particle's size, and hence its β, is constant, its orbit will be a conic section. Only if the particle's β changes abruptly, as when a small grain is ejected from a comet, or gradually as in the case of a subliming grain, will its orbital evolution be non-trivial (Burns *et al.* 1979). Radiation pressure, therefore, does not significantly alter the nature of most heliocentric orbits and, accordingly, it has received scant attention in the literature.

The situation is quite different for particles that orbit a planet rather than the Sun (Milani *et al.* 1987); since the planet itself is essentially uninfluenced by radiation pressure while small objects orbiting it may be, the problem cannot be treated by simply reducing the mass of the Sun as in the case of heliocentric orbits. Furthermore, the dominant forces are different in each problem; in the case at hand, the important forces are the planet's gravity and the solar tidal force rather than direct solar gravity as in the heliocentric problem. In many situations, therefore, radiation pressure produces stronger effects on circumplanetary orbits than on solar orbits; we will show the truth of this statement when the "planet" is actually a large asteroid with a radius of 100 km.

Since radiation pressure typically induces much smaller accelerations than the asteroid's gravity, an orbit-averaged perturbation technique is often appropriate. This analysis, leading to a simplified set of differential equations describing the evolution of the osculating orbital elements due to an external force which is constant in magnitude and direction, has been carried out by Burns *et al.* (1979) and Chamberlain (1979), among others. The semimajor axis of a circumplanetary orbit is found to be unchanged by radiation pressure. Burns *et al.* solved the planar system ($i = 0$) considering small eccentricity and weak radiation pressure, assumptions applicable to most situations arising in the solar system. Their solution was later extended to arbitrary eccentricities and moderate radiation pressure by Mignard (1982). Both Burns *et al.* and Mignard find periodic oscillations in the orbital eccentricity that, for weak radiation pressure, vary with the planet's orbital period. The solution to the full system with arbitrary inclination, as derived by Mignard and Hénon (1984), involves complicated coordinate transformations that render the study of an orbit with initial conditions expressed in orbital elements impractical. The planar solution shows, however, that if radiation pressure is sufficiently strong, it can induce eccentricities large enough that particles are forced to crash into the asteroid (*cf.* Peale 1966, Allan and Cook 1967). This mechanism, which provides the potential to efficiently remove tightly bound material from circum-asteroidal orbits, will be discussed further in the sections to follow.

4.3 Zero-Velocity urves

As we noted in Section 3.1.3 above, the existence of the Jacobi integral and its associated zero-velocity curves proves to be useful in addressing the eventual fate of loosely bound, prograde orbits. Accordingly, in this section we explore zero-velocity curves derived with the inclusion of solar radiation pressure; as a first approach to the problem and to avoid the difficulties encountered in Section 3.1.3, we treat only the case of circular asteroid orbits. For circular orbits, we will find that exact results exist; extending the results to eccentrically orbiting asteroids, however, entails the same approximations discussed in Section 3.1.3.

The existence of a Jacobi integral for the restricted three-body problem with radiation pressure is anticipated since radiation pressure in the rotating frame can be derived from a time-independent potential. Indeed, the addition of radiation pressure to solar gravity does not greatly complicate the problem since these forces are identical in both direction and radial dependence. In fact, the derivation of the Jacobi integral and the zero-velocity curves in the photogravitational, restricted, circular three-body problem proceeds along almost identical lines as the "classical" derivation (Schuerman 1980). Extensive analysis of the stability of the resulting equilibrium points has been carried out by Luk'yanov (1984,1986,1988). We now apply these ideas to Hill's problem, which, like the restricted problem, has an integral of the motion.

Incorporating radiation pressure into the equation of motion (Eq. 2.2), we obtain the following:

$$\frac{d^2\mathbf{r}}{dt^2} = -\frac{GM_A}{r^2}\hat{\mathbf{r}} + \frac{GM_\odot}{A^3}[3\mathbf{x} - \mathbf{z}] - 2\mathbf{\Omega} \times \mathbf{v_{rot}} + \beta\frac{GM_\odot}{A^2}\hat{\mathbf{x}}, \tag{4.2}$$

where we have taken incoming solar rays to be parallel, an assumption that is valid in the vicinity of our asteroid. Assuming β is time-independent, the final acceleration on the right-hand side of Eq. (4.2) can be integrated to give the potential $\beta(GM_{\odot}/A^2)x$. Taking the scalar product of Eq. (4.2) with $\mathbf{v}_{rot}$, integrating over time and non-dimensionalizing ($G = 1, \Omega = 1$, and $r_H = 1$), we find:

$$C = \frac{6}{r} + 3x^2 - z^2 + 2\beta \left(\frac{3}{\mu}\right)^{\frac{1}{3}} x - v_{rot}^2. \tag{4.3}$$

Equation (4.3) depends on the parameter $\beta\mu^{-1/3}$ and so, as in the case of non-zero orbital eccentricity, care must be exercised when scaling from one asteroid to another. In particular, results scale as the Hill sphere only if the parameter $\beta\mu^{-1/3}$ is kept constant. This can be shown more explicitly by examining Eq. (4.2) in the same manner that we studied Eq. (3.3) in Section 3.1.2. If Q_{pr} and the particle and asteroid mass densities are constant, then β is inversely proportional to the particle's radius r_g (Eq. 4.1) and $\mu^{-1/3}$ is inversely proportional to the asteroid's radius R_A. Thus, simply stated, results from a small asteroid can be applied to a larger one if the product of the asteroid's radius and the radius of the orbiting particle is kept constant (*i.e.*, $\beta\mu^{1/3} \sim (r_g R_A)^{-1}=$ constant).

We can derive zero-velocity curves from Eq. (4.3) by setting $v_{rot} = 0$ and choosing a particular value of C. For weak radiation pressure, the shape of the resulting zero-velocity curves differs only slightly from the more familiar ZVCs of Hill's problem; for stronger radiation pressure, however, the difference is marked. In an attempt to provide the reader with some insight into the constraints on escape imposed by the ZVCs, we discuss their shape for a moderate value of radiation pressure, namely that appropriate for 1-millimeter particles around Amphitrite. Several ZVCs are drawn in Fig. 4.1; these are simply plots of Eq. (4.3) with $v_{rot} = 0$ and $z = 0$ for different values of the Jacobi constant C. The small circles that closely surround the asteroid have large Jacobi constants; their shape is primarily determined by the asteroid's gravity (*cf.* discussion by Chauvineau and Mignard 1990a for ZVCs without radiation pressure). As C is decreased, the circles grow larger and begin to distort due to the tidal and radiation-induced accelerations. Because they both are directed along the x-axis, these perturbation accelerations cause a distortion of the ZVCs along that axis. The tidal potential is an even function of x and thus causes an elongation symmetric about $x = 0$ (see Figs. 2.5 and 2.10). In contrast, radiation pressure, because it always acts in the $\hat{\mathbf{x}}$ direction, causes a non-symmetric distortion, shifting the ZVCs away from the Sun. We see that radiation pressure is dominant for 1-mm particles since the outer curves of Fig. 4.1 are highly asymmetric. One consequence of this asymmetry is that as the Jacobi constant is decreased, the curves open away from the Sun before they open toward it. Radiation pressure allows sufficiently energetic particles to escape in the anti-sunward direction; escape in the sunward direction, which requires still more "energy," occurs more rarely.

Figure 4.1 Zero-velocity curves, including solar radiation, for a 1-mm particle around Amphitrite. The Sun is located far out along the negative x-axis and the asteroid is the solid circle (not drawn to scale) at (0,0). Associated with each curve is a unique value of the Jacobi constant; larger curves have smaller Jacobi constants. The four-pointed star, located at (370,0), denotes the equilibrium point where all forces balance for 1-mm particles; a second equilibrium point lies between the asteroid and the Sun at (-579,0).

When discussing Fig. 4.1, we carefully avoided quoting any actual numbers for the Jacobi constants or the location of the point at which the ZVCs open (see, however, the figure caption). This was done to keep the discussion general and therefore applicable to a large range of radiation pressure strengths. In reality, the Jacobi constant and the points where the ZVCs open are all functions of the relative strength of radiation pressure. To solve for the opening positions, which occur at the equilibrium points of Eq. (4.2), we set the partial derivatives of Eq. (4.3) (with $v_{rot} = 0$) equal to zero (*cf.* Danby 1988, p. 260). Thus, defining

$$\gamma = \frac{\beta}{3}\;\frac{3}{\mu}^{\frac{1}{3}}, \tag{4.4}$$

we find two solutions which lie on the x-axis ($y = z = 0$) at positions given by solutions to the cubic:

$$x^3 + \gamma x^2 \mp 1 = 0, \tag{4.5}$$

where the upper sign refers to the critical point furthest from the Sun and the lower sign to the one closest to the Sun. Solving Eq. (4.5) for $\gamma \lesssim 1$ (weak-to-moderate radiation pressure), we obtain $x_{crit} \approx \pm(1 \mp \gamma/3 + \gamma^2/9)$ and $C_{crit} \approx 9 \pm 6\gamma - \gamma^2$. We find that there are indeed two critical ZVCs, since the two opening points occur at different values of the Jacobi constant. Thus if more curves with ever-decreasing Jacobi constants were plotted in Fig. 4.1, we would eventually see a tunnel from the asteroid to heliocentric space opening up on the left side of the figure. For $\gamma >> 1$ and $x_{crit} > 0$, we find $x_{crit} \approx \gamma^{-1/2}$, which tends toward zero, and $C_{crit} \approx 12\gamma^{1/2}$.

4.4 Integrations

Our philosophy in adding the effects of eccentricity and radiation pressure to the escape problem is to separate the two so that a more direct comparison with the results of Chapter 2 is possible. Accordingly, in all subsequent numerical integrations, we place the asteroid on a circular orbit around the Sun. As before, we model the asteroid 29 Amphitrite with the parameters given in Table 2.1. We start particles out along the x-axis away from the Sun with a speed such that the orbit would be circular in the absence of all perturbations. As in Chapter 3, we allow the velocity vector to take on one of three inclinations relative to the orbital plane: prograde ($i = 0^o$), retrograde ($i = 180^o$), or inclined ($i = 90^o$). These inclinations are representative of the three basic classes of circum-asteroidal orbits in the case when radiation pressure is absent. The period of integration was set at five asteroid years (≈ 20 years) to facilitate comparison with our previous results.

Although these are the same initial conditions used in Chapters 2 and 3, they are particularly appropriate here for two reasons. First, the radiation and tidal potentials are maximum in the anti-sunward direction; thus circular orbits starting on the positive x-axis have larger Jacobi constants than circular orbits of the same radius starting elsewhere. Our results for circular orbits, therefore, are conservative in the sense that at each distance, we study the initially circular orbit that, energetically, has the least chance of escaping. The second reason that our initial conditions are reasonable is more physical. One of the most dangerous potential sources for material in the circum-asteroidal environment is a "feeder" satellite, a small body from which material can be efficiently removed by meteoroid bombardment (Burns and Hamilton 1991). In contrast to direct impacts on the central asteroid in which material generally escapes or is re-accreted, much of the debris blasted from a moonlet can end up in orbit around the asteroid. We envision the following scenario: a "feeder" satellite uninfluenced by radiation pressure is continually subjected to a flux of hypervelocity particles which blasts debris from its surface. Although sufficiently energetic to escape the weak gravity of the satellite, much of the debris cannot escape the asteroid. As the clumps of ejected material separate exposing small bodies to solar rays, radiation pressure begins to exert its influence, preferentially eliminating the smaller particles. Our integrations begin at the point when mutual gravitational and shadowing effects can be neglected; further evolution of the debris in the aftermath of an impact event is governed by Eq. (4.2).

Confining ourselves to an orbit of a given starting inclination, we still must fix the initial size of the particle's circular orbit as well as the strength of the radiation pressure as parameterized by γ; thus we have a two-parameter space to explore. In order to avoid confusion, we continue to display plots for Amphitrite with distances measured in asteroid radii and particle sizes measured in millimeters; to apply these plots to Gaspra (with $E = 0$) we simply multiply the vertical axis by the ratio of the semimajor axes $2.20/2.55 \approx 0.86$ (Table 2.1) and change "millimeters" to "centimeters." The change in the vertical axis comes from scaling distances with the size of the Hill sphere (Section 2.3.3) while that of the horizontal axis arises from the condition that the parameter γ, defined in Eq. (4.4), be unaltered; keeping γ constant is equivalent to requiring that the product of the asteroid and particle radii be constant as was discussed immediately following Eq. (4.3). Most of the equations to follow, however, depend on the dimensionless quantities r (measured in Hill radii) and γ; use of these quantities both simplifies the appearance of the equations and facilitates scaling to other asteroids. The size of the Hill sphere for Amphitrite and Gaspra in asteroid radii is given in Table 2.1; below we make the connection between γ and the particle's size more apparent. Assuming spherical particles with the same density as that assumed for the asteroid ($\rho_g = 2.38\,\mathrm{g/cm^3}$) and a radiation pressure coefficient of unity ($Q_{pr} = 1$), we find, using Eqs. (4.1) and (4.4) that γ is inversely proportional to the particle's radius; for Amphitrite

$$\gamma_A = 0.0673/r_g, \tag{4.6}$$

while for Gaspra

$$\gamma_G = 0.673/r_g, \tag{4.7}$$

where r_g is the particle's radius in centimeters.

4.4.1 Prograde Orbits

Figure 4.2 shows the fate of several hundred prograde paths followed for five orbits of Amphitrite around the Sun. The picture is remarkably regular; orbits that share a common fate cluster together in one of three distinct regions with few exceptions. The relative strength of radiation pressure increases from right to left as the particle's size is decreased: this causes the rapid disappearance of bound orbits. For 10-mm particles, the division between bound and escape orbits is in agreement with that found analytically (Section 2.4) and numerically (Fig. 2.15) in the absence of radiation pressure: initially circular prograde orbits are stable out to about $220R_A$, or about one-half the radius of the Hill sphere. In this region, radiation pressure is strong enough to perturb orbits, but does not have the power to alter the orbital fates of many particles. As particle sizes are decreased, increased radiation pressure is seen to cause only a few extra escapes at large distances from the asteroid until we consider particles with radii of a millimeter. In the 1-mm column of Fig. 4.2 an amazing transition takes place; bound orbits suddenly extend only half as far from the asteroid as they did for particles twice as large, their demise being due to the appearance of a large number of orbits doomed to strike the asteroid. For these particles, radiation pressure is large enough to induce major oscillations in orbital eccentricity, excursions so large that $e \rightarrow 1$ and a collision with the minor planet is likely. Even more startling is the disappearance of bound orbits in the next column to the left; all orbits beyond $20R_A$, with one exception, either impact the asteroid or escape from its gravitational grasp. Particles in this column have radii ≈ 0.5 millimeters; around Gaspra this corresponds to particles nearly a centimeter across! Decreasing particle sizes still further yields no surprises; bound orbits do not reappear, and the increasing radiation pressure causes escapes to occur ever closer to the asteroid. Recall that all of the points plotted in Fig. 4.2 correspond to the fates of particles followed for just over twenty years; for this problem, radiation pressure accomplishes much in extraordinarily short times!

Probably the most interesting portion in Fig. 4.2 is the transition region where orbits first begin to impact the asteroid. Examining the orbits of the eleven 1-mm grains that crash, we find all but three of them, the one closest to the asteroid and the two furthest from it, impact in about a third of an asteroid year. Orbital

Figure 4.2 The fate of approximately 200 particles of different radii started about Amphitrite on *prograde* circular orbits of various sizes that evolve under the influence of solar radiation pressure; solid circles, open circles, and small dots correspond to bound orbits, crash orbits, and escape orbits, respectively. The columns of initial conditions are evenly spaced along the horizontal axis. Orbits with the same fate tend to cluster, dividing the plot into three distinct regions. Note the rapid disappearance of bound orbits as the particle sizes are reduced to 1 mm and then to ≈0.5 mm. This, of course, is due to the increasing strength of radiation pressure relative to the asteroid's gravity. For Gaspra, corresponding particle sizes would be ten times larger.

eccentricities rise monotonically to a critical value near unity at which point the pericenter of the orbit dips below the surface of the asteroid and impact occurs. The three exceptions, however, show that this is not the full story. Two of these orbits survive one stint of large eccentricity after which the orbit circularizes and the process begins anew. These orbits crash when the eccentricity rises to values near one a second time. The third orbit, which is the furthest from the asteroid, survives no less than eight successive periods of large eccentricity before finally striking the asteroid during its ninth cycle.

Several effects can cause these deviations from the simple sinusoidal oscillations of eccentricity predicted by Mignard (1982). Since the orbits under discussion are large, the tidal force from the Sun is significant and cannot be ignored as it is in the idealized case. This force will also influence the orbital eccentricity and may either augment or detract from radiation-induced changes. Furthermore, even in the absence of the tidal force, orbits of this size have long periods for which the orbital averaging employed by Burns *et al.* (1979) and Mignard (1982) is generally inappropriate. This will be the case any time the particle's orbital elements change significantly during a single circuit around the asteroid. One important consequence of rapidly varying elements is that if a particle attains an eccentricity of one at some point far from the asteroid, the eccentricity may decrease below the critical value necessary for collision before the particle suffers a close approach. This, in fact, is the reason that the three orbits just discussed survive several close approaches. A final consideration that does not affect our integrations, but would alter orbits around a real asteroid, is the non-spherical shape of typical minor planets. Higher-order gravity terms can significantly alter the evolution of even a large orbit if, as in the case under discussion, the eccentricity of the orbit is near unity so that close approaches occur.

The fish shape plotted in rotating coordinates in Fig. 4.3 is the amazing orbit discussed above that narrowly avoids collision eight times only to impact the asteroid on the ninth pass. The heavy black line is the zero-velocity curve appropriate for the initial condition, a 1-mm particle starting on a circular unperturbed orbit around the asteroid Amphitrite at $190R_A$. Although the ZVC is open, the particle never had the chance to taste the freedom of heliocentric space. At first sight this is strange, since the orbit extends nearly to the Lagrange point where forces on a stationary particle balance; prograde orbits that reach this far invariably escape since the Coriolis acceleration is outwardly directed. Retrograde particles, however, are stabilized by the Coriolis acceleration and can safely wander in this region; closer inspection of Fig. 4.3 reveals that although the orbit begins prograde, it becomes retrograde when farthest from the Sun, at the very fringes of heliocentric space. In fact, the orbit switches from prograde to retrograde and back again periodically, as can be seen from the time history of the inclination displayed in Fig. 4.4. These transitions necessarily take place at $e = 1$, when the particle's velocity vector points either directly toward or away

Figure 4.3 A 1-mm particle on an initially circular *prograde* orbit started at $190R_A$ about Amphitrite. The initial position is marked with a solid triangle whose upper apex points in the direction of the initial velocity; a filled square marks the end of the integration, and a solid circle represents the asteroid itself. In this case, the square and the circle overlap since the grain ends its orbital evolution on the asteroid's surface. The four-pointed star is the equilibrium point, and the heavy curve partially enclosing the orbit is the zero-velocity curve appropriate for this initial condition; its asymmetry is due to radiation pressure. Although the ZVC shows that the particle is energetically able to escape, the grain suffers a more drastic fate.

Figure 4.4 The time histories of some of the osculating orbital elements for the path displayed in Fig. 4.3. Plotted are the orbit's semimajor axis, its eccentricity, and its inclination. These curves are calculated by integrating the equation of motion, and transforming the resulting velocity $\mathbf{v}$ and position $\mathbf{r}$ into orbital elements (Danby 1988). Note that the particle switches from prograde ($i = 0^{\circ}$) to retrograde ($i = 180^{\circ}$) and back again periodically each time the eccentricity reaches unity.

from the asteroid; thus the very fact that the orbit survives so long warns us of the dangers of taking the orbit-averaged equations too seriously.

The history of the osculating elements in Fig. 4.4 is also enlightening, especially when discussed along with the evolution of the actual orbit. After a single prograde loop, the orbit switches to retrograde as the eccentricity approaches one; this first occurs very near the upper part of the zero-velocity surface in Fig. 4.3. A change in inclinations from $i = 0^{\circ}$ to $i = 180^{\circ}$ or vice-versa can, but need not, involve closely approaching the ZVC; this only occurs if the particle is at the apocenter of a rectilinear ellipse ($e = 1$). Indeed, Fig. 4.3 has examples of transitions at varying distances from the ZVC. The particle then dives in for a close approach to the asteroid which occurs at the small dip in the center of the eccentricity peak. The small reduction in eccentricity, which manifests itself in less than an orbital period, is enough to allow the particle to successfully negotiate the treacherous region. The particle subsequently moves outward toward the lower part of the ZVC, finally returning to its prograde state to repeat the cycle anew. The entire cycle, in which the eccentricity changes from zero to unity and back to zero, takes only four orbits of the particle around the asteroid; clearly an orbit-averaging technique is invalid here! The inadequacy of orbit-averaging can also be seen in the semimajor axis history of Fig. 4.4. Orbit-averaging of both radiation pressure and the tidal acceleration lead to predictions that the semimajor axis, a, will remain constant on timescales larger than the particle's orbital period; these predictions rely on the fact that the orbital elements, a included, do not change much during a single orbit. Large variations in the semimajor axis should, therefore, not occur on any timescale; the extent to which this is untrue is a measure of the validity of the averaging approximation.

4.4.2 etrograde Orbits

Figure 4.5 is the retrograde counterpart to Fig. 4.2. Qualitatively the two plots are very similar since radiation pressure acts analogously on prograde and retrograde orbits as will be seen below; differences in the plots can be explained by the effects of the Coriolis acceleration. As in Fig. 4.2, orbits in Fig. 4.5 are segregated into three distinct regions containing bound, escape, and crash orbits. For weak radiation pressure, such as that acting on 10-mm particles, circular orbits are stable out to about the Hill sphere in accordance with the results of Chapter 2. The Coriolis acceleration exerts a powerful influence on these orbits, keeping them bound at twice the distance of the largest prograde orbits. Retrograde orbits, like their prograde counterparts, experience a slight degradation of stability as particle sizes are decreased; but, as with prograde orbits, an abrupt transition occurs for 1-mm particles: half of the bound orbits are replaced by those that crash! The rapid erosion of stability is continued for grains ≈ 0.5 mm in size for which bound orbits disappear entirely; comparing Figs. 4.2 and 4.5, we see that

Figure 4.5 The fate of about 300 particles of different radii started about Amphitrite on *retrograde* circular orbits of various initial sizes; solid circles, open circles, and small dots correspond to bound orbits, crash orbits, and escape orbits, respectively. The columns of As in Fig. 4.2, orbits sharing a common fate cluster into three distinct regions; bound orbits rapidly disappear as particle sizes are decreased to 1 mm and then to ≈ 0.5 mm. For particles smaller than 0.1 mm, the differences between Fig. 4.2 and this plot are slight.

the disappearance of bound orbits in each case occurs for particles of the same size. As radiation pressure is increased, crash orbits continue to yield to escape orbits. Comparing with Fig. 4.2 again, we find a few extra crash orbits in the retrograde case and these rapidly disappear as the strength of radiation pressure increases; the extra impact orbits can also be attributed to Coriolis effects.

Figure 4.6 shows the most distant bound orbit in the 1-mm column of Fig. 4.5; its initial conditions are appropriate for a grain started at $180R_A$ from Amphitrite. Although we show only the first eccentricity cycle, which occurs over about an asteroid year, this orbit was in fact followed for five circuits of the asteroid around the Sun. The eccentricity behavior is similar to that of the prograde orbits; it increases to a value near one, remains flat as the "ellipses" in Fig. 4.6 move slowly clockwise, then decreases back to zero as the particle returns roughly to its initial position. Because the orbit is almost periodic, subsequent evolution repeats that described above although the "ellipses" do not fall exactly atop those already present. In its five-year tour, the particle survives multiple close approaches, the closest a mere $1.9R_A$ above the asteroid's surface! All impact orbits for 1-mm particles in Fig. 4.5 have the same sunwardly directed petals and general characteristics as the orbit in Fig. 4.6; in the former cases, however, the close approaches dip below $1R_A$ abruptly cutting short the orbital evolution! As with the prograde orbits discussed above, most of these retrograde orbits impact midway through their first eccentricity oscillation, although three of the five furthest survive at least one cycle for reasons similar to those discussed in Section 4.4.1. Moving closer to the asteroid along the 1-millimeter column, we find that bound orbits have progressively more distant close approaches (corresponding to smaller eccentricities), although again the orbital shapes are reminiscent of Fig. 4.6. Finally, we note that all bound orbits, Fig. 4.6 included, are purely retrograde; further from the asteroid, however, we do encounter orbits that switch between the prograde and retrograde states. These outer orbits have short lifetimes since they invariably crash while traversing the often fatal $e = 1$ regime.

4.4.3 Inclined Orbits

The situation for inclined orbits (here the term inclined will refer to orbits with $i = 90^o$) is somewhat different than for planar ones. In the orbit-averaged equations of Burns *et al.* (1979) and Chamberlain (1979) there is a $\cos i$ term that is small for inclinations near 90^o but equal to ± 1 for planar orbits. The change in this term reflects simple differences in the orbital geometry which we will illustrate with discussion of a hypothetical circular orbit around the asteroid. Imagine that a grain is started on a circular orbit fairly close to the asteroid such that its period is much less than that of the asteroid around the Sun. If the grain is placed on either a prograde or a retrograde orbit, the angle between the Sun and the particle as measured from the asteroid will circulate between

Figure 4.6 A 1-mm particle on an initially circular *retrograde* path starting at $180R_A$; symbols are those defined in Fig. 4.3's caption. Although the initial conditions for Figs. 4.3 and this figure are quite similar, the orbital paths have a very different appearance. The zero-velocity curve for this initial condition is open even wider than the one in Fig. 4.3; the fact that the particle does not escape is an example of the poor constraint imposed by retrograde ZVCs. Only the first several loops of this orbit are shown, but subsequent motion repeats the pattern shown here.

0° and 360° every synodic period; recall that the synodic period is the period of the particle with respect to the Sun. For a path inclined 90° to the asteroid's orbital plane, however, the situation is quite different. If the particle is started on the positive x-axis, then after one quarter of an asteroid orbit, the direction to the Sun is everywhere perpendicular to our hypothetical unperturbed circular orbit; at this point, the angle which circulates for the planar cases is constant! Clearly radiation pressure will act differently on inclined orbits than on planar ones. Considerations of the averaged equations of motion and the fact that a perpendicular perturbing force does not affect the orbital eccentricity (Danby 1988) lead us to the conclusion that driving orbital eccentricities to large values will be more difficult in the inclined case.

Figure 4.7 verifies these ideas; bound orbits exist for particles approximately five times smaller than that where the last bound planar orbits are seen. These bound orbits in the transition region disappear even more abruptly than in the planar case; in the column for $\approx$ 0.2-mm particles, stable orbits abound and there are no crash orbits while in the next column to the left there are no bound ones! Impact orbits sprinkled throughout the region of weak radiation pressure are probably not associated with that force at all; recall the large number of such orbits for inclinations in the near 90° range for our integrations of the purely gravitational three-body problem (Figs. 2.15 and 3.3). Discounting these exceptions, the bound, escape and crash orbits separate nicely into three regions as before. In Fig. 4.7, as in the planar figures, the right side of the plot smoothly approaches results found in Fig. 2.15 in the absence of radiation pressure. For very small particles that are significantly influenced by radiation pressure, results are in accordance with the planar cases; there are a few more impact orbits than in the analogous columns in Fig. 4.2 and a few less than in Fig. 4.9 as could be predicted by considering the Coriolis acceleration. In this region of all three figures, orbits crash extremely rapidly; few survive more than the time necessary to increase the eccentricity to one.

4.5 nalytic onsi erations

4.5.1 Bound-Escape Division

For each of the orbital classes (prograde, retrograde, inclined) described above, we have found that – to a greater or lesser degree – particles with similar characteristics (particle radius, initial orbit size) share similar fates, and that the boundaries between these fates are sharply defined. This suggests that the outcomes for such particles are being determined by simple processes; hence we now seek the mechanisms that segregate orbits into the three separate regions noted above. In this section and the ones to follow, we discuss the factors that cause a particle to escape and to crash, and we develop analytical expressions that define

Figure 4.7 The fate of approximately 200 particles of different sizes started on circular paths initially *inclined at* 90°; solid circles, open circles, and small dots correspond to bound orbits, crash orbits, and escape orbits, respectively. The columns of These orbits jealously guard their stability until particle sizes are reduced to 0.1 mm; reasons for this are discussed in the text. Crash orbits in the upper right of the diagram are of the type seen in Fig. 3.5 and are caused by the tidal force; those to the lower left, however, are due to radiation pressure. Comparing this figure to Figs. 4.2 and 4.5, we see few differences for particles smaller than 0.1 mm.

the divisions separating these areas from each other and from the region of bound orbits.

We know – by analogy with the purely gravitational case – that, if orbits lie within closed ZVCs, they will remain bound. Hence, as a criterion for escape, the opening of the zero-velocity curves will prove to be useful, at least in the prograde case. To connect ZVCs to the orbits discussed above, we substitute the initial conditions, $y = z = 0$ and the initial circular velocity condition,

$$v_{rot}^2 = \left[\frac{3}{d_{BE}}^{1/2} \cos i - d_{BE} \right]^2 - \frac{3}{d_{BE}} \sin^2 i, \tag{4.8}$$

into Eq. (4.3) to obtain:

$$C_{BE} = \frac{3}{d_{BE}} + 2(3d_{BE})^{1/2} \cos i + 2d_{BE}^2 + 6\gamma d_{BE}, \tag{4.9}$$

where C_{BE} is the Jacobi constant for which the zero-velocity curves first open and d_{BE} is the critical distance at which we expect the bound-escape division to occur. We solve this equation numerically in each of the three inclination cases and plot part of the solution curve in Figs. 4.8, 4.9, and 4.10 (dashed line). If extended to smaller particles sizes, the curve would also separate the crash orbits that had the potential to escape from those that did not. Although the theoretical results in all three inclination cases correctly predict that orbital stability is lost as particle sizes are decreased, the curves only succeed in fitting the numerical results for prograde orbits; the match steadily worsens as the inclination is increased. The reason for this is, of course, that the derivation of Eq. (4.9) ignores the all-important Coriolis acceleration. Not surprisingly, the radial portions of the neglected Coriolis term, which has a $\cos i$ dependence, provides increasing stability as the inclination is raised from $i = 0°$ to $i = 180°$. The situation is complicated by non-radial parts of the Coriolis acceleration, with a $\sin i$ dependence, which tend to destabilize orbits. The two effects combine to explain why the division between bound and escape orbits, as numerically obtained, occurs at a similar distance in the prograde and $i = 90°$ cases but much further out for retrograde orbits (Section 2.6.1).

4.5.2 Bound-Crash Division

Particles risk collision with the asteroid once their orbital eccentricities become so large that at pericenter their orbits pierce the asteroid's surface: $r_p = a(1 - e) < R_A$. If we neglect the tidal acceleration – an approximation that is certainly valid for strong radiation pressure – we can apply Mignard's expression for the eccentricity produced by radiation pressure (1982, his Eq. 28) to determine when an impact can occur. More precisely, the tidal acceleration can be ignored in determining when escapes will occur for orbits with initial semimajor axes $\lesssim r_H/3$ since

Figure 4.8 *Prograde* orbits. Same as Fig. 4.2 but now including theoretical lines dividing bound, escape, and crash orbits. The dashed line, discussed in Section 4.5.1, presents a criterion that should separate particles that are bound from those that escape. Similarly, the heavy and lightweight solid curves are those that our theory predicts for the bound-crash (Section 4.5.2) and crash-escape (Section 4.5.3) divisions, respectively.

Figure 4.9 *Retrograde* orbits. Same as Fig. 4.5 with theoretical lines dividing bound, escape, and crash orbits. See Fig. 4.8 and the text for an explanation of the three curves.

Figure 4.10 *Inclined orbits.* Same as Fig. 4.7 with theoretical lines dividing bound, escape, and crash orbits. See Fig. 4.8 and the text for an explanation of the three curves.

tides cause only small eccentricity oscillations in this regime. Furthermore, for orbits much larger than this, the orbit-averaging procedure employed by Mignard (1982) is no longer valid. For initially circular prograde orbits, Mignard's result for the variation of eccentricity can be rewritten in the useful form

$$(1-e^2)^{1/2} = \frac{n_\odot^2}{\alpha^2 + n_\odot^2} + \frac{\alpha^2}{\alpha^2 + n_\odot^2}\cos[(\alpha^2 + n_\odot^2)^{1/2} t], \tag{4.10}$$

where α is related to β via the equations: $\alpha = 3/(2\tau_0)$, and τ_0 is Chamberlain's (1979) expression for the time it takes radiation pressure to produce the circular velocity, *i.e.,* $\tau_0 = (GM_A/r)^{1/2}/(\beta GM_\odot/R^2)$. Loosely, α is the strength of the solar radiation pressure relative to the asteroid's local gravity. For weak radiation pressure ($\alpha << 1$), the eccentricity simply varies with the solar period, while for strong radiation pressure e varies more rapidly. Although mathematically Eq. (4.10) predicts a complex eccentricity when the right-hand side of the equation is less than zero (*i.e.,* when $\alpha > n_\odot$ and $\cos < 0$), this does not actually occur in the orbit-averaged perturbation equations from which Eq. (4.10) is derived because e is prevented from exceeding unity by $1 - e^2$ terms in these equations. What, then, really happens as e approaches one? There are two possibilities: the particle either can collide with the asteroid, preventing further evolution of the orbital elements, or, for longer-lived orbits, a prograde-to-retrograde transition can take place. Because Mignard's solution is restricted to prograde orbits, it is unable to predict the prograde-to-retrograde transition and instead suggests a complex eccentricity.

It is not difficult to repeat Mignard's derivation for retrograde orbits. We begin with the orbit-averaged equations of motion and consider the planar limit $i = 180^\circ$ (instead of the $i = 0^\circ$ taken by Mignard). With an appropriate choice of variables, the form of the resulting pair of equations can be made identical to those for the prograde case; specifically, we find that Eq. (4.10) applies equally well to retrograde orbits. This is a single example of a more general result: if the orbital elements, evolving under some perturbation force, are taken to remain constant over a single sidereal period, then the resulting orbit-averaged equations will yield similar histories for prograde and retrograde orbits. According to this model of the effects of radiation pressure, therefore, there should be no difference in the fate of initially circular prograde and retrograde orbits since Eq. (4.10) governs the evolution of both. This is true, of course, only as long as the particle remains close to the asteroid where the Coriolis acceleration, which encapsulates the differences between prograde and retrograde orbits, can be ignored. Further from the asteroid, differences in the Coriolis acceleration manifest themselves in the increased stability of the retrograde particles noted in the discussion of Figs. 4.2 and 4.5. In these regions (*e.g.,* 1-mm crash orbits), Eq. (4.10) does not strictly apply.

A collision with the asteroid can occur when the pericenter of the osculating orbit dips below the asteroid's surface. For the large orbits under discussion, this requires an eccentricity that is nearly unity. Accordingly, we solve for the minimum α that allows $e = 1$ in Eq. (4.10); although e cannot exceed one, it must necessarily attain this value during a prograde-to-retrograde transition. The collision criterion is $\alpha = n_\odot$, which can be recast as

$$d_{BC} = \frac{4}{27\gamma^2}, \tag{4.11}$$

where d_{BC} is the critical distance at which the division between bound and crash orbits is located. Furthermore, near this division where, by definition, $\alpha \approx n_\odot$, we expect that collisions will occur in half the period given by Eq. (4.10), *i.e.*, $2^{-3/2} \approx 0.35$ asteroid years. This simple estimate is in very good agreement with our observations for most of the prograde orbits discussed in Section 4.4.1. Equation (4.11) is also plotted on each of Figs. 4.8, 4.9, and 4.10 as a solid, heavyweight curve. We find reasonable agreement in the prograde and retrograde figures, but a rather poor match for the inclined orbits. The fact that inclined orbits are more resistant to radiation pressure-induced impacts should not be surprising in light of the discussion in Section 4.4.3. For 1-mm particles around Amphitrite where the limits of the theory are stretched the most, we see that bound retrograde orbits extend further from the asteroid than expected (Fig. 4.9), while bound prograde orbits extend to distances less than predicted (Fig. 4.8). These differences, which are due to the neglected Coriolis acceleration, only appear for large orbits and add stability to retrograde orbits as discussed above.

For ≈ 0.5-mm particles, bound orbits do not extend as far as predicted in both the prograde and retrograde cases. This is due to the finite size of the asteroid which allows impacts to occur for eccentricities less than one. This effect can be derived from Eq. (4.10) by putting $e = e_{crash}$, where $e_{crash} = 1 - R_A/d_{BC}$. Setting the cosine to -1, and solving for $\alpha/n_\odot$ as before, we find:

$$d_{BC} = \frac{4f^2(e_{crash})}{27\gamma^2}, \tag{4.12}$$

where $f(e_{crash})$ is given by

$$f(e_{crash}) = \frac{1 - (1 - e_{crash}^2)^{\frac{1}{2}}}{e_{crash}}; \tag{4.13}$$

and $f^2(1) = 1$ so that Eq. (4.11) is recovered. The solution of Eqs. (4.12) and (4.13) is complicated since e_{crash} is a function of d_{BC}; in general, the equation must be numerically solved. In practice, however, an iterative procedure in which an initial value of d_{BC} is substituted into the right-hand side of Eq. (4.12) to compute an updated value, converges to a reasonable estimate relatively rapidly. As an example, consider the bound-crash division for ≈ 0.5-mm particles which Eq. (4.11) predicts will occur at about thirty asteroid radii. For this distance, a collision takes place when $e = e_{crash} = 29/30 \approx 0.97$ for which $f^2(0.97) \approx 0.59$! Thus instead of occurring at 30 asteroid radii, a single iteration of Eq. (4.12) predicts that the division should happen at about 18 asteroid radii; a few more iterations show that the division is actually nearer to $14R_A$, which is in good agreement with Figs. 4.8 and 4.9. The surprisingly large change in $f^2(e_{crash})$ for $e_{crash} \lesssim 1$ has its origins in the fact that radiation pressure takes a long time to further increase the eccentricity of an already highly-eccentric orbit.

To make our results more useful, we instead solve for the minimum-sized particle found in an asteroid's neighborhood by applying Eq. (4.12), and employing Eqs. (4.1) and (4.4) to return to more familiar dimensional units. We find that particles satisfying the following inequality

$$\frac{r_g}{1\,\mathrm{cm}} \lesssim \frac{1}{120 f(e_{crash})} \left(\frac{r}{1R_A}\right)^{1/2} \left(\frac{M_{Amphitrite}}{M_{Asteroid}}\right)^{1/3} \left(\frac{2.55\,\mathrm{AU}}{a}\right)^{1/2} \left(\frac{Q_{pr}}{1.0}\right) \left(\frac{2.38\,\mathrm{g\,cm^{-3}}}{\rho}\right) \qquad (4.14)$$

are removed from circum-asteroidal orbit. This formula is applicable only for strong radiation pressure where the bound-crash division exists (see Figs. 4.8 through 4.10), roughly where $\gamma \gtrsim 1$. We find for Gaspra that, outside of $10R_A$, no particles with $r_g \lesssim 0.45\,\mathrm{cm}$ should be found and at Galileo's fly-by distance of $\approx 200R_A$, all particles with $r_g \lesssim 1.4\,\mathrm{cm}$ should be absent. Subsequent to the submission of this work (Hamilton and Burns 1992), Grün *et al.* (1992) reported that Galileo's dust instrument, sensitive to particles larger than $0.1\,\mu\mathrm{m}$, detected no hits during its fly-by of Gaspra. There were also no detections during the Ida fly-by (E. Grün 1993, private communication).

4.5.3 Crash-Escape Division

Although the criteria described in the two preceding sections define the most interesting boundaries, namely those that separate regions where particles can freely orbit from regions where they cannot, we now derive, for completeness, an approximate argument to describe the curve separating orbits that crash from those that escape. Unlike the boundaries discussed in the previous sections, here there is no nice theory to appeal to so we make the following somewhat arbitrary choice. We say that if a highly-perturbed particle can complete a single orbit around the asteroid, its eventual fate will be to crash into the asteroid. While this is not always true (some orbits near the actual boundary complete a few loops before escaping), it does apply to most of our numerical results, especially those for strong radiation pressure. We approximate further by saying that if our particle has enough "energy" to complete a quarter of a hypothetical circular orbit, it will complete a full loop around the asteroid and hence will eventually crash. This statement is certainly approximate since the path actually followed by the particle is certainly not circular; a look at the shape of the orbits in Figs. 4.3 and 4.6, however, shows that the approximation is fairly reasonable. In any case, particles with significantly less energy have no hope of swinging around the asteroid, while those with more "energy" should be able to. Mathematically, we set the right-hand side of Eq. (4.3), evaluated at $(x, y, z) = (d_{CE}, 0, 0)$ with v_{rot} as given by Eq. (4.8), equal to the same expression evaluated at $(0, d_{CE}, 0)$ with $v_{rot} = 0$. The result is:

$$\frac{3}{d_{CE}} = 2(3d_{CE})^{1/2} \cos i + 2d_{CE}^2 + 6\gamma d_{CE}, \tag{4.15}$$

where d_{CE} is the distance to the division between crash and escape orbits. We numerically solve Eq. (4.15) to obtain the last defining curve which is plotted in Figs. 4.8, 4.9, and 4.10 as a solid, lightweight curve. This approximate division agrees remarkably well with the actual boundary for strong radiation pressure, deviating significantly only for large orbits along which the neglected tidal and Coriolis accelerations are important.

4.6 iscussion

The above calculations and those of other groups have been carried out not so much to solve new celestial mechanics problems but rather to address a practical question: will the circum-asteroidal environment be hazardous to a fly-by spacecraft? Accordingly, a reader might anticipate that we would conclude this chapter with a probability calculation determining the odds of finding debris of various types in the asteroid's neighborhood. Unfortunately such calculations are fraught with uncertainty since they involve complicated supply and loss mechanisms, many of which are poorly constrained. We therefore content ourselves with a qualitative description of this problem, summarizing possible supply and loss processes.

As pointed out in Chapter 2, by Chauvineau and Mignard (1990a), and by many others, the distance within which co-planar prograde material can remain trapped for short periods about an asteroid circling the Sun is roughly half the Hill radius; for co-planar retrograde particles the size increases to about a full Hill radius. In extending these ideas to three dimensions, we showed in Section 2.6.2 that bound out-of-plane material can only rise to about two-thirds of a Hill radius; we used these results to define a stability surface within which bound orbiting material might be found. This surface overestimates the zone of stability, however, because nearly all unmodeled processes, some of which operate on short timescales and others that take ages, are destabilizing. The former dominate, since they will overwhelm continuous supply mechanisms, which act on longer timescales. Accordingly, the focus of Chapters 2, 3, and this one has been to discuss the effects that cause changes to the stability of orbits in time intervals comparable to the asteroid's orbital period (*cf.* Burns and Hamilton 1991).

In assessing the importance of an asteroid's elliptical orbit on the size of the stability zone, we discovered that the dimensions of the zone are roughly proportional to the minimum asteroid-Sun distance. Since the effects of an elliptic orbit can be quantified, the safety of a passing spacecraft can be assured simply by avoiding an asteroid's calculated stability zone. We also found that radiation pressure was remarkably effective in sweeping small particles rapidly out of the circum-asteroidal environment. These grains would normally be expected to be the most numerous and, since the largest of them can severely damage a spacecraft, they pose the greatest threat to a fly-by mission. Since small grains are removed much more rapidly than they are resupplied, however, our results define a region of space in which small orbiting debris will not be found.

Many loss mechanisms operate over much longer timescales. In this category we include the long-term effect of the gravitational tugs of Jupiter and the other planets (Chauvineau and Mignard 1990b) as well as close approaches of other asteroids which can disrupt a binary pair (Chauvineau *et al.* 1991). These effects cause particles within the stability zone defined above to escape, but their efficiency is critically dependent on the unknown rate at which supply mechanisms populate the stability zone. Other long-term loss processes – notably Poynting-Robertson drag, catastrophic fragmentation and sputtering – act most effectively on small grains. These grains are more efficiently removed by radiation pressure; collisions, for example, set lifetimes at $\sim 10^4 - 10^5$ years for particles between tenths of millimeters and a few centimeters in radius while radiation pressure typically removes such grains in only a few years. The important point to make is that *all* of these loss processes cause the actual region of space filled by stable orbits to be smaller than a simple circular three-body model would suggest.

Several mechanisms (Weidenschilling *et al.* 1989, Burns and Hamilton 1991) might supply circum-asteroidal satellites or debris: *i*) primordial co-accretion processes like those that are believed to have produced most planetary satellites; *ii*) formation in a nearly catastrophic collision like the event thought to have generated Earth's Moon; *iii*) capture of interplanetary debris within the asteroid's stability zone; *iv*) a continuous flux of impact ejecta leaking off the asteroid itself as the latter is bombarded by micrometeoroids; and *v*) similar ejecta leaving an asteroidal "feeder" satellite. The last of these is thought to be the most feasible supplier of circum-asteroidal debris, since a significant fraction of the ejecta can remain trapped in this case in contrast to mechanism *iv*). Unfortunately it is also the least calculable!

Since none of these processes can be quantified well and since definitive observations of life-threatening debris can not be made from the ground, mission planners have been quite anxious about where in the vicinity of an asteroid a spacecraft could safely fly. Clearly this is a very difficult engineering question.

Nonetheless, within the assumptions of the models, the recent research summarized above shows that regions beyond a few hundred asteroid radii will not contain stably trapped particles and that small particles will be entirely absent from the asteroid's vicinity. In addition, it is encouraging that no schemes seem capable of populating the most distant stable orbits. Nevertheless, when entering unknown territory, one always has a nagging worry that something was ignored, perhaps a new mechanism to stabilize orbits or one to efficiently generate distant material. For that reason this Ph.D. candidate, at least, has greeted the unscathed flight of the Galileo spacecraft past 951 Gaspra and 243 Ida, at distances of $\sim 230R_A$ and $\sim 170R_A$, respectively, with sighs of immense relief.

r 5

r i l r r i r [1]

5.1 eneral e arks on ust an r ital Pertur ation eory

Although dust particles contain only a tiny fraction of the mass in orbit about a planet, they far outnumber their macroscopic companions. In planetary systems these tiny motes are ubiquitous, both interspersed with macroscopic bodies in optically thick rings and organized into tenuous structures of their own. Sensitive detectors aboard spacecraft have discovered dust strewn throughout planetary systems, albeit in quantities too faint to be visible (Gurnett *et al.* 1983, 1987, 1991). Clearly the overall distribution of dust in circumplanetary orbits is complex; yet the distribution, and the fact that it can indicate the presence of larger, perhaps unseen, source bodies is of interest to diverse groups of researchers (see Chapter 1). A necessary prerequisite for obtaining such knowledge is a good understanding of the orbital dynamics of an individual dust grain.

Micron-sized dust grains moving along circumplanetary orbits are subject to strong non-gravitational perturbations due to scattering of solar photons and due to Lorentz forces arising from the planet's rotating magnetic field. The effect of these perturbations on an orbiting dust particle can be determined by including the perturbation forces in the left-hand side of Newton's second law $\mathbf{F} = m\mathbf{a}$ (see Chapter 4). In general this equation cannot be solved analytically, so we are forced to resort to approximate or numerical methods (see Chapters 2–4, Horanyi *et al.* 1992, Schaffer and Burns 1992 among others). In many cases, however, we are interested not in the detailed information of how a particle's position and velocity change with time but only in how the character of its orbit varies. In these cases, the six *osculating* orbital elements (a, e, i, Ω,ω, and ν) defined in Figs.

[1] This chapter is based on the paper: Hamilton, D.P. (1993), Motion of dust in a planetary magnetosphere: Orbit-averaged equations for oblateness, electromagnetic, and radiation forces with application to Saturn's E ring. *Icarus* **101**, 244–264 [copyright 1993 by Academic Press, Inc.]

5.1 and 5.2 are particularly useful. Other choices for these elements, especially the sixth, are also possible (Danby 1988, p. 201). We shall often use $\mathcal{M} \equiv nt$, the *mean anomaly*, where $n = (GM_p/a^3)^{1/2}$ is the particle's mean motion, t is the time measured from the moment of pericenter passage, G is the gravitational constant, and M_p is the planetary mass.

If the perturbation forces are small compared to the planet's gravitational attraction, the first five osculating elements will change slowly over timescales much longer than the particle's orbital period. Therein lies the primary advantage of the orbital elements: because they are connected to the geometry of the orbit and because they vary slowly with time, the osculating elements allow a direct visualization of the orbital history of a perturbed body in a way that far surpasses that possible with a set of positions and velocities. Take, for example, the case of an orbit around an oblate planet which we will discuss below in Section 5.2. It is well known that the orbit-averaged solution to this problem is, to high accuracy, simply a precessing ellipse (Danby 1988, p. 345). The orbit retains its size, shape, and inclination off the equatorial plane while its node regresses and its pericenter precesses, each at a constant rate. By calculating these rates from equations given below and using Figs. 5.1 and 5.2, one can easily picture the resulting orbital evolution. Attaining the same picture from positions and velocities as functions of time requires more computation and considerably greater insight!

We note that, technically, the osculating elements differ slightly from *geometric elements* which describe the true shape of the orbit; these deviations are of order the dimensionless ratio (ϵ) of the perturbing force to gravity. For an oblate planet, therefore, the discrepancies are of order J_2 (see Greenberg 1981, Borderies and Longaretti 1987). These differences are especially important when true eccentricities and inclinations are small compared to ϵ (*e.g.,* the geometrically circular orbit discussed by Greenberg, 1981, has a small osculating eccentricity and appears as if it is always at its osculating pericenter), and when other perturbations do not strongly affect an element. Similarly, the rate of change of the mean anomaly is unequal to the mean motion for perturbed orbits; the deviations are of order ϵ and are due both to real changes in a particle's speed as well as differences between the osculating and geometric elements. Because we are primarily interested in how a particle's orbit evolves, we will not use the mean anomaly perturbations in this chapter, but merely include them in the equations to follow for completeness.

The fact that, for modest perturbations, the osculating orbital elements vary slowly in time is useful both numerically and analytically because it allows the effects of a perturbation to be averaged over a single (assumed constant) Keplerian orbit. The resulting averaged expressions describe how the osculating orbital elements change in time and are accurate to first-order in ϵ. In the following sections we treat the strongest perturbation forces acting on close circumplanetary dust grains – higher-order gravity, radiation pressure, and the electromagnetic

Figure 5.1 View of an elliptical orbit in the orbital plane. Three of the orbital elements – the semimajor axis a, eccentricity e, and true anomaly ν – are depicted. Simple geometry shows that the orbit center is offset from the planet by a distance ae, the semiminor axis is given by $b = a(1 - e^2)^{1/2}$ and the semi-latus rectum by $l = a(1 - e^2)$.

Figure 5.2 Three additional orbital elements that define the orientation of an elliptical orbit relative to a fixed plane and a reference direction in that plane. The longitude of the ascending node, Ω, measures the angle from the reference direction to the point where the orbit's plane intersects the reference plane; the argument of pericenter, ω, defines the angle between that intersection point and pericenter (the closest approach of the orbit to the central body); and the inclination, i, measures the angle between the orbital and reference planes. Inclination is defined such that $0^{\circ} \leq i \leq 180^{\circ}$.

force – and derive the appropriate orbit-averaged equations.

5.2 ig er- r er ravity

Treatments of the orbital perturbations arising from non-spherical terms in a planet's gravitational field can be found in many texts (*e.g.,* Danby 1988), but we include a short discussion of them in this section both for completeness and to provide a simple example of the orbit-averaging process that can be compared with the more complicated ones to follow.

Because a planet's spin is responsible for most of the distortion of its gravity field, the field can be well represented by adding an axially symmetric perturbing potential

$$V_{GR} = \frac{GM_p}{r} \sum_{j=2}^{\infty} J_j \left(\frac{R_p}{r} \right)^j P_j(\cos\theta), \tag{5.1}$$

to the standard point source potential; the perturbing force is obtained by taking the negative gradient: $\mathbf{F}_{GR} = -m_g \nabla V_{GR}$. Throughout this chapter the subscripts "p" and "g" stand for "planet" and "grain," respectively; here R_p is the planet's radius and m_g is the mass of the dust grain. The $P_j(x)$ are Legendre polynomials and the J_j are dimensionless coefficients that can be evaluated for a particular planet to describe its gravity field.

To derive the first-order, orbit-averaged equations, we rewrite the potential Eq. (5.1) in terms of the orbital elements and average it over time to obtain the negative of the disturbing function. Inserting the disturbing function into the potential form of the planetary equations (Danby 1988, p. 336) we find the following equations for the variation of the elements:

$$\left\langle \frac{da}{dt} \right\rangle_{J_2} = 0, \tag{5.2}$$

$$\left\langle \frac{de}{dt} \right\rangle_{J_2} = 0, \tag{5.3}$$

$$\left\langle \frac{di}{dt} \right\rangle_{J_2} = 0, \tag{5.4}$$

$$\left\langle \frac{d\Omega}{dt} \right\rangle_{J_2} = -\frac{3nJ_2R_p^2}{2a^2(1-e^2)^2} \cos i, \tag{5.5}$$

$$\left\langle \frac{d\omega}{dt} \right\rangle_{J_2} = \frac{3nJ_2R_p^2}{2a^2(1-e^2)^2} \left(2 - \frac{5}{2}\sin^2 i \right), \tag{5.6}$$

$$\left\langle \frac{d\mathcal{M}}{dt} - n \right\rangle_{J_2} = \frac{3nJ_2R_p^2}{2a^2(1-e^2)^{3/2}} \left(1 - \frac{3}{2}\sin^2 i \right), \tag{5.7}$$

where the angular brackets denote orbit-averaged quantities (*cf.* Danby 1988, p. 347). Technically, each of the orbital elements on the right-hand side of Eqs. (5.2–5.7) should be encased in angled brackets as we ignore their short-period fluctuations, but these brackets will be omitted for clarity since we refer only to the averaged elements throughout this chapter.

Notice that Eqs. (5.2–5.7) are trivially integrable even though the full problem is not (Kozai 1959). The first three expressions imply that the elements a, e, and i are constant and, consequently, the right-hand sides of the final three equations are also fixed. Thus the angles Ω and ω circulate, having values that change linearly in time. Since the circulation times are $\sim (a/R_p)^2/J_2$ times longer than the orbital period, the solution to Eqs. (5.2–5.6) is simply a slowly rotating ellipse.

The final equation merely expresses the average rate at which a particle completes a single osculating orbit from pericenter to pericenter; the rate differs slightly from the mean motion both because the particle's average angular speed is changed and because the position of pericenter slowly shifts. For equatorial orbits, the right-hand side of Eq. (5.7) is positive and the particle completes its radial pericenter-to-pericenter oscillation slightly faster than its unperturbed Keplerian counterpart. This is an expected result since, in the equatorial plane, planetary oblateness augments the inward pull of point-source gravity; the increased force effectively raises the oscillator's "spring constant" and hence its frequency.

5.3 a iation Pressure

For micron-sized grains in circumplanetary orbit, solar radiation pressure is a strong perturber. In its simplest form, radiation pressure imparts a force on a grain given by:

$$\mathbf{F}_{RP} = -\beta \frac{GM_\odot}{R^2}\hat{\mathbf{s}}, \tag{5.8}$$

where $M_\odot$ is the solar mass, R is the Sun-planet distance, $\hat{\mathbf{s}}$ is a unit vector pointing from the planet toward the Sun, and β is the dimensionless ratio of the radiation force to solar gravity given by Eq. 4.1.

This simple expression ignores the anisotropy of re-radiated photons (Poynting-Robertson drag), grain rotation (Yarkovsky effect), the planetary shadow, and complications arising from the rotation and finite angular size of the Sun. These effects are quite small compared to the main force of radiation pressure and can usually be neglected in a first approximation. Dissipative forces, such as Poynting-Robertson drag, however, can be important even if they produce slow changes because they affect the semimajor axis, an element unperturbed by direct radiation pressure. Similarly, effects of the planet's shadow makes it possible for radiation pressure to alter the semimajor axis of an orbit (Mignard 1984, Horanyi and Burns 1991). Over the short times considered here, however, the weak drag forces cannot cause appreciable orbital evolution; and we can ignore the shadow effects since these are small and periodic.

Orbit-averaged solutions to a particle moving around a spherical planet subject to that planet's gravity and solar radiation pressure have been derived by several authors, including Burns *et al.* (1979) and Chamberlain (1979), both of whom used Gauss' form of Lagrange's planetary equations for a force constant in magnitude and direction, and Mignard (1982), who used a disturbing function approach and included the effects of solar motion. All of the above authors did their analysis in the plane defined by the orbital motion of the planet around the Sun (hereafter called the ecliptic plane[2]) and measured their inclinations from that plane. In the case of motion about an oblate planet, however, the planet's equatorial plane is also important - this is especially true since most sources of circumplanetary dust (planetary rings and inner satellites) reside near this plane. It is natural, therefore, to seek an orbit-averaged solution to the orbital evolution caused by radiation pressure that can be expressed in the planet's equatorial plane; this is equivalent to adding a non-zero planetary obliquity, γ, to the previously derived solutions. In this section we discuss two approaches to obtaining equatorial equations and then we derive analytical expressions valid for all obliquities.

One approach, motivated by the fact that orbit-averaged equations valid in the ecliptic plane are already available (Mignard 1982), is to simply translate these equations into the equatorial plane; this task can only be accomplished if the orbital elements themselves can be converted. Since the new set of elements describe the same elliptical orbit from a different reference plane, only the angles (i, Ω, and ω) that define the orbit's orientation relative to the plane will be altered (Fig. 5.2) - the other elements (a, e, and ν) will be identical in both frames. We seek, therefore, functions that relate the new orientation angles to the old. These can be obtained either by simple rotations or from spherical trigonometry. As seen from the planet's center, the equatorial plane, the ecliptic plane, and a particle's orbital path all appear as great circles on the sky (Fig. 5.3); for simplicity we have chosen to measure each orbital node from the ascending node of the ecliptic on the equatorial plane. The spherical triangle formed by the intersections of these great circles implicitly define the equatorial elements in terms of the ecliptic elements. Unfortunately, the expressions resulting from translating to equatorial elements are cumbersome enough to defeat the main purpose of orbit-averaging which is to obtain simple equations for analytic work. Accordingly, we try a different tack.

Since we hope to combine the effects of radiation pressure with other perturbations, we need orbit-averaged expressions referenced to the equatorial plane

[2]Here we use the term ecliptic somewhat loosely to avoid confusion between the planet and particle orbital planes. Strictly speaking, the ecliptic refers only to Earth's orbital plane.

Figure 5.3 The planet's equatorial plane, the ecliptic plane, and the dust grain's orbital plane, each as projected onto the sky from the planet's center. Elements describing the orientation of the grain's orbital plane relative to either of the two reference planes are shown; in each case, the inertial reference direction is the ascending node of the ecliptic on the equatorial plane. Primed quantities are elements referenced to the ecliptic plane, unprimed ones are measured along (or from) the equatorial plane, and γ is the planet's obliquity. The spherical triangle formed by the intersections of these three planes implicitly define one set of elements in terms of the other $[(i, \Omega, \omega) \rightarrow (i', \Omega', \omega')]$.

and to an inertial direction. We choose a right-hand coordinate system centered on the planet with $\hat{\mathbf{x}}$ pointing to the ascending node of the ecliptic on the equatorial plane (Fig. 5.3), $\hat{\mathbf{y}}$ in the equatorial plane, and $\hat{\mathbf{z}}$ along the spin axis. Since, to a good approximation, the Sun is motionless during the time it takes the particle to complete a single orbit (this will only be inaccurate for very distant orbits), we can average over an orbital period while holding the Sun's position constant. The problem breaks down into three pieces: 1) determine the response of the orbit to a constant force along each of the coordinate axes; 2) solve for the Sun's motion in the equatorial frame; and 3) linearly combine these solutions.

Starting the first task, we resolve the solar position in the equatorial frame into components with magnitudes s_x, s_y, and s_z, the time-variable values of which will be determined shortly. The solar position as seen from this frame is then simply: $\hat{\mathbf{s}} = s_x\hat{\mathbf{x}} + s_y\hat{\mathbf{y}} + s_z\hat{\mathbf{z}}$. The perturbing potential V_{RP} is obtained from Eq. (5.8) via the relation $\mathbf{F}_{RP} = -\nabla V_{RP}$. Since the magnitude and direction of radiation pressure changes only slightly over a single orbit of the dust grain, we treat the right-hand side of Eq. (5.8) as a constant and find $V_{RP} = F_{RP}(s_x x + s_y y + s_z z)$. To average the disturbing function, $-V_{RP}$, over time, we first need to express the cartesian coordinates x, y, z in terms of orbital elements:

$$x = r\sin\theta\cos\phi = r(\cos\Omega\cos u - \sin\Omega\sin u\cos i), \tag{5.9}$$

$$y = r\sin\theta\sin\phi = r(\sin\Omega\cos u + \cos\Omega\sin u\cos i), \tag{5.10}$$

$$z = r\cos\theta = r\sin i\sin u, \tag{5.11}$$

where

$$r = \frac{a(1-e^2)}{1+e\cos\nu}, \tag{5.12}$$

and $u \equiv \omega + \nu$ is the *argument of latitude*. Since s_x, s_y, and s_z are nearly constant during the time it takes to make a single circuit around the planet, only the following orbital time-averages are needed:

$$\langle x\rangle = -\frac{3}{2}ea(\cos\Omega\cos\omega - \sin\Omega\sin\omega\cos i), \tag{5.13}$$

$$\langle y\rangle = -\frac{3}{2}ea(\sin\Omega\cos\omega + \cos\Omega\sin\omega\cos i), \tag{5.14}$$

and

$$\langle z\rangle = -\frac{3}{2}ea\sin i\sin\omega. \tag{5.15}$$

We note that $\langle y \rangle$ can be obtained from $\langle x \rangle$ by subtracting 90° from the node in Eq. (5.13) and that all expressions reduce appropriately if e or i equals zero. Inserting these expressions into the potential formulation of the planetary equations, we obtain (after some algebra) the following expressions for the variation of the orbital elements:

$$\left\langle \frac{da}{dt} \right\rangle_{RP} = 0, \tag{5.16}$$

$$\left\langle \frac{de}{dt} \right\rangle_{RP} = \alpha(1-e^2)^{1/2}[s_x(\cos\Omega\sin\omega + \sin\Omega\cos\omega\cos i)$$
$$+s_y(\sin\Omega\sin\omega - \cos\Omega\cos\omega\cos i) - s_z\cos\omega\sin i], \tag{5.17}$$

$$\left\langle \frac{di}{dt} \right\rangle_{RP} = \frac{\alpha e}{(1-e^2)^{1/2}}(s_x\sin\Omega\cos\omega\sin i$$
$$-s_y\cos\Omega\cos\omega\sin i + s_z\cos\omega\cos i), \tag{5.18}$$

$$\left\langle \frac{d\Omega}{dt} \right\rangle_{RP} = \frac{\alpha e}{(1-e^2)^{1/2}}(s_x\sin\Omega\sin\omega$$
$$-s_y\cos\Omega\sin\omega + s_z\sin\omega\cot i), \tag{5.19}$$

$$\left\langle \frac{d\omega}{dt} \right\rangle_{RP} = \frac{\alpha(1-e^2)^{1/2}}{e}[s_x(\cos\Omega\cos\omega - \sin\Omega\sin\omega\cos i)$$
$$+s_y(\sin\Omega\cos\omega + \cos\Omega\sin\omega\cos i) + s_z\sin\omega\sin i] - \cos i\left\langle \frac{d\Omega}{dt} \right\rangle_{RP}, \tag{5.20}$$

$$\left\langle \frac{d\mathcal{M}}{dt} - n \right\rangle_{RP} = -\frac{\alpha(1+e^2)}{e}[s_x(\cos\Omega\cos\omega - \sin\Omega\sin\omega\cos i)$$
$$+s_y(\sin\Omega\cos\omega + \cos\Omega\sin\omega\cos i) + s_z\sin\omega\sin i]. \tag{5.21}$$

Here $2\alpha/(3n) = \beta M_\odot a^2/(M_p R^2)$ is the ratio of the radiation force to the planet's gravity at a given semimajor axis. In terms of previous expressions $\alpha = 3/(2\tau_0)$ in Chamberlain (1979)'s notation and $\alpha(1-e^2)^{1/2} = 3HF/(2m\mu)$ in Burns *et al.* (1979)'s notation. Eqs. (5.16–5.21) are fully three-dimensional, valid for all eccentricities and inclinations.

It remains only to determine the coefficients s_x, s_y, and s_z. Imagine a rotating ecliptic coordinate system such that the Sun remains fixed along the x_R-axis. In general, at time $t = 0$, the Sun is located at an angle δ from the inertial reference direction in Fig. 5.3. To find the coordinates of a unit vector pointing toward the Sun in the equatorial frame, we apply two rotations: first a rotation of $-n_\odot t - \delta$ around the normal to the ecliptic back to the mutual node, then a rotation around the inertial direction by minus the obliquity to align the reference planes. In matrix notation, the transformation is:

$$\hat{\mathbf{s}} = R_x(-\gamma)R_z(-n_\odot t - \delta)\hat{\mathbf{x}}_R, \tag{5.22}$$

where $R_x(\theta)$ and $R_z(\theta)$ are rotation matrices around the x and z axes, respectively (see Danby 1988, p. 425). Performing the multiplication, we find:

$$s_x = \cos(n_\odot t + \delta), \tag{5.23}$$

$$s_y = \cos\gamma \sin(n_\odot t + \delta), \tag{5.24}$$

and

$$s_z = \sin\gamma \sin(n_\odot t + \delta). \tag{5.25}$$

Notice that for $\gamma = 0, n_\odot = 0, \delta = 0$, we have $s_x = 1, s_y = s_z = 0$ and Eqs. (5.16–5.20) reduce to those of Burns *et al.* (1979) or Chamberlain (1979). Mignard (1982)'s equations are obtained after a little trigonometry, by letting $\gamma = 0, \delta = 0$ and employing the transformation $\Omega = \Omega_R + n_\odot t$. Here Ω_R is Mignard's longitude of the nodes which differs from Ω because the former is measured from a direction that rotates at an angular speed $n_\odot$. With a little trigonometry, we find that Eqs. (5.16–5.21 and 5.23–5.25) are also in agreement with expressions derived independently by Smyth and Marconi (1993).

We have derived Eqs. (5.23–5.25) last to emphasize the fact that the orbit-averaging can be performed for arbitrary s_x, s_y, and s_z as long as their time dependence is slow compared to the particle's orbital period. For example, we could easily treat the problem of motion around a planet which orbits the Sun on an elliptical path by simply replacing the argument in the R_z rotation matrix in Eq. (5.22) by an expression valid for the Sun's non-uniform rate. Of course, in such a case it would also be necessary to add a time dependence to α to allow for the more important fact that radiation pressure weakens as the planet moves away from the Sun (*cf.* discussion following Eq. 4.10).

5.4 lectro agnetic Forces

5.4.1 General emarks

The rings and small satellites of the outer planets lie close to their primaries in environments characterized by swarms of energetic charged particles trapped by strong magnetic fields. Immersed in this sea of particles, a dust grain quickly acquires an electric charge by a number of mechanisms (Goertz 1989), the most important of which are the electron and ion charging that occur as the grain sweeps up these gyrating particles. Uncharged dust grains are impacted by electrons more frequently than by ions because the thermal speed of the former far exceeds that of the latter - in essence, the electrons get to the grains before the ions do. As a grain becomes more negatively charged, it is able to electrostatically ward off some electrons while simultaneously attracting a comparable number of ions until a balance is attained (Burns and Schaffer 1989, their Fig. 1). For typical magnetospheric parameters and micron-sized grains, equilibrium is established in a fraction of an orbital period. The addition of other charging mechanisms, such as photoelectron currents and secondary electron emission, usually only perturbs the equilibrium grain charge, although for high secondary yields such processes can lead to multiple equilibria (Meyer-Vernet 1982). Finally, even the equilibrium charge may gradually change as the grain's orbit takes it into regions where plasma populations differ and as the grain's velocity relative to the plasma varies (*cf.* Burns and Schaffer 1989). Stochastic variations of the grain's charge, which are generally relatively small and occur swiftly, have little effect on orbital evolution (Schaffer and Burns 1994).

Despite the complexity of these charging mechanisms, it is often a good approximation to assume that the equilibrium charge on a grain is constant. Take, for example, the orbital elements displayed in Fig. 5 of Horanyi *et al.* (1992) which show an eccentric orbit that ranges from 1 out to 7 saturnian radii. Although the relative velocity between the grain and the co-rotating plasma varies tremendously, changes in the equilibrium potential are limited to $\pm 5\%$. This is in agreement with Burns and Schaffer (1989)'s Fig. 1 which shows a weak dependence of the equilibrium potential on velocity. Potentially more serious are the fluctuations in a grain's charge cause by spatial and temporal variations in the density and temperature of the magnetospheric plasma. Because the plasma density in the E ring is relatively large, a grain's charge adjusts to its surroundings much more rapidly than it orbits the planet (Horanyi *et al.* 1992). If we make the reasonable assumption of cylindrically symmetric spatial variations in the plasma parameters, it can be shown, with the formalism to be introduced below, that the semimajor axis and eccentricity of the grain's orbit change in almost the same way as they do for a constant charge. Since the purpose of this section is to account for the first-order effects of the Lorentz force, henceforth we will make the simplifying assumption of constant charge. In Chapter 6, we will return to comment further on the validity of this approximation for the specific case of Saturn's E ring.

Planetary magnetic fields are responsible not only for trapping the electrons and ions that charge up a dust grain, but also for the resulting orbital perturbations suffered by such grains. In the standard model, these fields are assumed to arise from two sources: currents interior to a given radial distance from the planet and currents exterior to this distance; connections between the regions are ignored. Because of the assumed lack of currents in the region of interest

($\mathbf{J} \sim \nabla \times \mathbf{B} = 0$), the magnetic field can be derived from a scalar magnetic potential Φ in analogy with the electric potential. The j, k component of the scalar magnetic potential in the frame rotating with the planet is given by the usual spherical harmonic expansion:

$$\Phi_{j,k} = R_p \left(\frac{R_p}{r}\right)^{j+1} [g_{j,k} \cos(k\phi_R) + h_{j,k} \sin(k\phi_R)] P_j^k(\cos\theta), \tag{5.26}$$

where j is an integer ranging from one to infinity, k is an integer ranging from zero to j, and $\phi_R = \phi - \Omega_p t$, with the subscript 'R' denoting the rotating coordinate system. Here θ and ϕ are the angular spherical coordinates defined in the non-rotating frame. The $g_{j,k}$ and $h_{j,k}$ are field coefficients with units of gauss which can be evaluated for each planet (for Saturn see Connerney *et al.* , 1984; Schaffer and Burns, 1992, tabulate values for the giant planets and give additional references). In Eq. (5.26) we have ignored the (usually small) contributions from the exterior currents; their effects can be readily included (Acuña *et al.* 1983a) when necessary (*e.g.,* beyond a few planetary radii in the Jovian system). The Schmidt-normalized associated Legendre polynomials $P_j^k(x)$ are defined in terms of the regular Legendre polynomials; the relevant expressions can be found in Schaffer and Burns (1992). Finally, the magnetic field contribution from the j, k component of the potential is

$$\mathbf{B}_{j,k} = -\nabla\Phi_{j,k}, \tag{5.27}$$

while the total field in the rotating frame is obtained by summing all of the individual components

$$\mathbf{B} = \sum_{j=1}^{\infty} \sum_{k=0}^{j} \mathbf{B}_{j,k}. \tag{5.28}$$

Two ways exist to obtain the Lorentz force valid in a non-rotating frame centered on the planet. Although the methods give identical results, they are conceptually quite different and it is instructive to go through each argument. In the first method, we calculate the force in the rotating frame as $\mathbf{F}_{EM} = q(\mathbf{v}_{rel}/c \times \mathbf{B})$ with $\mathbf{v}_{rel} = \mathbf{v} - (\mathbf{\Omega}_p \times \mathbf{r})$, where $\mathbf{v}_{rel}$ is the orbital velocity of the dust grain relative to the rotating frame, $\mathbf{v}$ is its velocity relative to a non-rotating planetocentric coordinate system, $\mathbf{\Omega}_p$ is the spin vector of the planet, c is the speed of light, and q is the charge on the grain. Employing special relativity to transform the force back to the non-rotating frame, we find that it is unaltered to first-order in v/c and hence:

$$\mathbf{F}_{EM} = \frac{q}{c}\{[\mathbf{v} - (\mathbf{\Omega}_p \times \mathbf{r})] \times \mathbf{B}\}. \tag{5.29}$$

The preceding discussion makes it quite clear that the Lorentz force vanishes for an equatorial circular orbit at the synchronous distance: there the velocity relative to the magnetic field is zero and thus no force is present.

The second way to treat the problem is to transform the magnetic field from rotating coordinates to non-rotating ones before calculating the force. Utilizing special relativity again, we find that the magnetic field is unchanged (neglecting terms of order $\Omega_p r/c << 1$), and that an electric field $\mathbf{E} = -(\mathbf{\Omega}_p \times \mathbf{r}) \times \mathbf{B}$ is present in the non-rotating frame. This is the so-called "co-rotational electric field" discussed by Burns and Schaffer (1989) among others. The Lorentz force is then calculated from $\mathbf{F}_{EM} = q[\mathbf{E} + (\mathbf{v}/c \times \mathbf{B})]$ and Eq. (5.29) is obtained once more. This discussion highlights the role of the magnetic field; it illustrates that part of the Lorentz force does no work and, as we shall see, is less able to influence the orbital elements.

Although the magnetic field can be expressed as a gradient of a potential, the electromagnetic force, because of its velocity dependence, cannot. We are therefore unable to use the disturbing function approach that was applied to radiation pressure and instead must use Gauss' form of the planetary equations. These equations are given in orbital coordinates where the acceleration at a particular point on the orbit is resolved into orthogonal components which are radial ($\hat{\mathbf{R}} = \hat{\mathbf{r}}$), normal to the orbit ($\hat{\mathbf{N}}$), and tangential ($\hat{\mathbf{C}}$) to a circle in the orbital plane that passes through the point. The Lorentz force, Eq. (5.29), is written in equatorial spherical coordinates which are converted into the orbital coordinates by use of Eqs. (5.9–5.11) and the following expressions:

$$\hat{\theta} = -\frac{\cos i}{\sin\theta}\hat{\mathbf{N}} - \frac{\sin i \cos u}{\sin\theta}\hat{\mathbf{C}}, \tag{5.30}$$

and

$$\hat{\phi} = -\frac{\sin i \cos u}{\sin\theta}\hat{\mathbf{N}} + \frac{\cos i}{\sin\theta}\hat{\mathbf{C}}. \tag{5.31}$$

Carrying out the transformation and keeping track of all terms, we find that the normal, radial, and tangential components of the Lorentz acceleration can be represented as:

$$N = \frac{q}{cm_g} \quad -\frac{v_r B_\theta \sin i \cos u}{\sin\theta} + \frac{v_r B_\phi \cos i}{\sin\theta} - B_r v_C + B_r \Omega_p r \cos i \quad , \tag{5.32}$$

$$R = \frac{q}{cm_g} \quad -\frac{v_C B_\theta \cos i}{\sin\theta} - \frac{v_C B_\phi \sin i \cos u}{\sin\theta} + B_\theta \Omega_p r \sin\theta \quad , \tag{5.33}$$

$$C = \frac{q}{cm_g} \quad \frac{v_r B_\theta \cos i}{\sin\theta} + \frac{v_r B_\phi \sin i \cos u}{\sin\theta} + B_r \Omega_p r \sin i \cos u \quad , \tag{5.34}$$

where v_r and v_C are the radial and circular parts of the velocity and the B_i are the appropriate magnetic field components. These three equations are valid for any magnetic field. Finally we need to express the radial and circular velocity components of a Keplerian elliptical orbit in terms of the orbital elements; from conservation of orbital energy and angular momentum we have:

$$v_r = \frac{GM_p}{a}{}^{1/2} \frac{e \sin \nu}{(1-e^2)^{1/2}}, \tag{5.35}$$

and

$$v_C = \frac{GM_p}{a}{}^{1/2} \frac{1 + ecos\nu}{(1-e^2)^{1/2}}. \tag{5.36}$$

5.4.2 The Aligned Dipole

We begin by discussing the axisymmetric ($k = 0$) terms in the magnetic field expansion given by Eqs. (5.26–5.28) as they have no time dependence and can be readily orbit-averaged. Of these, the $j = 1$ term is the strongest so we focus on it first. The magnetic field produced by this $g_{1,0}$ term is a spin-axis aligned dipole which has the following components:

$$B_r = 2g_{1,0} \frac{R_p}{r}{}^3 \cos\theta, \tag{5.37}$$

$$B_\theta = g_{1,0} \frac{R_p}{r}{}^3 \sin\theta, \tag{5.38}$$

$$B_\phi = 0. \tag{5.39}$$

for convenience, and in analogy with Eq. (4.1), we define a dimensionless parameter, L, as a rough measure of the strength of the electromagnetic force relative to the planet's gravity. We take the $g_{1,0}$ term of the magnetic field given in Eqs. (5.37–5.39), evaluate Eq. (5.29) in the equatorial plane with $\mathbf{v} = 0$, and divide by the planet's gravitational force (note this is similar to the parameter ϵ defined by Schaffer and Burns 1987). The result is:

$$L = \frac{q g_{1,0} R_p^3 \Omega_p}{c G M_p m_g}. \tag{5.40}$$

Inserting the force resulting from Eqs. (5.37–5.39) into the planetary perturbation equations and performing the time-averages we obtain:

$$\left\langle \frac{da}{dt} \right\rangle_{g_{1,0}} = 0, \tag{5.41}$$

$$\left.\frac{de}{dt}\right|_{g_{1,0}} = -\frac{nL}{4}e(1-e^2)^{1/2}\sin^2 i \sin(2\omega), \tag{5.42}$$

$$\left.\frac{di}{dt}\right|_{g_{1,0}} = \frac{nL}{4(1-e^2)^{1/2}}e^2 \sin i \cos i \sin(2\omega), \tag{5.43}$$

$$\left.\frac{d\Omega}{dt}\right|_{g_{1,0}} = \frac{nL}{(1-e^2)^{1/2}}\left[\cos i - \frac{1}{(1-e^2)} \quad \frac{n}{\Omega_p} \quad \right], \tag{5.44}$$

$$\left.\frac{d\omega}{dt}\right|_{g_{1,0}} = \frac{nL}{(1-e^2)^{1/2}}\left[-\cos^2 i + \frac{3\cos i}{(1-e^2)} \quad \frac{n}{\Omega_p} \quad \right], \tag{5.45}$$

$$\left.\frac{d}{dt} \quad - n \quad \right|_{g_{1,0}} = -2nL. \tag{5.46}$$

These expressions have been simplified from exact formulae by dropping terms of high order in e and i. Nevertheless, Eqs. (5.41–5.46) are quite accurate even for very large inclinations and eccentricities as we shall soon see.

The electromagnetic force, like radiation pressure, makes non-zero contributions to variations in the eccentricity and inclination but these contributions depend on powers of $\sin i$ and e; they are therefore quite small unless the orbit under consideration is both highly-eccentric and significantly-inclined. Thus, at least for small inclinations and eccentricities, the effect of the planet's dipolar magnetic field is not unlike that of J_2 (the planet's quadrupole gravitational field) since both forces primarily cause precession. This crude similarity should not be surprising since, at least near the equator plane, both forces have strengths that diminish rapidly with distance and directions that are predominantly radial. For electromagnetism, the nodal and apsidal precession rates are dependent on inclination, eccentricity, and the semimajor axis as are their J_2 counterparts. Unlike the gravitational case, however, the electromagnetic rates vary considerably relative to one another for circular orbits of different sizes near the equatorial plane (compare Eqs. 5.44–5.45 with 5.5–5.6). Close to synchronous orbit ($n = \Omega_p$), for example, the nodal rate vanishes, while the apsidal rate is zero further from the planet near the place where $3n = \Omega_p$. Incidentally, as synchronous orbit is approached in the limit ($n \to \Omega_p, e \to 0, i \to 0$), the Lorentz force vanishes as does the nodal rate, but the pericenter rate does not. How can a force which is zero all along an orbit cause orbital evolution? The solution to this apparent paradox is, of course, that it does not; a circular orbit has no unique pericenter so the fact that an ill-defined angle fails to vanish is unimportant. For small eccentricities, pericenter exists, the Lorentz force is non-zero, and Eq. (5.45) gives the correct precession rate.

5.4.3 The Aligned Quadrupole

As in the gravitational case, inclusion of the higher-order axisymmetric ($k = 0$) terms in the magnetic field expansion requires that the lower-order terms be treated more carefully (*i.e.,* taken out to the next order in L), a task that rapidly increases in algebraic complexity. For gravity, a treatment including just the J_2 term is a good approximation because J_3 and the other odd harmonics are all exceedingly small for the giant planets, and because the fields produced by the larger, even harmonics fall off very quickly with increasing distance. Accordingly, we might hope that higher-order symmetric terms in the electromagnetic expansion could be ignored as well. We find, however, that the axisymmetric quadrupole has a non-trivial influence on orbital dynamics; its importance can be easily understood by noting that near the equator, the radial component of the dipole magnetic field is small (of order i). In contrast, the quadrupole field is primarily radial and its magnitude actually exceeds the radial dipolar field for orbits with small inclinations. When crossed into a transverse velocity, the radial field produces a strong normal force which perturbs the inclination, node and pericenter; hence we expect that quadrupole effects will be important for these elements. Further expansion to include the symmetric octupole and higher $k = 0$ terms is unnecessary, as the radial and theta components of the combined dipole and quadrupole magnetic field dominate contributions from higher-order terms.

Rather than repeating the derivations of Section 5.4.2 for the symmetric quadrupole term (an arduous task!), we will treat only the case of small inclinations which is of the most interest for planetary applications. For inclinations smaller than 30°, the theta component of the magnetic field is dominated by the dipole term and so we ignore the small quadrupole contributions to that component. The radial component of the quadrupole field

$$B_r = \frac{3}{2} g_{2,0} \left(\frac{R_p}{r} \right)^4 (3\cos^2\theta - 1), \tag{5.47}$$

however, is important. The largest effect of the radial quadrupole field on a slightly inclined orbit is to produce a normal force; consequently, we ignore Eqs. (5.33) and (5.34) and consider only Eq. (5.32); this force affects only the inclination, node, and pericenter derivatives. The first two terms of Eq. (5.32) are identically zero because we have ignored the theta component of the quadrupole field and there is no phi component. Furthermore, it turns out that the final term also contributes nothing. Performing the much-simplified averaging calculation, we obtain:

$$\left(\frac{di}{dt} \right)_{g_{2,0}} \approx \frac{3}{2} nL \left(\frac{g_{2,0}}{g_{1,0}} \right) \left(\frac{R_p}{a} \right) \left(\frac{n}{\Omega_p} \right) \frac{e\cos\omega}{(1-e^2)^{5/2}}, \tag{5.48}$$

$$\left(\frac{d\Omega}{dt} \right)_{g_{2,0}} \approx \frac{\tan\omega}{\sin i} \left(\frac{di}{dt} \right)_{g_{2,0}} \tag{5.49}$$

$$\left.\frac{d\omega}{dt}\right|_{g_{2,0}} \approx -\cos i \left.\frac{d\Omega}{dt}\right|_{g_{2,0}} \tag{5.50}$$

where approximation signs have been used instead of equal signs to remind the reader that these equations represent only part of the quadrupole perturbations, albeit the most important contributions for low-inclination orbits. In this limit, additional quadrupole perturbations are insignificant when compared to the effects of the aligned dipole.

Notice that, with a little manipulation, the form of Eqs. (5.48–5.50) is identical to the s_z component of the radiation pressure equations (Eqs. 5.18–5.20). This occurs because both sets of equations arise from small, nearly constant vertical forces being applied to an orbit; we will take advantage of this similarity in Section 5.5 to follow. The $\sin i$ in the denominators of Eqs. (5.19) and (5.49) simply expresses the fact that, for low inclinations, the orbital node is poorly defined and small perturbations can force large changes in that element.

5.4.4 Asymmetric Terms

Up to now, we have ignored the non-axisymmetric ($k \neq 0$) terms in the magnetic field expansion; this is a good approximation for the almost perfectly aligned saturnian field (Connerney *et al.* 1984), but not for the magnetic fields of the other giant planets. Thus non-axisymmetric magnetic field terms merit a brief discussion. First of all, we note that asymmetric terms are more difficult to treat than the symmetric ones since the planet's rotation causes the field's orientation to change rapidly. Typically, a planet's spin period is comparable to the orbital period of an inner orbit; for this type of orbit there is not a unique timespan over which to average. In contrast, distant orbits have periods that are so long that we can average the magnetic field first over a single spin period and then over the orbital motion. Following this procedure, we find that all of the non-axisymmetric terms in Eq. (5.26) average to zero (*i.e.,* they give no contribution to orbital evolution in this limit).

Encouraged by this result, we might be tempted to ignore the effects of the non-axisymmetric terms even close to the planet, arguing that they will only induce small periodic oscillations. For most orbits this is true, but at specific locations orbital and spin frequencies are commensurate and the averaging process is invalid (*cf.* Schaffer and Burns 1992, Burns *et al.* 1985). These "Lorentz" resonances can be treated by isolating the commensurate terms in Eqs. (5.32–5.34) for use in the planetary equations, but for now we ignore the non-axisymmetric terms and accept the fact that the orbit-averaged equations derived in this section will not be valid near resonant locations. We analyze Lorentz resonances in Chapter 7.

5.5 ouple ertur ations

In the circumplanetary environment, all three perturbation forces discussed above (higher-order gravity, the electromagnetic force, and radiation pressure) conspire to perturb the orbits of micron-sized dust grains. Since the forces are generally small, the orbit-averaged equations derived in the previous sections can be simply summed to account for the cumulative effect of all perturbations:

$$\frac{d\chi}{dt}_{total} = {}_{j} \ \frac{d\chi}{dt}_{j}, \tag{5.51}$$

where χ is any of the six osculating orbital elements. The resulting expressions are cumbersome, but several hundred times faster to numerically integrate than their Newtonian counterparts. In addition, the output of the Newtonian equations (vector position and velocity) must be translated into osculating orbital elements. As a demonstration of the validity of our derivations, we compare numerical integration of the Newtonian (Fig. 5.4) and orbit-averaged (Fig. 5.5) equations for a 1-micron grain charged to -5.6 Volts, approximately the potential expected in the saturnian environment (Horanyi *et al.* 1992, Fig. 1). The initial conditions in both cases are appropriate for a grain launched from the moon Enceladus on an initially circular, coplanar orbit at $3.95R_p$; the Sun is initially at its maximum height above the equatorial plane (90° past the ascending node of the ecliptic on Saturn's equator - Fig 5.3). Plotted are the five osculating orbital elements and the *solar angle* $\phi_\odot$ (defined below). All six panels of the two plots agree quite well which reassures us that the approximations made in the previous sections are valid.

The most notable difference between Figs. 5.4 and 5.5 appears in the semimajor axis traces; in the first figure, the semimajor axis displays a peculiar "fuzziness," while no evolution whatsoever of this element is apparent in the second. The discrepancy is due to effects that occur during a single orbital period; in Fig. 5.4 these effects are clearly visible while in Fig. 5.5 they do not exist because they have been averaged out. These short-period terms arise when the vector position and velocity are translated into the osculating orbital elements; in the presence of perturbations, the values of the elements depend on the point along an orbit at which they are calculated. The difference in these values over a single orbit is first-order in the small dimensionless quantities J_2, α/n, and L, and the oscillations in osculating semimajor axis are greater for larger eccentricities as can be readily seen in the plot. By noting the value of a in Fig. 5.4 at points where $e \approx 0$, however, we see that no long-term change occurs in the semimajor axis in agreement with Fig. 5.5. The fact that the rapid and sometimes discontinuous changes in ω and Ω that occur at low e and i are not perfectly reproduced (see Figs. 5.4 and 5.5 at $t \sim 8$ years for example) is unimportant since these variables become singular as e and i, respectively, tend toward zero.

Figure 5.4 Osculating orbital elements plotted against time from integrations of the full Newtonian equations of motion. The integrations shown in this figure and in the following one used identical initial conditions: a spherical grain 1 micron in radius ($\rho_g = 1g/cm^3, Q_{pr} = 1$) charged to a potential of -5.6 Volts initially released from Enceladus at $3.95R_p$ on a circular Keplerian orbit in the equatorial plane. Forces operating on the orbit include: the monopole and J_2 components of the gravity field, radiation pressure (no shadowing), and the Lorentz force from an aligned dipole. The agreement between the two methods is very good and is discussed further in the text. Other values of interest are the three dimensionless parameters $J_2 = 0.01667$, $\alpha/n = 0.00012$, $L = -0.00295$; the ratio $n/\Omega_p = 0.32439$, and the initial precession rates, as given by Eqs. (5.59) and (5.60), $\dot{\Omega}_{xy}(e \sim 0) = -345^o/\text{year}$ and $\dot{\omega}_{xy}(e \sim 0, \phi_\odot = 90^o) = 315^o/\text{year}$. The difference between the two rates is roughly the initial slope of the solar angle trace in the sixth panel.

Figure 5.5 Osculating orbital elements plotted against time from integrations of the orbit-averaged equations of motion. This integration followed the evolution of an orbit subject to the identical forces, and starting with the same initial conditions, as the orbit in Fig. 5.4. The agreement between the two figures is impressive. The most noticeable discrepancy, the semimajor axis history, is discussed further in the text.

The agreement between Figs. 5.4 and 5.5 also encourages qualitative and quantitative descriptions of evolution based on the orbit-averaged equations. To assist this endeavor, we write out Eq. (5.51) explicitly. As noted above, most sources for circumplanetary dust are thought to be the small moons and rings orbiting close to their planet; these objects move on nearly circular orbits in the equatorial plane. While radiation pressure can cause eccentricities of some initially circular orbits to grow quite large as we will see in Chapter 6, it is often difficult for grains to attain orbits significantly inclined to the equatorial plane. This supposition holds for orbits close to the planet and away from the Lorentz resonant locations. A useful limit for analytic work, therefore, is that of nearly equatorial orbits. We keep only the leading terms in $\sin i$ for each element and also take $i < \gamma$, which is a reasonable assumption in most cases (Jupiter is a possible exception as $\gamma \sim 3^{\circ}$). Additionally, we assume that $\gamma \lesssim 30^{\circ}$ so that $\cos\gamma \approx 1$; this, of course, is not a good approximation for Uranus with $\gamma \approx 98^{\circ}$! Using Eq. (5.51) to sum the effects of all of the forces discussed in the previous sections, we find, with the help of some trigonometric identities:

$$\frac{da}{dt} = 0, \tag{5.52}$$

$$\frac{de}{dt} = \alpha(1-e^2)^{1/2}\sin\phi_{\odot}, \tag{5.53}$$

$$\frac{di}{dt} = Z\cos\omega, \tag{5.54}$$

$$\frac{d\Omega}{dt} = Z\frac{\sin\omega}{\sin i} + \dot{\Omega}_{xy}, \tag{5.55}$$

$$\frac{d\omega}{dt} = -Z\frac{\sin\omega}{\sin i} + \dot{\omega}_{xy}, \tag{5.56}$$

where

$$\phi_{\odot} \equiv \Omega + \omega - n_{\odot}t - \delta \tag{5.57}$$

is the *solar angle*, approximately the angular difference between the longitudes of the Sun and the orbit's pericenter measured in the equatorial plane; the change in this angle is given by:

$$\dot{\phi}_{\odot} = \dot{\Omega}_{xy} + \dot{\omega}_{xy} - n_{\odot}. \tag{5.58}$$

The precession rates arising from oblateness, electromagnetic, and radiation forces (excluding the terms proportional to Z, defined immediately below) are:

$$\dot{\Omega}_{xy} = -\frac{3nJ_2R_p^2}{2a^2(1-e^2)^2} + \frac{nL}{(1-e^2)^{3/2}} \quad 1 - e^2 - \frac{n}{\Omega_p} \quad , \tag{5.59}$$

and

$$\dot{\omega}_{xy} = \frac{3nJ_2R_p^2}{a^2(1-e^2)^2} - \frac{nL}{(1-e^2)^{3/2}} \quad 1 - e^2 - \frac{3n}{\Omega_p} \quad + \frac{\alpha(1-e^2)^{1/2}\cos\phi_\odot}{e}; \qquad (5.60)$$

the Z terms are excluded for reasons that will become apparent in the following sections. Finally, the quantity

$$Z = \frac{e}{(1-e^2)^{1/2}}\left[\alpha s_z + \frac{3}{2(1-e^2)^2} nL \quad \frac{g_{2,0}}{g_{1,0}} \quad \frac{R_p}{a} \quad \frac{n}{\Omega_p} \quad \right], \qquad (5.61)$$

represents the contributions of the two vertical forces in the problem; the out-of-plane component of radiation pressure and the force arising from the aligned quadrupole field. These two forces are small and do not cause substantial orbital evolution; notice that the terms proportional to Z in Eqs. (5.55–5.56) are equal and opposite so that for small i, the absolute longitude of pericenter $\varpi = \Omega + \omega$ is unaltered. Nevertheless, the forces are important because they influence the vertical extent of an orbit as will be discussed in greater detail below.

The presence of J_2, L, and α in all of the above expressions indicate the effects of oblateness, electromagnetism, and radiation pressure, respectively; Eq. (5.53), for example, shows that eccentricity, in this low inclination limit, is driven solely by radiation pressure. Additional approximations to the set of equations (5.52–5.56) can be made for specific situations. For example, we can drop the electromagnetic terms for orbits around bodies with insignificant magnetic fields (certainly Venus and Mars; and presumably Pluto, asteroids, and comets) or for small uncharged objects (atoms and molecules) around any planet. In the latter case, Eq. (4.1) would need to be altered since geometrical optics are not valid for atoms and molecules (*cf.* Smyth and Marconi 1993).

In this chapter we have set up a framework within which the most powerful non-gravitational forces can be treated. The expressions that we derived are general and applicable in numerous locales throughout the solar system. In the next chapter we apply our results to one of these objects: Saturn's diffuse E ring.

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6.1 Intro uction

It is important to understand the dynamics of the very faint rings surrounding the giant planets since, owing to the rarity of collisions, such entities offer an excellent opportunity to learn the fundamental processes affecting the motion of individual ring particles. Because the particles comprising the ethereal rings are usually small, however, the orbital evolution of even a single particle can be quite complex: in addition to the usual gravitational perturbations (*e.g.,* due to planetary oblateness and embedded satellites), small grains are also subject to radiation pressure and electromagnetic forces (Chapter 5) as well as weaker variations due to drags, and charge variations (Burns 1991). All these processes are active to some extent in the dense rings as well, but they are obscured by other perturbations, especially collisions and collective effects.

Perhaps the best studied of all the ethereal rings is Saturn's E ring. Much of the interest in this three-dimensional structure arises because the Cassini spacecraft will make many passes through this region. Recently, Showalter *et al.* (1991) have combined spectrophotometric data of the E ring from ground-based measurements with that from the Pioneer 11 and Voyager encounters. Their most important findings are: the ring extends from $\lesssim$3 to $\gtrsim 8R_S$ (the equatorial radius of Saturn R_S=60,330 km); its optical depth profile peaks sharply near the orbit of Enceladus ($a_E = 3.95R_S$), making this satellite the suspected source of the ring, with a simple power law decay that is sharper inward [$\tau \sim (r/a_E)^{15}$] than outward [$\tau \sim (a_E/r)^7$] of Enceladus' orbit; in general, the ring shows a gradual in-

[1]This chapter is based on three papers: Horanyi, M., J.A. Burns, and D.P. Hamilton (1992), The dynamics of Saturn's E ring particles. *Icarus* **97**, 248–259 [copyright 1993 by Academic Press, Inc.], Hamilton, D.P. (1993), Motion of dust in a planetary magnetosphere: Orbit-averaged equations for oblateness, electromagnetic, and radiation forces with application to Saturn's E ring. *Icarus* **101**, 244–264 [copyright 1993 by Academic Press, Inc.], and Hamilton, D.P., and J.A. Burns (1993), The origin of Saturn's E ring: Self-sustained, naturally. *Science*, submitted.

crease in vertical thickness with distance from Saturn, ranging from $\simeq$ 6,000 km at its inner boundary to $\simeq$ 40,000 km at its outer edges but has a local minimum at the orbit of Enceladus, where the thickness is only $\simeq$ 4,000 km; and, perhaps most puzzling of all, the size distribution of the particles is very narrow, being composed mainly of particles with $1(\pm 0.3)\,\mu$m radii. In this chapter we suggest that many of these observations can be understood in terms of the short-term dynamics of single particles injected at Enceladus; a schematic of the E ring, showing its relation to Enceladus and the main rings, is given in Figure 6.1. We will demonstrate that, to some degree, the E ring's optical depth profile results from the competing effects of the perturbations due to planetary oblateness and the Lorentz force, which allow solar radiation pressure to induce quite large eccentricities for a selected particle size range including the micron-sized grains thought to be present in the ring. This mechanism is capable of spreading material quite quickly across large radial distances from Saturn and producing a sharply peaked optical depth profile; its effectiveness is found to be strongly size-dependent, which is consistent with the E ring's very narrow particle size distribution. In the following three sections we use analytic and numerical methods to understand the radial, azimuthal, and vertical structure of Saturn's E ring.

6.2 a ial Structure

In order to understand the first-order radial structure, we begin with a simple analytical model based on the results of the previous chapter. In order that such a model be analytically tractable, we make a number of simplifying assumptions. First, we include only the perturbations from oblateness, solar radiation pressure and the Lorentz force, neglecting all weaker perturbations (*e.g.,* drag forces). In addition, we assume that the charge on a dust grain orbiting in the E ring is nearly constant which is in accord with the findings of Horanyi *et al.* (1992), although with a more extreme plasma model, this need not be the case. In this section, we also neglect variations in the inclination and the node, since for small inclinations, these elements do not affect the ring's radial structure. Finally, we initially assume low eccentricities, although this assumption will be relaxed shortly. For small eccentricities, Eqs. (5.53), (5.58), (5.59), and (5.60) become

$$\frac{de}{dt} = \alpha \sin \phi_{\odot} \tag{6.1}$$

$$\frac{d\phi_{\odot}}{dt} = \frac{\alpha}{e} \cos \phi_{\odot} + \dot{\varpi}_{xy} \tag{6.2}$$

where $\dot{\varpi}_{xy}$, defined to be

$$\dot{\varpi}_{xy} = \frac{3nJ_2R_p^2}{2a^2} + \frac{2n^2L}{\Omega_p} - n_{\odot}, \tag{6.3}$$

Figure 6.1 The saturnian system. The solid central disk represents Saturn and immediately exterior to it are the optically thick A and B rings (hatched). The E ring covers the stippled region outside the main rings and encompasses the orbits of at least four major saturnian satellites: Mimas, Enceladus, Tethys, and Dione. For clarity, we show only Enceladus' orbit (the circle of radius $a_{moon} \sim 3.95R_S$) and that of an E-ring grain which originated on Enceladus (ellipse with $a_{dust} = 3.95R_S$ and $e_{dust} = 0.5$). The orbital turning points A (apocenter) and P (pericenter) of the particle's orbit are located at distances $a_{dust}(1 + e_{dust})$ and $a_{dust}(1 - e_{dust})$ from Saturn. E-ring particles cross the orbit of Enceladus at the points I_1 and I_2 and can venture within the radial distance of the opaque main rings only if they fly above or below them.

represents the uniform motion of pericenter relative to the Sun in the absence of radiation pressure. Before solving Eqs. (6.1) and (6.2), let us qualitatively discuss the solutions.

The first two terms on the right side of Eq. (6.3) are much larger than $n_\odot$ and, since the grain charge and hence L is expected to be negative (Horanyi *et al.* 1992), these two terms compete against one another. Thus, the pericenter can be gravitationally dominated (precession with $\dot{\varpi}_{xy} > 0$), stopped ($\dot{\varpi}_{xy}$=0) or electromagnetically dominated (regression with $\dot{\varpi}_{xy} < 0$). Which of these situations occurs will depend on the particle's size, charge and its position in the magnetosphere. We now discuss the simple case in which $\dot{\varpi}_{xy} = 0$. Since the second term on the right side of Eq. (6.3) is strongly size-dependent (see Eq. 5.40), near cancellation of the two terms will occur only for a narrow range of particle sizes. For the expected conditions in the E ring ($\Phi \sim -5$V, α=0.2 yr^{-1}), the critical grain size is $r_g = 1\,\mu$m, very similar to the size of the grains actually observed in the ring.

In connection with initial conditions, presuming the E-ring particles originate on Enceladus, we make three observations: *i*) the escape velocity from the satellite is probably less than 10^{-2} times the satellite's orbital velocity; *ii*) electromagnetic perturbations alone do not introduce large orbital velocity changes (Schaffer and Burns 1987); and *iii*) Enceladus' orbit is nearly circular. Accordingly we assume that the grain is launched at 3.95 R_S onto an approximately circular Keplerian orbit. From such a starting condition ($e \sim 0$), Eq. (6.2) shows that $\phi_\odot$ will swiftly turn to $\pi/2$ and then will stay locked; simultaneously, by Eq. (6.1), the eccentricity grows at the constant rate α (Horanyi *et al.* 1990).

Of course the eccentricity can only increase until the orbit intersects the outer edge of the A ring at $2.27 R_S$ where collisions with the opaque ring will eliminate the particle; written in terms of orbital eccentricity, this condition is $e_{coll} \approx 0.43$. (Naturally, this applies only to particles staying in the equatorial plane whereas below we will find that collisions with the main rings are less likely once the inclination is allowed to be non-zero.) According to Eq. (6.1), such an eccentricity will be achieved in a little more than 2 years. To summarize, one-micron particles injected at Enceladus with $\Phi \approx -5$ volts, will be rapidly dispersed owing to their eccentric orbits and then will be lost by collisions with the A ring. We must recall, however, that it is the fine balance between the perturbations due to oblateness and the Lorentz force that anchors the pericenter in this case, thereby allowing solar radiation pressure to induce large eccentricities.

For the general case, where $\dot{\varpi}_{xy} \neq 0$, one can most readily solve Eq. (6.1) and Eq. (6.2) by transforming to the variables $P \equiv e \sin\phi_\odot$ and $Q \equiv e \cos\phi_\odot$, which are found to describe simple harmonic oscillations. In terms of the original variables, the solution is

$$e = \frac{2\alpha}{\dot{\varpi}_{xy}}\left|\sin \; \frac{\dot{\varpi}_{xy}}{2}t \;\right| \tag{6.4}$$

$$\phi_{\odot} = \text{modulo} \;\; \frac{\dot{\varpi}_{xy}}{2}t, \pi \;\; + \frac{\pi}{2}, \tag{6.5}$$

assuming the initial condition $e(t = 0) = 0$. The eccentricity changes periodically as the pericenter moves at a constant rate from $\pi/2$ to $3\pi/2$ (for $\dot{\varpi}_{xy} > 0$), at which point $\phi_{\odot}$ jumps back to $\pi/2$ again (for a geometrical representation of this solution, see Horanyi and Burns 1991). The period of the eccentricity variation is $P = 2\pi/\dot{\varpi}_{xy}$ and the maximum eccentricity (within the approximation of small e) is $e_{max} = 2\alpha/\dot{\varpi}_{xy}$.

As seen in Eqs. (5.40) and (6.3), $\dot{\varpi}_{xy}$, and therefore e_{max}, is very sensitive to the grain size. For a specific particle size, one can compute the range of voltages that will produce precession rates such that e_{coll} is achieved. Larger voltages cause the Lorentz precession rate to dominate that from the planet's oblateness while for smaller voltages the converse holds; both cases mitigate the ability of solar radiation pressure to produce high eccentricities. Figure 6.2 displays the maximum eccentricity e_{max} achieved by particles of three sizes and various voltages near those of the nominal E-ring grains. Particles on 2-D orbits are lost to the main rings when the pericenter dips into the A ring, which occurs for $e_{coll} = 0.43$. As we will see below, three-dimensional orbits survive until the *orbital nodes* intersect the A ring (this always happens before the orbit intersects the planet) which occurs for $e'_{coll} = 0.65$ (Section 6.4). The curves to the left (right) of the flat tops in Fig. 6.2 correspond to $\dot{\varpi}_{xy} < 0$ ($\dot{\varpi}_{xy} > 0$). Because particles of different sizes are spread in such dramatically different ways, the population of grains that is present at the outskirts of the E ring could differ considerably from that introduced at Enceladus. As an illustration of this effect, consider a population consisting of three sizes (0.5, 1.0, and 1.5 μm) injected at Enceladus. The eccentricity histories of these particles, plotted in Fig. 6.3, differ significantly and the maximum values achieved are in agreement with Fig. 6.2.

An excellent test of our model can be made by the Cassini spacecraft which will carry out complete photometric observations of the E ring and in-situ detections of the dust particles composing it. These data sets will constrain particle size distributions across the E ring. Indeed, the importance of radiation pressure will be shown if a wide distribution of particle sizes is found to be present near the orbit of Enceladus but only a very selected size range is seen elsewhere. A more direct test involves using Cassini's dust detector to see whether the particles sensed at distances from Enceladus are on eccentric orbits.

We now compute the radial optical depth distribution due to grains moving on elliptical orbits. For diffuse structures like the E ring, the optical depth is proportional to the time a grain spends within any given radial interval, r to $r+\Delta r$, which in turn is inversely proportional to rv_r, where v_r is the average radial

Figure 6.2 The maximum eccentricity $e_{max} = 2\alpha/\dot{\varpi}_{xy}$ that is achieved according to Eq. (6.4), as a function of the assumed (constant) surface potential for various size grains (heavy lines) introduced at Enceladus (at $3.95 R_S$). The results from numerical integration are also shown (dashed lines); the differences between the curves at large eccentricities clearly signal the breakdown of the assumption that $e \ll 1$. The curves are truncated at $e'_{coll} = 0.65$, the eccentricity at which all three-dimensional orbits with $a = 3.95 R_S$ will intersect Saturn's A ring; particles confined to the ring plane will be lost once they reach e_{coll}=0.43 when the orbital pericenter dips into the outer A ring.

Figure 6.3 History of orbital eccentricities for 0.5, 1.0, and 1.5 μm particles evolving under oblateness, electromagnetism, and radiation pressure as they move about Saturn with orbital semimajor axes of $3.95 R_S$. In each case, particles are taken to be icy spheres of density 1.0g/cm^3 at a potential of -5Volts. For 0.5 μm particles, the Lorentz force dominates orbital precession and the orbit spins too rapidly for radiation pressure to create substantial eccentricities. Similarly, for 1.5 μm particles, oblateness dominates and the orbit precesses swiftly in the opposite direction with the same outcome. For 1.0 μm particles, however, the Lorentz and oblateness precessions largely cancel, allowing radiation pressure to greatly perturb the orbital eccentricity; note that a single eccentricity oscillation occurs in one precession period. The symbols D, T, and M identify the orbital eccentricities at which particles launched from Enceladus will cross the orbits of the satellites Dione, Tethys, and Mimas, respectively.

velocity over the interval considered; the extra r in the denominator appears because the area of an annulus of width Δr over which these particles are spread is $2\pi r\Delta r$. In terms of the orbital elements the radial velocity can be written as

$$v_r = {\frac{GM_p}{a}}^{1/2} \frac{[a^2e^2 - (r-a)^2]^{1/2}}{r}. \tag{6.6}$$

The radial optical depth profile due to a single particle moving along a Keplerian orbit of a given eccentricity is then

$$\tau^e(r) = \frac{T_o}{[a^2e^2 - (r-a)^2]^{1/2}}, \tag{6.7}$$

where T_o is a normalization constant; clearly this is valid only for distances between the orbit's radial turning points [*i.e.*, for $a(1-e) \leq r \leq a(1+e)$]; elsewhere $\tau^e(r) = 0$. Fig. 6.4 plots Eq. (6.7) for several eccentric orbits; note the symmetry about $r = a$ and the enhanced optical depth at the orbital turning points.

A particle evolving under radiation pressure, however, does not have a constant orbital eccentricity as assumed immediately above, but by combining Eqs. (6.4) and (6.7), and integrating over a full cycle of the eccentricity variation, we find that a single particle contributes to τ as

$$\tau(r) = T_1 \int_0^{2\pi/\dot{\varpi}_{xy}} \tau^e(r)dt \ , \tag{6.8}$$

where T_1 is another normalization constant.

Equation (6.8) describes a distribution sharply peaked at the radial distance of the source itself (Fig. 6.5). This occurs because the particle *i*) spends substantial time at low eccentricity, and *ii*) even when at higher e, always passes twice through its initial radius on each orbit. We note that the optical depth distributions in Figs. 6.4 and 6.5 are each symmetric about the source's orbit despite the fact that each particle spends more time at apocenter of its orbit than at pericenter; this possibly counterintuitive result arises because the apocenter particles are spread over a proportionally larger annulus.

Figure 6.2 shows that our low-eccentricity approximation is reasonable for $e \lesssim 0.3$ but may not be good for larger eccentricities where nonlinear effects become important. In Fig. 5.5, for example, the sinusoidal oscillations of the eccentricity are noticeably distorted. To extend our results to higher eccentricities, we numerically integrate particle orbits and use these to infer the ring's optical depth as a function of radial distance. In order to construct these ring profiles, we followed grains of three characteristic sizes (0.5, 1.0, and 1.5μm) with identical initial conditions and noted their radial positions every 10 days. We then constructed radial optical depth profiles (Fig. 6.6) from the resulting orbits, normalizing the former in the same manner as in Fig. 6.5. For illustrative purposes,

Figure 6.4 The profile of optical depth vs. radius plotted for grains with orbits of semimajor axis = 3.95 R_S and various eccentricities. The curves, which are undefined at each orbit's pericenter and apocenter, are truncated there for clarity. The reason for the symmetry about $3.95 R_S$ is discussed in the text.

Figure 6.5 The radial optical depth contribution of a single particle during a full period of its eccentricity oscillation for $e_{max} = 0.3,\ 0.5$ and 0.7 (solid lines). These curves were constructed by first subtracting a constant value from the solution of Eq. (6.8) and then normalizing the peak at Enceladus' position to unity; this procedure is similar to the background sky subtraction performed on photographic plates. This normalization process causes the area under each curve to differ, but does preserve the symmetry around $r = a$ in each case. Also plotted (dotted line) is the inferred radial brightness distribution based on the observations and represented by two power-law drop-offs from Enceladus (Showalter *et al.* 1991).

Figure 6.6 The optical depth profiles (continuous lines) for grains of radii 0.5 (top), 1.0 (middle), and 1.5 (bottom) microns. All grains were given the same initial conditions, the orbits were sampled every 10 days for 90 years, and the curves were normalized as in Fig. 6.5. Also plotted for comparison are the Showalter *et al.* (1991) observations (dashed line). The plot clearly demonstrate the enhanced mobility enjoyed by the one micron-sized grains. The three maxima clustered near $4R_S$ in the central panel are due to the fact that the grain's orbital eccentricity does not decrease to exactly zero on every cycle (see second panel of Fig. 5.5).

we ignored possible collisions with the inner saturnian rings even though some of our orbits attain maximum eccentricities dangerously close to $e'_{coll} \simeq 0.65$. As with our analytic result (plotted in Fig. 6.5), the optical depths in the three simulations (Fig. 6.6) have sharp peaks near the source with steep drop-offs on either side.

The radial range covered by one-micron grains matches the observed width of the E ring well, arguing convincingly for a population of one-micron grains. Both our analytic and numerically derived optical depth profiles are symmetric about Enceladus' orbit, however, in contradiction to the asymmetry displayed by the observed ring (Showalter *et al.* 1992). We will return to address this point in Section 6.5.

6.3 zi ut al Structure: ccentricity an Solar ngle

6.3.1 Low Eccentricity Case

As discussed above, the large, almost-periodic variation in the eccentricity displayed in Figs. 5.4 and 5.5 is responsible for most of the E ring's structure. In contrast, the semimajor axis remains essentially constant and the inclination stays small. Due to the latter fact, substantial variations in Ω and ω do not significantly affect the radial or azimuthal structure of the ring. Furthermore, since the governing equation (Eq. 5.53) for eccentricity in this low-inclination limit depends only on the solar angle and the eccentricity itself, these two variables can be decoupled from the rest, as in Section 6.2. Accordingly we discuss the eccentricity and the solar angle in this section and the elements i, Ω, and ω in the next.

Ideally, we would like to find an exact solution for Eq. (5.53) and Eq. (5.58) valid for arbitrary eccentricities but, due to the presence of nonlinear $1 - e^2$ terms, we have been unable to do so. By contrast, for small eccentricities $\dot{\phi}_\odot$ is nearly a constant, and a solution, in which the eccentricity varies sinusoidally, can easily be found (Burns *et al.* 1979, Horanyi *et al.* 1992). In the present case, however, we are interested in highly-eccentric orbits and so are forced to content ourselves with a qualitative description of the orbital evolution based on these two equations. First, as predicted by Eq. (5.53) and seen in Fig. 5.5, the eccentricity always grows when $0° \lesssim \phi_\odot \lesssim 180°$ and shrinks when $180° \lesssim \phi_\odot \lesssim 360°$. When $\dot{\phi}_\odot \approx 0$, $\phi_\odot$ in Eq. (5.53) remains nearly constant, the elliptical orbit keeps a given orientation with respect to the Sun, and the eccentricity changes monotonically. In the low-eccentricity solution, an exact cancellation of the precession rates with an associated permanent growth of eccentricity is possible but, as seen in Eqs. (5.59) and (5.60), the rates actually depend on different powers of $1 - e^2$

which cause an imperfect cancellation as the eccentricity varies. At large e, these nonlinear effects are important and significantly influence the azimuthal structure of the ring.

In order to study the nonlinear effects, we must first understand the simple case when these terms are absent; this situation is approximated in Fig. 5.5 where e^2 is always relatively small. As Fig. 5.5 shows, at $t = 0$ the solar angle $\phi_\odot$ is immediately driven to 90° by radiation pressure; this occurs because, for small eccentricities, the final term in Eq. (5.60) dominates $\dot{\phi}_\odot$. After the eccentricity rises slightly, the final term is less important so that the solar angle *regresses* continually from $t = 0$ to $t \approx 8.5$ years under the gravitational and electromagnetic terms in Eqs. (5.59) and (5.60); in this example, the regression rate is nearly uniform because, for these relatively low eccentricities, nonlinear terms are small. The vertical "jumps" in $\phi_\odot$, an example of which occurs at $t \approx 5$ years in Fig. 5.5, are simply due to the fact that the angle is plotted modulo 360°. As soon as the solar angle crosses zero, the eccentricity begins decreasing until eventually it is sufficiently small that the final term in Eq. (5.60) dominates again. As before, this term attempts to drive $\phi_\odot$ to 90° causing the angle to become positive and the eccentricity to increase. The cycle repeats almost periodically with departures from periodicity arising from the sensitive dependence of $\dot{\phi}_\odot$ on e.

Because of the coupling between e and the solar angle, the largest eccentricities in Fig. 5.5 are attained when the pericenter of the orbit is pointed toward the Sun ($\phi_\odot = 0°$). At this time, apocenter is directed away from the Sun and, accordingly, particles reach their maximum distance from the planet in this direction [$r = a(1 + e)$ - see Fig. 5.1]. Thus, if the E ring were composed solely of such particles, it would be asymmetric in azimuth, extending further in the antisolar direction than in the solar direction. A less negatively charged grain or, alternatively, a slightly larger particle, would have an initially *precessing* solar angle so that the maximum eccentricity would occur when the apocenter of the orbit points toward the Sun ($\phi_\odot = 180°$). Since the true E ring is likely composed of an ensemble of grains with slightly different sizes, shapes, and/or charges, it will probably include both precessing and regressing orbits with some apocenters pointing toward and others away from the Sun. This ensemble predicts that the E ring will be shaped like a Saturn-centered ellipse, extending to equal distances in the solar and antisolar directions and less far in the perpendicular directions. Figure 6.7 plots such an ensemble.

A fore-aft bulge of this type could not be identified in the available Voyager images (M.R. Showalter 1991, private communication). In addition, inbound - outbound differences in Voyager plasma absorption detections, which have been interpreted as caused by an asymmetric E ring (Sittler *et al.* 1981), could not be due to the E ring studied here because our particles are too small and too widely separated to be effective absorbers. Although such a distribution would minimize the ring's apparent radial extent as viewed from Earth, little or no asymmetry

Figure 6.7 The Saturn-centered ellipse. The Sun lies off along the negative x-axis and Saturn (not to scale) is at (0,0). To form this plot, we calculated the orbits of two different grains (0.97 μm and 1.2 μm, each charged to -5.5Volts) and plotted them together on this figure. The sizes were chosen so that the smaller particle's solar angle regresses while the larger particle's precesses; each attains a similar maximum eccentricity ($\sim$ 0.4). Each grain's orbit lies within a circular shaped region with its center offset from Saturn. Large grains extend further toward the Sun while small grains are found preferentially on the far side of Saturn.

would be visible to terrestrial observers since Saturn's phase angle (Earth-Saturn-Sun angle) cannot exceed six degrees. We emphasize, however, that these results do not include the effects of $1 - e^2$ terms that we now consider.

6.3.2 High Eccentricity Case

To demonstrate how larger eccentricities produce nonlinear effects, we consider the orbital evolution of a 1 micron grain charged to -5.4 Volts (as opposed to -5.6 Volts for Fig. 5.5); all other initial conditions as well as the operating forces remain unchanged. The resulting orbital evolution, obtained from numerical integrations of Eqs. (5.52–5.56), is displayed in Fig. 6.8. The only difference between the two cases is the slightly altered grain charge, yet striking dissimilarities are apparent in both the eccentricity and solar angle traces. In Fig. 6.8, the solar angle initially regresses as it does in Fig. 5.5, but the regression is slightly less rapid; this allows the eccentricity to grow large enough to reverse the sign of $\dot{\phi}_\odot$ before the solar angle dips below zero. As a result, $\sin\phi_\odot$ is larger for a longer period of time permitting the eccentricity to increase substantially. The augmented eccentricity causes the solar angle to *precess* through 180^o, at which point the eccentricity finally begins to decrease and a cycle similar to that discussed above is established. Azimuthal asymmetry arises because the stronger $1-e^2$ dependence of the gravitational precession terms in Eq. (5.59) and (5.60) causes the orbit to *precess* for large e which always leads to a maximum extension in the solar direction ($\phi_\odot = 180^o$). Although for this orbit, e is large enough that the grain would actually be lost to the main saturnian ring system, the orbital evolution displayed here is typical for a large range of similar initial conditions.

We now summarize the relevance of these results to the E-ring problem. Consider an ensemble of grains with slightly different sizes and voltages, but all launched from Enceladus on initially circular orbits. A fraction of the grains in this ensemble will have $e_{max} \lesssim 0.4$; these will be relatively uninfluenced by the nonlinear $1-e^2$ terms and will lead to a "Saturn-centered ellipse" like Fig. 6.7. In addition, however, our ensemble will contain grains that achieve large eccentricities. These particles all precess eventually (the orbit in Fig. 5.5 almost attains the largest eccentricity possible with a strictly regressing solar angle), and so the maximum eccentricity always occurs when the apocenter is pointed toward the Sun. Furthermore, because precession is rapid for very large eccentricities, the most elongated orbits sweep through a large range of pericenter angles (see the solar angle and eccentricity panels between $t = 6$ and $t = 7$ years in Fig. 6.8), resulting in distant particles in all directions on the sunward side of Saturn. Since the real ring contains both dynamical classes, an Enceladus-derived E ring might be expected to extend ~ 1.2 times as far in the solar and perpendicular directions as in the antisolar direction. In contrast to low eccentricity, this distribution displays nearly its full radial extent when viewed from Earth.

Figure 6.8 Osculating orbital elements plotted against time from integrations of the orbit-averaged equations of motion; these agree well with full Newtonian integrations (not shown). The same forces operating in Fig. 5.5 are present here, and the initial conditions are identical to those in Fig. 5.5 except the grain's voltage has been changed slightly to -5.4 Volts. This small change in the voltage decreases the strength of the electromagnetic force ($L = -0.00284$) which in turn, changes the precession rates to $\dot{\Omega}_{xy}(e \sim 0) = -338^{o}$/year and $\dot{\omega}_{xy}(e \sim 0, \phi_{\odot} = 90^{o}) = 315^{o}$/year; all other quantities retain the values noted in Fig. 5.5's caption. These slightly different precession rates drastically affect the eccentricity history.

The above discussion explicitly assumes that the charge on a grain remains constant throughout its orbital evolution. Could a varying particle charge disrupt the behavior seen here? Realistically, small rapid fluctuations in a grain's voltage occur as the grain's position in the magnetosphere (where plasma densities and temperatures might vary) and its velocity relative to the plasma change. A delay in the response of the grain's voltage to local conditions can affect long-term evolution of semimajor axes (Burns and Schaffer 1989), but over the short times considered here, this process is unlikely to be important. Because the charge fluctuations are fast compared to the orbital period, however, they should be treated before averaging the perturbation equations over an orbit. As argued above, this will not seriously influence the orbital semimajor axis and eccentricity. The inclination and precession equations, however, are more strongly affected. A difference in the inclination equations only adjusts the magnitude of Z (Eq. 5.61), however, which does not seriously alter the behavior of Eqs. (5.54–5.56). Slightly different electromagnetic precession rates would still cancel the gravitational rates, although at a minutely different grain size. Most importantly, the $1 - e^2$ dependence of the electromagnetic precession rate could be changed significantly (exponent < -2); in this case, the nonlinear effect that favors an E ring with a minimum extension in the antisolar direction would be reversed; the E ring would then have its small dimension in the solar direction. In any case, an asymmetry of some sort is likely to persist.

Finally we point out that the surface brightness of the E ring depends not only on these dynamic considerations but perhaps even more on the distribution of particle sizes and shapes present in the E ring. This distribution determines the number of grains in each of the dynamical classes discussed two paragraphs back. If orbits with lower eccentricities (Fig. 6.8) are most prevalent, for instance, then the surface brightness will be dominated by the "Saturn-centered ellipse." Whatever the size distribution though, the E ring's surface brightness should display measurable azimuthal asymmetry.

6.4 ertical Structure: Inclination, No e, an ericenter

Having completed our discussion of the components responsible for azimuthal variations, we now focus on the smaller perturbations to the E ring's vertical structure. These perturbations arise from weak normal forces which influence only the elements i, Ω, and ω. We start by discussing Figs. 5.5 and 6.8, simulations that do not include the effects of the aligned quadrupole, for simplicity (*i.e.*, $g_{2,0}$ is artificially set to zero), although our derivations are general and will allow us to return to the important influence of the quadrupole term shortly. Perhaps the most unusual behavior displayed by the elements i, Ω, and ω in Figs. 5.5

and 6.8 is the fact that the argument of pericenter locks, alternately to $\omega = +90^{\circ}$ when the physical location of pericenter is above the equatorial plane (Fig. 5.2), and to $\omega = -90^{\circ}$ when pericenter is below the plane. This locking is correlated with the solar position such that the orbital pericenter is always displaced to the same side of the equatorial plane as the Sun. In all figures, the Sun starts at its maximum elevation above the equatorial plane (the summer solstice in Saturn's northern hemisphere) and remains above the plane for one quarter of its orbital period of $\sim$ 29.5 years crossing the equatorial plane at $t \approx 7.4, 22.1$, and 36.9 years.

At first sight this locking may seem unimportant: since inclinations are small, what difference does it make that pericenter is always elevated out of the equatorial plane by a few tenths of a degree? There are several answers to this question. First, since these orbits periodically attain highly-eccentric orbits, an E-ring particle can dip in very close to the main saturnian ring system. Because the main rings are so thin (Cuzzi *et al.* 1979, Sicardy *et al.* 1982), however, even small inclinations cause E-ring grains to rise well above the main rings and hence collisions with these rings can only take place at orbital nodes. Locking the pericenter to $\pm 90^{\circ}$ puts both the nodes along the latus-rectum of the ellipse (Fig. 5.1), maximizing the ability of an orbit of a given eccentricity to avoid intersecting the inner rings. Such orbits can spread E-ring material across the maximum radial range. A collision with the A ring is inevitable when $a(1-e^2) = 2.27 R_S$ from which, for $a = 3.95 R_S$, $e'_{coll} \simeq 0.65$; this always occurs before collision with Saturn [$a(1 - e''_{coll}) = 1$ or $e''_{coll} \simeq 0.75$.]

Additionally, pericenter locking alters the probability for an impact into a saturnian satellite since most moons lie at low inclinations relative to Saturn's equator. Most notably, this phenomenon enhances the probability of reimpact into Enceladus since an E-ring particle's node lies at a radial distance $a(1-e^2)$ which, for small e, is very close to Enceladus' orbit at $r = a$. Finally, and perhaps most interestingly, this dynamical effect suggests that the vertical structure of the E ring is time-variable over a single orbit of Saturn around the Sun. Before discussing the ramifications of this time variability, we wish to understand the locking analytically.

The behavior of ω suggests that the angle is attracted to a stable equilibrium point, and so we seek such a solution. First, however, we note that there are several places in Fig. 6.8 (*e.g.,* near $t = 7, 13, 22$ ... years) where the argument of pericenter is not strongly locked to its equilibrium value; in these locales, oscillations in ω are large and circulation can occur. These deviations happen either when the Sun passes through the equatorial plane (roughly every 15 years) and the argument of pericenter begins its transfer from one equilibrium value to the next or when the orbital eccentricity is small, in which case the pericenter is poorly defined and can circulate rapidly as predicted by the final term in Eq. (5.60). To avoid these problems, we choose to initially study the locking effect for non-zero and constant values of e and s_z, ignoring the time-dependence of these parameters. We will return to justify and relax this approximation shortly. Setting $\omega_{eq} = \pm 90^{o}$ (the subscript "eq" stands for equilibrium) and remembering that inclination must be positive, we find that Eq. (5.56) is zero only when

$$\sin i_{eq} = \left| \frac{Z}{\dot{\omega}_{xy}} \right| \tag{6.9}$$

which, from Eq. (5.55) leads to

$$\left. \frac{d\Omega}{dt} \right|_{eq} = \dot{\Omega}_{xy} + \dot{\omega}_{xy}. \tag{6.10}$$

Finally, setting Eq. (5.56) equal to zero and utilizing Eq. (6.9) yields an improved determination of ω_{eq}:

$$\sin \omega_{eq} = \mathrm{sign}(\dot{\omega}_{xy}/Z). \tag{6.11}$$

We check the solution given by Eqs. (5.54–5.56) for stability by linearizing it about the equilibrium point. Here we set $\chi = \chi_{eq} + \Delta\chi$, where χ is any of i, Ω, and ω, to find:

$$\frac{d\Delta i}{dt} = -i_{eq}\dot{\omega}_{xy}\Delta\omega, \tag{6.12}$$

$$\frac{d\Delta\Omega}{dt} = -\frac{\dot{\omega}_{xy}\Delta i}{i_{eq}}, \tag{6.13}$$

$$\frac{d\Delta\omega}{dt} = \frac{\dot{\omega}_{xy}\Delta i}{i_{eq}}, \tag{6.14}$$

which can be trivially solved to yield:

$$\Delta i = i_{eq}\omega_0 \cos(\dot{\omega}_{xy}t + \zeta_0), \tag{6.15}$$

$$\Delta\Omega = -\omega_0 \sin(\dot{\omega}_{xy}t + \zeta_0), \tag{6.16}$$

$$\Delta\omega = \omega_0 \sin(\dot{\omega}_{xy}t + \zeta_0), \tag{6.17}$$

where the initial conditions ω_0 and ζ_0 are independent of i, Ω, and ω. Thus oscillations about the equilibrium point are stable and have frequency $\dot{\omega}_{xy}$ which, for the parameters of Fig. 6.8, corresponds to a period of ~ 1 year. The fact that the oscillation period is short compared to the characteristic periods of e and s_z justifies our earlier treatment of these latter parameters as constants; since e and s_z both change slowly with time, the rapid oscillations are able to stay centered on the slowly drifting equilibrium value. These results, Eqs. (6.9–6.17), seem to be in good agreement with Figs. 5.5 and 6.8. Eq. (6.11) correctly predicts that pericenter and the Sun always lie on the same side of the equatorial plane since $\dot{\omega}_{xy} > 0$ and, with no quadrupole term, Z changes sign every time the Sun crosses the equatorial plane. Furthermore, Eq. (6.9) shows that the inclination approaches zero when Z is small which occurs either when the Sun is in the ring plane or when $e \to 0$, as we already inferred from Figs. 5.5 and 6.8. Additionally, several features of Eqs. (6.15–6.17) can be checked against the full numerical integrations. As expected, the oscillations in all three elements have eccentricity-dependent periods of approximately one year and, as predicted by Eq. (5.60), this period decreases for large eccentricities (the inclination trace in Fig. 6.8 provides a nice example). Furthermore, since no discernible oscillations appear in the solar angle, which is basically the sum of Ω and ω, the oscillations in these angles must be equal in magnitude and 180° out of phase as predicted by Eqs. (6.16) and (6.17). Additionally, we find that the i oscillations peak one-quarter of a period before the ω oscillations as predicted by Eqs. (6.15–6.17), although the phase difference is difficult to detect in these figures.

We now construct vertical profiles in the same manner as the optical depth profiles (Fig. 6.6) of Section 6.2 and display the results in Fig. 6.9. The characteristic wedge-shape of each plot is due to pericenter locking which keeps orbital nodes near Enceladus. By definition, vertical offsets are minimum near orbital nodes and hence the ring is thinnest there. The radial dependence of the ring thickness from our simulation for one-micron grains (Fig. 6.9) qualitatively imitates Showalter *et al.* (1991)'s interpretation of the Baum *et al.* (1981) ground-based observations described in Section 6.1. Like the actual E ring, our model for solely one-micron grains has a greater thickness at its outer edge than close to the planet, and is thinnest at its source. Although the relative proportions are roughly correct, the magnitude of the predicted thickness is ~ 10 times less than the observed thickness. In fact, the problem is even worse since these plots are time averages; a snapshot of the ring at a particular instant in time will find all orbital apocenters either above or below Saturn's equatorial plane and hence the ring will be less thick. We will have more to say about this discrepancy in Section 6.5.

We now add the effects of the aligned quadrupole term to the array of forces influencing the dust grain. Figure 6.10 shows the orbital history of a grain with identical properties and initial conditions as the particle in Fig. 6.8; the only change is that the magnetic field from which the Lorentz force is calculated now includes the aligned quadrupole component. The eccentricity and solar angle traces in Fig. 6.10 are basically unchanged from Fig. 6.8, thus the results of Sections 6.2 and 6.3 stand, but the i, Ω, and ω traces are substantially altered. Inclinations of nearly a degree (three times larger than in Fig. 6.8) are attained - the effects of the quadrupole term are definitely important for Saturn's E ring!

Figure 6.9 A scatter diagram in the $r = (x^2 + y^2)^{1/2}, z$ plane for the orbits discussed in Fig. 6.6. The vertical structure for the one-micron grains is similar to the structure displayed by the actual E ring, although the heights attained in our simulations are a factor of ~ 10 too small.

Figure 6.10 Osculating orbital elements plotted against time from integrations of the orbit-averaged equations of motion. Again, the results agree well with the full Newtonian integrations which are not shown. Initial conditions and numerical quantities are the same as in Fig. 6.8, but the additional effects of the aligned magnetic quadrupole have been included. Note the striking difference in the i and ω traces in the two figures. The magnetic field coefficients used for Saturn are $g_{1,0} = 0.2154$ gauss and $g_{2,0} = 0.0164$ gauss (*cf.* Connerney *et al.* 1984).

Furthermore, the pericenter favors locking to -90° over locking to 90°; this can be easily explained by considering how the addition of the quadrupolar term changes Z. Using the values given in the figure captions, we find that the second term in Eq. (5.61) is always negative and, for small e, its magnitude is less than the maximum value of the first. Thus Z will be predominantly negative and, as predicted by Eq. (6.11), ω will usually be found near -90°. When the Sun is high above the equatorial plane, however, Z is positive and ω locks to 90° as observed in Fig. 6.10. Since the two terms in Z have different eccentricity dependencies, the time spent with $\omega \sim 90^{\circ}$ will vary from one occasion to the next. This same sharp eccentricity dependence of the quadrupole contribution to Z is also responsible for the difference in maximum inclinations observed in Figs. 6.8 and 6.10.

If all E-ring particles originated from Enceladus and had parameters like those chosen for Fig. 6.10, we would expect that inner portions of the ring (the pericenter sides of instantaneously elliptical orbits) would be offset to the south of the equatorial plane when the Sun is not too far to the north. The outer portions of the ring, of course, would be offset in the opposite direction. There is not a single solar position at which orbits transfer from one equilibrium to the next; as the Sun rises in the northern sky, orbits with low eccentricities switch first, followed by those with greater eccentricities. In addition, a more realistic ensemble of different particle sizes and shapes would cause further smear in the time when orbits switch equilibria since α and L vary significantly with particle properties. So when the Sun is to the north of the equatorial plane, the situation is difficult to assess. Conversely, when it is to the south, Z is negative and all orbits in the ensemble should have their pericenters depressed toward the south. We see that this is indeed the case in Fig. 6.11 which, like Fig. 6.9, is a time-averaged plot.

The effects of different initial conditions and additional satellite sources for E-ring particles further complicates the issue; these factors can cause the initial conditions to be far from the equilibrium point. When this is true, the oscillations in ω can be large enough to cause circulation of that element and this washes out the asymmetry discussed above. Assuming that dust grains originate from satellites, they will always start on nearly circular orbits for which the equilibrium inclination is $i_{eq} = 0$ (Eqs. 5.61 and 6.9). Several effects can cause initial inclinations to differ from zero most notably the small underlying inclination of the source satellite itself, and the dispersion of grain launch velocities. We find, numerically, that initial inclinations of more than about 0.5° for grains launched from either Enceladus or Tethys cause oscillations large enough to destroy the locking. This cutoff can also be found analytically from Eq. (5.56). It is reasonable to assume that most of the grains escape from their source moon with the minimum possible energy; in this case escape will occur along the Saturn-satellite line (*cf.* Figs. 2.10 and 2.11) with minimal change to the initial inclination. The orbits of Enceladus and Dione are negligibly inclined, but those of Tethys and

Figure 6.11 Vertical scatter plot. The actual three-dimensional path traced out by the particle of Fig. 6.10 has been collapsed into this two dimensional figure. Note that vertical structure has been dramatically altered from that displayed in Fig. 6.9's central panel. The asymmetry arises from Saturn's non-zero symmetric quadrupolar term ($g_{2,0}$).

Mimas have inclinations that exceed a degree; thus, nominally, grains launched from Enceladus will have their pericenters locked while those from Tethys will not. Summing the contributions of several source satellites and different initial conditions complicates the picture, but we believe that some vertical asymmetry and time-variability are likely to remain.

6.5 vi ence for itional Satellite Sources

The main problem with our simple model, which assumes a single source at Enceladus, is the fact that we find, in contrast to the actual ring, a radial distribution that is symmetric about the source satellite (see Fig. 6.6). This symmetry is quite robust since it arises directly from the geometry of elliptical orbits (Section 6.2). One possible solution is that drag forces, which cause outward evolution of orbits, are responsible for the asymmetry. This is unlikely, however, since these forces operate timescales much longer than the typical lifetimes of E-ring grains (see Section 6.6.1). A more promising hypothesis is that there are additional sources of E-ring material further out in the ring. Besides Enceladus, the moons Mimas, Tethys, Dione, and the Lagrangian companions of the two latter satellites all lie within the E ring. Micrometeoroid collisions or impacts of E-ring particles themselves into the moons could loft material off these small bodies. Since micron-sized particles originating from nearby satellites will most likely have equilibrium potentials similar to that of grains from Enceladus, pericenter precession rates will match for particles similar in size to those considered here. As eccentricities grow and material spreads radially, these grains will merge with those emanating from Enceladus. These sources would create distributions similar to those in Fig. 6.5 but peaking at different distances from Saturn; the sum of these distributions would necessarily be asymmetric and might better match the observations.

Additional source satellites alleviate another problem, namely that material introduced from Enceladus cannot reach the outer limits of the known E ring ($8R_S$) because, with the orbit's fixed semimajor axis (see Eq. 5.52), any eccentric path that reaches beyond about $6.5R_S$ would also penetrate the opaque inner rings. Material from outer satellite sources, however, can easily reach the nominal outer limit of the E ring. Furthermore, in contrast to the linear model of Section 6.3.1, the nonlinearity of the orbit-averaged equations considered in Section 6.3.2 causes some orbits with nearly maximum eccentricities to be oriented perpendicular to the Sun-Saturn line. Thus our model suggests that the E ring, as seen from Earth, could display nearly its full breadth. This correction may be enough for a primary source of dust grains at Enceladus and a weaker source at Tethys or one of its Lagrangian companions ($a \approx 4.89R_p$) to account for the full width of the E ring as observed from Earth. This additional source at Tethys' distance is also consistent with the extra material seen in the vicinity of that moon (Showalter *et al.* 1991's Fig. 11).

Finally, the vertical structure discussed in Section 6.4 may provide further dynamical evidence for a secondary source of particles from Tethys. If Enceladus were the ring's only source, our numerical simulations would predict maximum thicknesses of about 7,500 km (if orbital pericenters are locked) and 15,000 km (if pericenters are not locked - see below). As noted above, however, the E ring is about 40,000 km thick at its outer edge, still quite a bit broader than our predictions based on Fig. 6.10. Grains launched from Tethys, however, attain inclinations of $\sim 2.5^{\circ}$ and, because of Tethys' relatively large orbital inclination, the orbital pericenters are not locked. When combined, these effects lead to a predicted thickness of $\gtrsim 40,000$ km at the outer edge of the E ring, a figure that is in agreement with the observations.

Could other mechanisms, most notably Lorentz resonances, provide the increase in thickness without an additional Tethys source? While most of the strongest Lorentz resonances lie very close to Saturn, we note that just interior to Enceladus there is an important second-order 3:1 resonance driven by the tilted dipolar field whose strength is proportional to $eig_{1,1}$ (Chapter 7). If we assume a 0.8° tilt in Saturn's magnetic dipole, which is that initially proposed by Ness *et al.* (1982) (*cf.* Acuña *et al.* 1983b), we find that the resonance is sufficiently strong to break pericenter locking for some orbits. Numerical simulations indicate that inclinations can subsequently be pumped up to a few degrees. Thus we conclude that while a Tethys source accounts nicely for the observed inclinations, the 3:1 Lorentz resonance acting on material launched from Enceladus may also be able to do so. In either case, however, the breadth of the E ring and its radial asymmetry about Enceladus' orbital radius still argue for a Tethys source.

6.6 onsequences of ig ly ccentric r its

6.6.1 Collisions with Embedded Satellites

After the preceding discussion of the detailed dynamics of individual grains launched from Enceladus, we turn now to the question of the ring's origin. Several mechanisms have been suggested for lofting small dust grains off Enceladus into orbit around Saturn, including volcanoes (Pang *et al.* 1984a,b) and/or geothermal activity (Haff *et al.* 1983), as well as the impact of a comet (McKinnon 1983). Evidence supporting either of the first two suggestions is scant; the Voyager spacecraft found no indications of volcanoes or geysers on Enceladus, although the satellite does have a relatively young surface ($\lesssim 1$ billion years). Furthermore, the suggestion that Tethys contributes material to the E ring is difficult for all of these models to accommodate, as it requires activity undetected by Voyager on *two* satellites in the former cases, and an improbable pair of cometary impacts in the latter. In the next few sections, we propose a self-sustaining model of the E ring which follows naturally from considering the consequences of highly-eccentric orbits.

The E ring shares the region between 3 and $8R_S$ with an ensemble of moons that travel along nearly circular paths (Table 6.1); accordingly, once the orbits of E-ring particles become moderately eccentric, they will cross the paths of these satellites (see Figs. 6.1 and 6.3). Given a satellite of radius R_{moon} on a low-eccentricity orbit at radial distance a_{moon}, a grain on a "crossing" orbit will strike the moon with an e-folding timescale of:

$$T_{col} \approx \pi(\sin^2 i_{dust} + \sin^2 i_{moon})^{1/2} \left(\frac{a_{moon}}{R_{moon}}\right)^2 \left(\frac{U_r}{U}\right) T_{orb}, \tag{6.18}$$

where $T_{orb} = 2\pi a_{dust}/v_{dust}$ is the dust grain's orbital period, a_{dust} is its semimajor axis, and v_{dust} is its orbital velocity (Öpik 1976). In Eq. (6.18), U is the relative velocity between the moon and the dust grain, U_r is its radial component, and the orbital inclinations i_{dust} and i_{moon} (Table 6.1) are measured relative to the plane of the main ring system. The ratio U_r/U is nearly independent of e_{dust} and, to within $< 20\%$, equals one. Typical times for dust grains launched from moons within the E ring to reimpact the source satellite are given in Table 6.1 assuming, for illustrative purposes, that $i_{dust} = 0.1^{\circ}$ and $a_{dust} = a_{moon}$. The value chosen for i_{dust} does not significantly affect collision timescales for Mimas and Tethys since these satellites are significantly inclined. Moreover, the orbital nodes of grains launched from the uninclined satellites (Enceladus, Dione) become locked near the radial position of the source (Section 6.4), making Eq. (6.18) somewhat of an overestimate. Since orbital locking tends to enhance impact probabilities onto the source satellites, we choose a value for i_{dust} that is somewhat lower than typical E-ring inclinations (see Fig. 6.10). Thus the entries in (Table 6.1) apply reasonably well to the actual E ring.

The albedo patterns of the saturnian satellites may support the notion that eccentrically-orbiting E-ring grains commonly strike these bodies. A distribution of eccentric orbits having Enceladus' semimajor axis will preferentially strike the leading (trailing) face of exterior (interior) satellites since collisions will occur near apocenter (pericenter). Hence, assuming that impacts cause surface brightening, one can explain why Tethys and Dione, satellites exterior to Enceladus, have brighter leading hemispheres while Mimas, which lies interior to Enceladus, has a brighter trailing hemisphere (Verbiscer and Veverka 1992). Erosional brightening of the leading hemisphere is consistent with an enhanced meteoroid flux to the front faces of the exterior satellites (Clark *et al.* 1986, Veverka *et al.* 1986, Buratti *et al.* 1990), but cannot account for Mimas' brighter trailing side. Furthermore, Enceladus itself is photometrically similar across diverse geologic zones suggesting the presence of a ubiquitous surface layer of micron-sized grains (Buratti 1988), perhaps due to a long history of sandblasting by E-ring material.

The grain-moon collision timescales in Table 6.1 are extremely rapid: Ence-

Table 6.1 Satellites within the E ring

Name	a_{moon} (R_S)	e_{moon}	i_{moon} (^{o})	R_{moon} (km)	ρ_{moon} (g/cm^3)	v_{escape} (km/s)	v_{moon} (km/s)	T_{col} (years)
Mimas	3.08	0.02	1.53	195	1.17	0.16	14.3	200
Enceladus	3.95	0.00	0.02	250	1.24	0.21	12.6	19
Tethys	4.89	0.00	1.09	525	1.26	0.44	11.4	98
Telesto (T+)	4.89	0.00	0.00	12	(1.0)	0.009	11.4	17,000
Calypso (T-)	4.89	0.00	0.00	12	(1.0)	0.009	11.4	17,000
Dione	6.26	0.00	0.02	560	1.44	0.50	10.0	19
Helene (D+)	6.26	0.01	0.20	16	(1.0)	0.012	10.0	51,000
Rhea	8.74	0.00	0.35	765	1.33	0.66	8.5	120

Physical and orbital properties of the satellites are from Burns (1986). The final three columns are calculated from $v_{escape}^2 = 2GM_{moon}/R_{moon}$, $v_{moon}^2 = GM_s/a_{moon}$, and Eq. (6.18), respectively; the mass of Saturn is $M_s = 5.688 \times 10^{29}$g. The mass densities of the leading and trailing Lagrangian companions of Tethys (T+, T-) and the leading companion of Dione (D+) are unmeasured.

ladus, immersed in the heart of the E ring, sweeps up the entire ring in a characteristic time of 20 years. Without a supply of new material, the E ring should have lost more than 50% of its mass in the interval between its discovery (Feibelman 1967) in 1966 and the Voyager fly-bys in 1981. Since it is unlikely that the E ring is disappearing so quickly, a mechanism that continuously replenishes the ring must exist. In particular, a burst of activity in the distant past, – through volcanism, geysers, or large impacts – is incapable of accounting for the E ring that we observe today. Whatever process creates E-ring particles must be occurring *now*.

6.6.2 Collisional Yield; a Self-Sustaining ing

What are the consequences of these frequent grain-moon collisions? First we note that impacts are energetic since dust grains on highly-eccentric orbits strike embedded satellites at large relative velocities. From expressions for the radial and tangential velocity components of an elliptic orbit, the relative speed between a particle traveling on a low-inclination, arbitrarily-sized, eccentric orbit and a moon moving along a circular, nearly equatorial path is approximately

$$v_{col} \approx e v_{moon}, \tag{6.19}$$

where v_{moon}, the orbital speed of the moon, is roughly 10km/s (Table 6.1). Remarkably, this simple expression is accurate to about 10% for particle orbits of all sizes and shapes as long as the collision does not occur too near an orbital turning point (Fig. 6.1). Owing to the large eccentricities of E-ring grains, collision velocities often surpass 5 km/s, a value far in excess of satellite escape velocities (Table 6.1). These hypervelocity impacts eject an amount of mass greatly exceeding that of the impactor (O'Keefe and Ahrens 1977) into circum-saturnian orbit where it merges with the E ring.

Since micron-sized impacts add material to the E ring, the ring may sustain itself with these collisions; only a small fraction of the collisional ejecta, however, is composed of the dynamically-favored, micron-sized grains. As Fig. 6.3 demonstrates, grains that fall outside this special size window never attain the high eccentricities necessary for energetic collisions; instead they eventually encounter the source satellite in low-velocity collisions that liberate little or no mass. Thus a self-sustaining E ring requires that, on average, the collision of a micron-sized grain must eject at least one micron-sized fragment.

Can a micron-sized impact accelerate a comparable-sized fragment to escape velocity? Clearly the answer to this question depends on the nature of the collision and on the escape velocity of the impacted satellite. The collisional fragments of interest are similar in size to the projectile and they must survive intact, which suggests spallation (Melosh 1989). Due to cancellation of the initial compressional and the reflected rarefaction stress waves, spall fragments are only lightly shocked and can exceed the projectile in size. A few experiments (Frisch 1990, 1991, Eichhorn and Koschny 1992), in which small hypervelocity projectiles collide with icy targets, yield large, rapidly-moving spall fragments. Unfortunately, these experiments give inconclusive answers to our question for Enceladus since the measured speeds of the fastest projectile-sized collisional fragments are similar to that satellite's escape velocity. Perhaps the micron-sized yield of a typical collision is unusually high owing to Enceladus' surface regolith of micron-sized debris (Buratti 1988).

Energetic grain-moon collisions are also suggested by the vast quantity of OH molecules observed in the E-ring region which seems imply a quantity of H_2O twenty times more than traditional sources can supply (Shemansky *et al.* 1993). The collisions that we have argued are capable of lofting a micron-sized object into space liberate many times that much mass in the form of water molecules and tiny aggregates. The E-ring source is 10-100 times more efficient than others (*e.g.,* micrometeoroid bombardment, sputtering) and can easily generate the observed population of OH molecules (Hamilton and Burns 1993a).

6.6.3 Collisions with ings

As noted above, some fraction of E-ring material will also interact with the interior G, F, and A rings of Saturn (Table 6.2) (Showalter and Cuzzi 1993, Showalter *et al.* 1992, Dones *et al.* 1993). The resulting collisions deplete the E ring and create many small ejecta fragments in the target rings. However, in addition to bombardment by E-ring grains, the rings of Saturn are also struck by interplanetary meteoroids. Which source dominates? We calculate that the mass flux of E-ring grains onto Enceladus exceeds the interplanetary flux (taken to be 4.5×10^{-17}g/cm^{2}s from Cuzzi and Durisen (1990) by 10^4 (Hamilton and Burns 1993a). At the F and G rings, where E-ring fluxes are reduced, this ratio drops to 10-100, and for the outer 100km of the A ring, to 1-10. Regions interior to the outermost A ring are shielded from E-ring grains, and so the interplanetary flux dominates there.

Most collisionally-dominated rings, like the main rings of Saturn, have power law size distributions with $q \sim 3.0$ (Zebker 1985). Interestingly, both the F and G rings display anomalously high q's, 4.6 and 6, respectively (see Table 6.2), suggesting an excess of very small dust particles. The dustiness of these rings may be augmented by high velocity impacts of E-ring motes into the dusty components of the F and G rings. Such collisions are catastrophic and act to steepen the size distribution. We suggest that the unique size distributions of the F and G rings are determined, in part, by the influx of E-ring grains.

The outer few hundred kilometers of the A ring are brighter and dustier than its inner parts (Dones *et al.* 1993). The additional flux of small particles to the exterior part of the ring may brighten large ring members (in the same manner

Table 6.2 Properties of the outer saturnian rings

Name	inner edge (R_S)	outer edge (R_S)	τ_{small}	τ_{large}	q	comment
A	2.03	2.27	$\lesssim$0.03	0.7	$\sim$3	$\sim$ 100m thick
F	2.32	2.32	0.1	0.02	4.6$\pm$0.5	core of $>$cm objects
G	2.75	2.88	2 e-6	2.5 e-8	6.0$\pm$0.2	core of $>$cm objects
E	$<$3.00	$>$8.00	1 e-5	?	-	peak at 3.95 R_S

Here τ_{small} is the optical depth in dust particles with radii in the micron and submicron range while τ_{large} is the optical depth in particles larger than 1mm. Distributions of particle sizes within a ring are usually well approximated by power laws of the form r_g^{-q}, where r_g is the particle size and q, the power law index, is listed for the A, F, and G rings. Saturn's E ring is not well represented by a power law; it seems to have a monodisperse size distribution (Showalter *et al.* 1991) that it seems to brighten Mimas, Enceladus, Tethys, and Dione) and/or augment the production of dust (as in the F and G rings).

6.6.4 Intraparticle Collisions

As mentioned above, the E ring gains mass during typical impacts of micron-sized grains with embedded satellites. Since the rate of mass increase is proportional to the number of ring members, the ring would increase in mass exponentially with time without a mechanism to quench this growth. Intragrain collisions, where the loss rate is quadratic in the number density of ring members, will eventually overwhelm a linear source and will stabilize the ring at a particular optical depth. If the E ring is marginally self-sustaining and intragrain collisions are catastrophic, then at steady state grain-grain and grain-moon collisions should occur with roughly similar probabilities. In the actual ring, the cross-sectional area of E-ring particles is a few times that of Enceladus, and so the intraparticle collision rate is similar to the grain-moon collision rate as expected.

6.6.5 Computer Simulations

In order to test whether the E ring might be self-sustaining, we use a computer simulation containing more sophisticated versions of the simple ideas discussed above. Our model includes all of the moons and rings listed in Tables 6.1 and 6.2, and considers a discrete spectrum of seven particle sizes ($0.4\,\mu$m - $1.6\,\mu$m in steps of $0.2\,\mu$m). Prior to running our models, we numerically followed the orbital histories of grains of all sizes launched from each moon, and recorded the average inclination, maximum eccentricity, and the period of the eccentricity oscillation (*cf.* Fig. 6.3). These three parameters are used to approximate the effects of orbital perturbations and hence serve as the dynamical inputs to our model.

The collisional yield for a hypervelocity impact into a moon depends only on the target's escape velocity and surface properties, and the impactor's kinetic energy. We assume that in order to send one projectile-sized fragment into space, the impact energy must exceed the kinetic energy of the escaping fragment by a factor of 100-400. We further consider that the amount of escaping ejecta scales with the impact energy and that intragrain collisions are entirely catastrophic. Finally, individual collision rates and yields are folded together with dynamical evolution to calculate a matrix of transition rates (*e.g.,* the rate at which $1.2\,\mu$m grains from Enceladus create $0.8\,\mu$m grains at Tethys). The differential equations governing the population of grains of given sizes associated with specific moons are then numerically integrated (*cf.* Colwell and Esposito 1990).

Table 6.3 shows the results of one of our simulations; we see that, as in the actual E ring, most particles in the micron-range are localized at Enceladus.

Enceladus is selected as the dominant source for several reasons: the non-zero inclinations of its neighbors substantially reduce their collision probabilities (Table 6.1); the large escape velocities of Tethys, Dione, and Rhea limit their collisional yields; and Mimas-derived particles are quickly lost to the inner saturnian rings. Besides mimicing the radial structure of the true E ring, our simulated ring has a mass and a peak optical depth within a factor of three of the observed values; this agreement reinforces our assertion that intraparticle collisions in fact determine these quantities. The results presented in Table 6.3 are typical; better apparent agreement could be forced by tweaking parameters.

The only element of our model that stands in contrast to observations is the size distribution which shows an excess of submicron grains rather than being monodisperse at $1\,\mu$m (Showalter *et al.* 1991). Such a result is not unexpected; it follows clearly from our assumptions that micron-sized impacts create equal amounts of mass in each size bin and that smaller grains are swept up at roughly the same rate as their more massive brethren. Furthermore, since a substantial population of submicron dust is observed in the F and G rings, it might also be expected in the E ring. Nevertheless, the presence of many small particles in the model's output disagrees with the most straightforward interpretation (Showalter *et al.* 1991) of Voyager observations, namely that the E ring has an appreciably lower optical depth in submicron particles than in micron grains (M.R. Showalter 1993, private communication). The above model is so successful in accounting for various features of the E ring and its embedded satellites that one might wonder how this discrepancy can be explained. Our model may incorrectly reproduce the actual ejecta size distribution or may underestimate loss rates for submicron grains. Furthermore, the interpretation of the observations assumes that particle sizes are uniform throughout the E ring (M.R. Showalter 1993, private communication) and yet we have seen that submicron dust remains confined to a narrow band around the source's radial location (Fig. 6.3). Perhaps, micron-sized dust appears to dominate because it alone is found across the entire E ring.

6.6.6 Implications for Other ings

In the previous sections, we have shown how dusty rings may be generated and have suggested that the E ring is one member of this class. High-velocity impacts into satellites sustain the E ring through the addition of ejecta, and ring particles are lost in catastrophic grain-grain collisions. The resulting steady-state ring has a calculable mass and optical depth that agree with the measured quantities. In general a self-sustaining ring of this type requires only *i*) advantageously-located source satellites that are proper sizes and *ii*) a mechanism for increasing ring-particle orbital eccentricities and thereby enhancing collisional yields. The small jovian satellites Metis and Adrastea satisfy these two conditions; thus the faint ring surrounding and inward of these two satellites may be similarly generated.

Table 6.3 Steady-state particle population for the E ring

	0.4 μm	0.6 μm	0.8 μm	1.0 μm	1.2 μm	1.4 μm	1.6 μm
Mimas	9.3 e+22	1.1 e+22	1.3 e+21	1.4 e+20	3.4 e+20	2.6 e+20	2.5 e+20
Enceladus	7.5 e+22	4.3 e+22	1.1 e+22	1.4 e+21	1.5 e+21	5.6 e+20	2.1 e+20
Tethys	6.0 e+22	9.8 e+21	2.1 e+21	2.1 e+19	3.8 e+18	3.6 e+16	7.2 e+14
Dione	4.9 e+21	4.0 e+20	5.9 e+19	1.1 e+16	2.1 e+14	3.2 e+13	2.5 e+12
Rhea	4.7 e+20	1.6 e+18	4.3 e+14	7.7 e+12	3.2 e+11	0	0
Total	2.3 e+23	6.4 e+22	1.5 e+22	1.6 e+21	1.8 e+21	8.2 e+20	4.6 e+20

The final, near steady-state population of a simulated E ring. Each size/moon bin was started with a population of 10^{18} particles and the ring was allowed to evolve toward collisional steady state for 500 years. Most changes in the population occurred over the first hundred years when grain-grain collisions were rare. The total volume of our steady-state E-ring model is equivalent to a sphere of radius 35 meters which corresponds to a maximum optical depth of about 30% that of the true E ring. The cumulative cross-sectional area of these ring particles is 1.1 times that of Enceladus. Although we neglect the Lagrangian companions of Tethys and Dione in this simulation for simplicity, their contributions may be important.

6.7 uture servations an re ictions

6.7.1 Ground-Based

The most favorable time for ground-based observations of the E ring occurs when Saturn's main rings, as seen from Earth, appear edge-on; this last occurred in 1979-80 and will next happen in 1995-96. At these times, the signal from the E ring is strong due to the long optical path length through the region, and scattered light from the main rings is dramatically decreased. Sensitive observations made during this period should be capable of extending the known inner and outer limits of the ring since these apparent boundaries (Table 6.2) are most likely due to the weakening of signal relative to background. This is especially true in the inner region where the bright glare from the main rings complicates interpretation. Our model predicts that material should be present inward to the edge of the A ring.

Because the predicted azimuthal structure of the E ring is symmetric as viewed from the Sun, it is very unlikely that any asymmetry will be seen from Earth. The vertical asymmetry should, however, technically be visible to terrestrial observers, although the magnitude of the effect may be too small to be noticeable. When the Sun is nearly in the ring plane, the quadrupole dominates pericenter locking and dust exterior to Enceladus' orbit ($a = 3.95R_p$) should be offset slightly to the north; interior to Enceladus it should be found slightly to the south (Fig. 6.11). The magnitude of the offset depends on the unknown properties of the ensemble of grains that make up the E ring; offsets should increase, however, with radial distance from Enceladus. Grains originating from Tethys, however, will be distributed more symmetrically about the equatorial plane and so the vertical offset may disappear at large distances where Tethys-derived grains predominate.

6.7.2 From Spacecraft

There are definite hints of vertical asymmetry from the Voyager fly-by missions. Showalter *et al.* (1991) cite evidence from Voyager images centered at about $4R_p$ for a northern offset of several hundred kilometers - larger than that expected by differences between the equatorial and Laplace planes. The offset predicted by Eqs. (6.9–6.11) is small at this distance because it is just outside the position where the orbital nodes lie. Since the Sun was elevated only $\sim 4^o$ north of the equatorial plane at the time of the fly-by, the quadrupole term should still dominate the solar term and material exterior to the nodes should be elevated slightly to the north as observed. In addition, Voyager 1 swept through the E ring at a distance of about $6.1R_p$, near the Dione "clear zone," and returned data from its PWS instrument, which was discovered to be sensitive to dust impacts. These data imply an offset to the south (W. Kurth 1992, private communication). Most of the material in this region probably originates from Tethys, in which case the orbital pericenters are not locked; thus we cannot easily predict the sense of the observed offset.

Questions about the sources of dust and possible asymmetries in the E ring's structure are difficult to answer from ground-based observations alone. Because single particle dynamics dominate collective effects in the E ring, detailed information on individual particle orbits, which can most easily be obtained from spacecraft observations, are desirable. The sources of E-ring material should be easily identified when the Cassini orbiter, with its sophisticated dust detector, arrives at Saturn and makes repeated passes through the region. The mission should also be able to determine the nature and extent of any azimuthal and vertical asymmetry.

r 7

s s[1]

7.1 Intro uction

Gravitational orbital resonances, in which the frequency of a perturbing force is commensurate with a natural orbital frequency, have fundamental importance in the solar system. Satellites resonate with one another as in the saturnian Mimas-Tethys and Enceladus-Dione pairs as well as the famous jovian Io-Europa-Ganymede triple. At resonant locations in the main rings of Saturn, satellites cause density and bending waves, and sometimes form gaps and ringlets. Some features in the saturnian rings have even been ascribed to tiny perturbations from axially asymmetric terms in the planet's gravitational field (Franklin *et al.* 1982, Marley and Porco 1993). Since gravitational resonances are so common in the solar system, might non-gravitational resonances also be prevalent? This is almost certainly true; however examples of such resonances will only be found by looking in the right places. Since non-gravitational forces can compete with gravitational ones solely when particles are small, we expect these resonances for particles with radii less than a few microns. The faint ring systems of the giant planets are composed primarily of tiny particles and so such locales are ideal sites to seek out signs of non-gravitational resonant interactions.

These signs are clearly present both in the main jovian ring (Burns *et al.* 1985) and in Saturn's E ring (Chapter 6). In the former location, Lorentz (electromagnetic) resonances, which arise from Jupiter's spinning magnetic field, are capable of pumping up the eccentricity and inclination of ring particles. In particular, the transition between the main ring and the vertically extended halo occurs at a location where the ratio of the orbital frequency to the planet's spin rate is nearly 3:2 (Burns *et al.* 1985). Particles drifting inward and across this strong resonant location increase their inclinations by a factor of several hundred (see Schaffer and Burns 1992). As we have seen in Chapter 6, the particles in

[1]This chapter is based on the paper: Hamilton, D.P. (1993), A comparison of Lorentz, planetary gravitational, and satellite gravitational resonances. *Icarus*, submitted.

Saturn's diffuse E ring are also in nearly resonant orbits although this time the driving force is radiation pressure instead of electromagnetism. Because E-ring orbits retain a given orientation with respect to the Sun for an extended period of time, radiation pressure is able to build up large orbital eccentricities and spread material across the full breadth of the E ring (Chapter 6).

Other non-gravitational resonances have also been identified, among them shadow resonances (Horanyi and Burns 1991, Mignard 1984) and resonant charge variations (Burns and Schaffer 1989, Northrop *et al.* 1989). In the former, conditions change in the planetary shadow (radiation pressure and the photoelectric current shut off) which occurs naturally once per orbit; such orbits are thus intrinsically resonant. Shadow resonances may be responsible for the strange azimuthal asymmetry seen in the main jovian ring and in its halo (for a description of the asymmetry, see Showalter *et al.* 1987). Resonant charge variations occur when the charge on a dust grain changes with a period that is commensurate with the grain's orbital period; the termination of the photoelectric current during shadow passage provides a simple example, while another depends on variations in the current flow to a grain as its position and velocity change along its orbit.

Because gravitational resonances have been extensively studied, it is valuable when studying non-gravitational effects to draw from the body of knowledge already amassed. Accordingly, the primary emphasis of this work is to explore the similarities of non-gravitational and gravitational orbital resonances by comparing and contrasting their structure and effects on orbiting particles. We choose to look at two different types of gravitational resonances – those due to an orbiting satellite and those due to the "lumpiness" of an arbitrarily shaped planet – and we pick Lorentz resonances both because of their importance at Jupiter and because of their similarity to gravitational resonances (Hamilton and Burns 1993b). In the interest of brevity, henceforth we adopt the following notation: LR = Lorentz resonance, SGR = satellite gravity resonance, and PGR = planetary gravity resonance. By comparing three different types of orbital resonances, we progress in understanding the traits that underlie all orbital resonances and those that are unique to particular ones.

A second goal of this chapter is the mathematical characterization of the Lorentz perturbation which is useful for several applications. As noted above, Lorentz resonances are known to play a key role in the jovian ring (Burns *et al.* 1985). They are also suspected of being important elsewhere, perhaps accounting for dust found over the Neptunian pole (Hamilton *et al.* 1992), causing larger inclinations in the saturnian E ring (Chapter 6), and accounting for curious phenomena at the corotation distance (Showalter *et al.* 1985). These resonances have been analytically treated by Schaffer and Burns (1987) and more recently by Schaffer and Burns (1992) who used a perturbed harmonic oscillator model of resonance. Here we instead follow the standard celestial mechanics approach; since gravitational perturbations have been treated in this way, similarities and differences between resonances might be more readily apparent. Furthermore, the celestial mechanics approach has several advantages over the harmonic oscillator approach, the most obvious of which is that the results of perturbations are described by slowly-varying orbital elements which allow graphic visualization of orbital evolution.

The importance of Lorentz resonances in many of the above applications remains speculative because resonant strengths are poorly known; indeed, even the structure of these resonances is not well understood. In Section 7.2, we attempt to rectify this situation by expanding the Lorentz force out to second-order in small quantities e and i. In Section 7.3, we compare SGRs, PGRs, and LRs and discuss underlying symmetries contained in their expansions. We add the important dissipative effects of drag forces in Section 7.4, following which we present our conclusions.

7.2 xpansion of ertur ing orces

7.2.1 Planetary Gravity

We begin by discussing perturbations to two-body motion arising from small deviations in a planetary gravity field. This well-studied problem shares many aspects with the Lorentz perturbation and, accordingly, facilitates our later discussion of that force. Because we consider only small perturbations, solutions to the full problem differ only slightly from the exact solution to the two-body problem. Accordingly, we make use of the orbital elements since these will change relatively slowly in time. The basic task then, is to write the perturbation in terms of osculating orbital elements so that the time rate of change of each of these elements can be determined. We now sketch the derivation following the comprehensive treatment of Kaula (1966).

Working in a planet-centered reference frame rotating at the planet's spin rate $\mathbf{\Omega}_p$, the gravitational potential Φ outside an arbitrarily-shaped body can be shown to satisfy Laplace's equation, $\nabla^2\Phi = 0$ (Danby 1988). The solution of Laplace's equation in spherical coordinates for a cylindrically-symmetric planet, is given by Eq. 5.1. For an asymmetric body, solving Laplace's equation leads to the standard spherical harmonic expansion of the gravitational potential:

$$\Phi = -\frac{GM_p}{R_p} \sum_{j=0}^{\infty} \left(\frac{R_p}{r}\right)^{j+1} \sum_{k=0}^{j} [C^*_{j,k}\cos(k\phi_R) + S^*_{j,k}\sin(k\phi_R)] P_j^k(\cos\theta), \qquad (7.1)$$

where, as before, G is the gravitational constant, M_p and R_p are the planetary mass and radius, and r, θ, ϕ_R are the usual spherical coordinates defined in the rotating frame. These coordinates can be translated into the non-rotating frame by the identity $\phi_R = \phi - \lambda'$, where $\lambda' = \Omega_p t$ is the longitude of a reference point on the rotating planet. The $P_j^k(x)$ are associated Legendre polynomials (Kaula 1966, Schaffer and Burns 1992). Finally, the coefficients $C_{j,k}^*$ and $S_{j,k}^*$ are dimensionless quantities whose values are set by the mass distribution within the planet. Note, however, that several conventions exist for normalizing the associated Legendre polynomials (Stern 1976); because the choice of normalization alters the numerical values of $C_{j,k}^*$ and $S_{j,k}^*$, care must be taken when these coefficients are evaluated. Kaula (1966) for instance, uses unnormalized polynomials in the main text, but quotes numerical values for the Earth (through $j = k = 6$, his Tables 3 and 4) in which a spherical harmonic normalization (his Eq. 1.34) has been used. To further complicate matters, the same polynomials arise when the magnetic field is expanded, but these are conventionally Schmidt-normalized which differs from both of the above choices (see Schaffer and Burns 1992). We choose to Schmidt-normalize the gravity coefficients to facilitate the comparison of PGRs and LRs, and place asterisks on the coefficients as a reminder of this unconventional choice.

The disturbing function, *i.e.,* the negative of Eq. 7.1 rewritten in terms of orbital elements, is found by converting the spherical coordinates to orbital quantities and substituting into Eq. 7.1; the relevant expressions, Eqs. (5.9–5.11), allow r, θ, and ϕ to be replaced by a, e, i, Ω, u, and ν. As in the previous chapters, a and e are the semimajor axis and eccentricity of the elliptical orbit, i is the orbital inclination, and Ω is the longitude of the ascending node; the argument of latitude, u, and the true anomaly, ν, vary rapidly and nonlinearly in time (Fig. 7.1). We therefore replace these latter two quantities with the longitude of pericenter ϖ, which changes slowly, and the mean longitude of the particle λ, which varies nearly linearly in time. In addition, this choice causes all reference angles to be measured from the same zero-point in space which makes the symmetries of the expansion most apparent (see Section 7.3.1 below). The elements employed in our expansions are therefore: a, e, i, Ω, ϖ, and λ.

We eliminate the argument of latitude with the expression

$$u = \varpi - \Omega + \nu \tag{7.2}$$

(Fig. 7.1), leaving only the true anomaly ν, which always appears inside trigonometric functions, to be translated. Because expressions relating $\cos\nu$ and $\sin\nu$ to trigonometric functions of the mean anomaly M are available (*e.g.,* Smart 1953, p. 41), we proceed by using multiple-angle identities to first write our series (Eq. 7.1) in terms of sums and products of $\cos\nu$ and $\sin\nu$. We do this using a symbolic algebra program (MACSYMA), although with care it can be done analytically (Kaula 1966). Next the substitutions for $\cos\nu$ and $\sin\nu$ are employed; these expressions are complex, involving Bessel functions and their derivatives, but can be reduced to the form $\sum_j B_j e^j \cos(jM)$ where the B_j are constants (Smart 1953, p. 41). These expressions converge only for $e < 0.66$, a constraint of little importance since most applications are to low-eccentricity orbits. Finally,

Figure 7.1 Orbital elements. The symbols A and P stand for apoapse and periapse, respectively, while AN and DN refer to the ascending and descending nodes. Longitude angles (*e.g.,* λ, ϖ, and Ω) are measured **from** a specified reference direction in space. Node angles (*e.g.,* Ω) are measured **to** the ascending node while arguments (*e.g.,* u and ω) are measured **from** this point. Similarly, pericenter angles (*e.g.,* ϖ and ω) are measured **to** periapse while anomalies (*e.g.,* ν) are measured **from** there.

we complete the transformation to our orbital elements by replacing M via the identity $M = \lambda - \varpi$.

The resulting expression is quite complex, containing products and quotients of infinite power series in the eccentricity. We simplify by formally multiplying and dividing the various series so that each term in the full expression contains only a single power of the eccentricity. Next, we replace all products and powers of trigonometric functions with multiple-angle expressions; these steps are computationally intensive and tedious, and therefore are best left to symbolic programs. The final result is the disturbing function, a series containing terms of the following form:

$$f(a, e, i, ...) \cos(A_\lambda \lambda + A_{\lambda'} \lambda' + A_\varpi \varpi + A_\Omega \Omega + A_0), \tag{7.3}$$

where the A_j are integer constants and f is a function of a, e, i and the field coefficients $C^*_{j,k}$ and $S^*_{j,k}$. Readers interested in more explicit analytic results for the disturbing function relevant to planetary gravity fields should consult Kaula (1966)'s Section 3.3.

We wish to compare these results with those that arise from the Lorentz force considered in the next section, but because a disturbing function cannot be defined for the Lorentz force, we must derive time rates of change of the orbital elements in both cases. These rates are obtained by inserting the disturbing function into Danby (1988)'s Eq. (11.9.9) which gives six new series, one for the rate of change of each orbital element, each of which contains terms of the form of Eq. (7.3). We use expressions for dn/dt, de/dt, di/dt, $d\Omega/dt$, $d\varpi/dt$, and $d\epsilon/dt$ where the mean motion is given by

$$n = \left(\frac{GM_p}{a^3}\right)^{1/2}. \tag{7.4}$$

The variable $d\epsilon/dt$ encapsulates all perturbative changes to a particle's orbital mean motion; it is equivalent to Danby (1988)'s $d\epsilon_1/dt$, and satisfies $d\epsilon/dt = d\lambda/dt - n$. To facilitate the comparison of inclination and eccentricity resonances, we Taylor-expand the six series in e and i and truncate so that only terms second-order in small quantities remain. Our results for selected quadrupole and octupole components of the planetary gravity field are presented in Table 7.1. Many of the patterns seen in Table 7.1 [*e.g.,* the similarity of the coefficients of the time rates of change of the eccentricity (inclination) and the pericenter (node)] follow from the fact that these expressions are derived from a single disturbing function.

7.2.2 The Lorentz Force

In addition to planetary gravity, a charged dust grain in orbit around a planet responds to the Lorentz force arising from the rotating magnetic field associated with the planet (Section 5.4.1). Close in, the magnetic field $\mathbf{B}$ rotates at the

Table 7.1 The second-order expansion of perturbations due to the $C^*_{2,2}$ and $C^*_{3,2}$ components of the planetary gravitational field. The first column contains the resonant argument, $\Psi = A_\lambda \lambda + A_{\lambda'} \lambda' + A_\varpi \varpi + A_\Omega \Omega$ [see Eqs. (7.3) and (7.8)]. When the disturbing function is expanded to second-order in the small quantities (e and i), dn/dt is given to second-order, de/dt and di/dt to first-order, and the angular quantities $d\Omega/dt$, $d\varpi/dt$, and $d\epsilon/dt$ to zeroth-order. The response for the $S^*_{2,2}$ and $S^*_{3,2}$ components (Eq. 7.1) are obtained from these by the transformation $C^* \rightarrow S^*$ and $\Psi \rightarrow \Psi - \pi/2$. Section 7.4.1 gives an example of how to use this table.

planet's constant spin rate $\mathbf{\Omega}_p$, and the Lorentz force is given by Eq. 5.29. Assuming that the magnetic field is evaluated in a current-free region ($\mathbf{J} \sim \nabla \times \mathbf{B} = 0$), the only remaining constraint that must be satisfied is Maxwell's equation $\nabla \cdot \mathbf{B} = 0$ (Stern 1976, Section 5.4.1). Taking $\mathbf{B} = -\nabla \Phi_{mag}$, we find that $\nabla \times \mathbf{B} = 0$ is automatically satisfied, and $\nabla^2 \Phi_{mag} = 0$ with solutions like Eq. (7.1) above. Hence

$$\mathbf{B} = -R_p \nabla \sum_{j=1}^{\infty} \left(\frac{R_p}{r}\right)^{j+1} \sum_{k=0}^{j} [g_{j,k} \cos(k\phi_R) + h_{j,k} \sin(k\phi_R)] P_j^k(\cos\theta), \tag{7.5}$$

which is merely a combination of Eqs. (5.26–5.28). Recall that the $g_{j,k}$ and $h_{j,k}$ are planetary magnetic field coefficients with units of gauss [Schaffer and Burns (1992) tabulate values for the giant planets and give additional references].

A measure of the relative strength of the Lorentz force is given by the parameter L defined in Eq. 5.40; the Lorentz force can be treated as a perturbation to gravity for grains satisfying $L << 1$. Assuming typical grain potentials of a few Volts (*e.g.,* Horanyi *et al.* 1992), this inequality translates to grains larger than several tenths of a micron in radius. For many applications, including the jovian ring (Showalter *et al.* 1987) and the saturnian E ring (Showalter *et al.* 1991), dust grains are inferred to be micron-sized and gravitationally dominated; hence a perturbation approach is appropriate.

Since the Lorentz force depends on velocity, it cannot be written as the gradient of a potential and thus an electromagnetic disturbing function does not exist. Therefore, in order to obtain the time rates of change of the orbital elements for a general force, we proceed as follows:

1. Resolve the force into three orthogonal components: one normal to the orbital plane, the second oriented radially, and the third perpendicular to the others.
2. Insert these components into the perturbation equations of celestial mechanics (*e.g.,* Danby 1988, Eq. 11.5.13).
3. Convert all quantities into orbital elements.

The first step has already been accomplished for the Lorentz force in Section 5.4.1 (Eqs. 5.32–5.34). Next we insert the expressions for B_r, B_θ, and B_ϕ from Eq. (7.5) into the force components which are in turn substituted into the perturbation equations.

Finally, we rewrite the perturbation equations in terms of our set of orbital elements; this step closely parallels that discussed above for the planetary gravity disturbing function. For each of the six perturbation equations, we first convert the spherical quantities (r, θ, ϕ) to orbital elements using Eqs. (5.9–5.11), after which we replace u and ν with Ω, ϖ, and M (see Eq. 7.2 and the following discussion). After simplification, we are again left with a series of terms of

Eq. (7.3)'s form. Our result for the response of a charged grain to magnetic dipole, quadrupole, octupole, and select higher-order terms, truncated to second-order in e and i, is given in Table 7.2. The $(g_{1,0} : \Psi = 0)$ and $(g_{2,0} : \Psi = 0)$ terms agree with low-order expansions of Eqs. (5.41–5.46) and Eqs.(5.48–5.50) as they should.

7.3 roperties of t e xpansions

7.3.1 Orbital Symmetries

Despite the fact that the expansions listed in Tables 7.1 and 7.2 arise from very different perturbations, remarkably similar patterns are evident in each case. For instance, in both expansions the power of the eccentricity in a given term is related to the coefficient of the pericenter angle, and the same holds for inclinations and nodes. Furthermore, in both cases, the coefficients of the angular quantities in every resonant argument sum to zero. These patterns are reminiscent of d'Alembert relations which constrain the form of SGRs, imposing symmetries that have been recognized for as long as the satellite disturbing function has been expanded. According to Brown and Shook (1933), the relation between pericenter coefficients and eccentricity powers was first discussed by d'Alembert (1754); a more complete list of symmetries present in the secular part of the disturbing function can be found in Applegate *et al.* (1986). We now present simple physical arguments for the origin of four of these symmetries; our first argument is not new (*e.g.,* Applegate *et al.* 1986, Message 1991), but we have found no reference for the following three. The constraints imposed by the symmetries are quite general, applying not only to SGRs, PGRs, and LRs, but to any orbital perturbation and, indeed, to any quantity that can be written in terms of orbital elements.

Any physical quantity, Q, (*e.g.,* a position, velocity, or perturbing force component, the disturbing function, a perturbation equation, etc.) that is expressed in terms of orbital elements can be written as a function of many variables,

$$Q = F(\chi_1, \chi_2, \chi_3, ..., \phi_1, \phi_2, \phi_3, ...), \tag{7.6}$$

some of which are longitude angles (ϕ_j) and some of which are not (χ_j). For the Lorentz perturbation (see Section 7.2.2 and Table 7.2), the set of χ_j include the quantities $\{a, e, i, L, g_{j,k}, h_{j,k}\}$ while the set of ϕ_j is simply $\{\Omega, \varpi, \lambda, \lambda'\}$. Since the longitudes are angular quantities, F must be periodic in each of them. A well-behaved periodic function can be expanded as a Fourier series in each of its cyclic variables; performing this expansion of Eq. (7.6) yields a series whose terms have the following form:

$$f(\chi_1, \chi_2, \chi_3, ...) \cos(\Psi), \tag{7.7}$$

Table 7.2 The second-order expansion of perturbations due to the Lorentz force with $\xi = n/\Omega_p$. All dipole, quadrupole, and octupole as well as a few of the important higher-order terms are given in separate subtables. The first column contains the resonant argument, $\Psi = A_\lambda \lambda + A_{\lambda'} \lambda' + A_\varpi \varpi + A_\Omega \Omega$ [see Eqs. (7.3) and (7.8)]. As with planetary gravity, we expand dn/dt to second order in e and i, de/dt and di/dt to first order, and the angular quantities $d\Omega/dt$, $d\varpi/dt$, and $d\epsilon/dt$ to zeroth order. By convention, the $h_{0,k}$ are taken to be zero; the response to the other $h_{j,k}$ terms can be obtained from this table by substituting $h_{j,k}$ in for $g_{j,k}$ and subtracting $\pi/2$ from Ψ. An example illustrating the proper use of this table for the $g_{3,3}$ component is given in Eqs. (7.26–7.28) of Section 7.4.1.

where the function f plays the role of an amplitude and

$$\Psi = A_0 + \sum_j A_j \phi_j. \tag{7.8}$$

This series is summed over all possible unique sets of integer A_j's. Now, although all quantities pertaining to an arbitrary orbit may be expressed in the general form of Eq. (7.7), the converse is not true; not all functions of this form represent valid physical quantities. We now discuss constraints on the form of Eq. (7.7) that **all** physical quantities must obey.

The first and best-known constraint arises from the fact that all longitude angles are measured from the same reference direction, or zero-point, in space (*e.g.,* Applegate *et al.* 1986). Because space is isotropic, the choice of reference direction is arbitrary, and hence its selection can in no way affect a given orbit or the perturbations acting on it. We choose a new zero-point of longitude by adding an angular quantity δ to each of the longitude terms and require that Eq. (7.6) be invariant under the transformation

$$\text{longitude angles} \rightarrow \text{longitude angles} + \delta \tag{7.9}$$

(Fig. 7.1). Since the invariance holds for arbitrary values of the variables χ_j and ϕ_j, the constraint applies separately to each term in the Fourier series. If Q is unaltered by Eq. (7.9), then combining Eqs. (7.7–7.9) yields:

$$f(a, e, i, ...) \cos(\Psi) = f(a, e, i, ...) \cos(\Psi + \delta \sum_j A_j). \tag{7.10}$$

Now since δ is arbitrary,

$$\sum_j A_j = 0. \tag{7.11}$$

Thus the longitude coefficients contained within each term of any physical quantity **must** sum to zero. Notice in particular that this rule is strictly obeyed by each term of the perturbation expansions listed in Tables 7.1 and 7.2.

Unlike the zero-point of longitude, the line of nodes for a given orbit is uniquely determined by the intersection of the orbital plane with a given reference plane. Nevertheless, it is an arbitrary choice to measure angles with respect to an orbit's ascending node rather than its descending node. If we adopt the unconventional choice of using the descending node, the following modifications must be made to the usual orbital elements:

$$\text{node angles} \rightarrow \text{node angles} + \pi$$

$$\text{arguments} \rightarrow \text{arguments} - \pi$$

$$i \to -i. \tag{7.12}$$

The first two transformations adjust the angles so that they are measured relative to the new reference point, the **descending** node (Fig. 7.1). As seen from the descending node, the orbit dips below the reference plane in the direction of orbital motion and thus the new inclination is negative. Since the transformation merely amounts to describing the same orbit from a different reference point, as with the zero-point of longitude, no analytic expression can depend on this choice.

In an entirely analogous manner, the line of apsides is determined for an eccentric orbit, but one can measure angles from either pericenter or from apocenter. By choosing to measure from apocenter, the usual orbital elements must be modified as follows:

$$\text{periapse angles} \to \text{periapse angles} + \pi$$

$$\text{anomalies} \to \text{anomalies} - \pi$$

$$e \to -e. \tag{7.13}$$

As with the node above, the first two transformations adjust angles so that they are measured relative to the new reference point (Fig. 7.1). The third transformation, in which the sign of the eccentricity is reversed, is necessary so that the transformed distance and velocity components along the elliptical orbit retain their original values. If Q is unaltered by the transformations, as are dn/dt, $d\Omega/dt$, $d\varpi/dt$, and $d\epsilon/dt$, then the following expressions constrain the form of Eq. (7.7):

$$f(a, e, i, ...) \cos(\Psi) = f(a, e, -i, ...) \cos(\Psi \ + \ \pi \ A_\Omega) \tag{7.14}$$

and

$$f(a, e, i, ...) \cos(\Psi) = f(a, -e, i, ...) \cos(\Psi \ + \ \pi \ A_\varpi), \tag{7.15}$$

which reduce to

$$f(a, e, i, ...) = (-1)^{A_\Omega} f(a, e, -i, ...) \tag{7.16}$$

and

$$f(a, e, i, ...) = (-1)^{A_\varpi} f(a, -e, i, ...). \tag{7.17}$$

When Q is de/dt [di/dt], it changes sign under the transformation Eq. (7.13) [Eq. (7.12)] and an extra minus sign appears on the left-hand side of Eq. (7.17)

[Eq. (7.16)]. Thus the function f is not arbitrary; indeed, it must be either even or odd in each of the variables e and i. Furthermore, the parity of f with respect to e or i determines the parity of the corresponding angular quantity's coefficient. This constraint is clearly evident for each of the entries in Tables 7.1 and 7.2; the time derivatives of the mean motion and the angular quantities obey Eqs. (7.16) and (7.17) while de/dt and di/dt differ by a minus sign). The symmetries also require that power series expansions of f contain all even (or all odd) powers of e and i, a fact that is apparent in high-order expansions of SGRs (Murray and Harper 1993), PGRs (Kaula 1966), and LRs (Hamilton, unpublished).

The final simple symmetry that we discuss arises from reflection of a system through the xy plane. Imagine working in a left-handed coordinate system in which angles are measured from the negative $\hat{\mathbf{z}}$ axis rather than the positive one (Fig. 7.2). The orbital elements are affected by the change; the ascending node of an orbit in the original xyz coordinate system becomes the descending node in the new system. Since the usual orbital elements include the longitude of the ascending node, changing to the new system necessitates adding π to angles that measure the location of the node and subtracting π from arguments measured from that location (*i.e.,* the first two lines of the transformation given by Eq. 7.12). With this transformation, we succeed in describing the same orbit from two different reference frames. For SGRs, the transformation must be performed on all satellite orbits and the requirement that Eq. (7.7) be unaltered by the transformation implies that the sum of the node coefficients must be even. Taken together with Eq. (7.16), this in turn implies the well-known result that no first-order inclination resonances exist for SGRs.

For PGRs and LRs, the situation is more complicated since the gravitational and magnetic fields must also be described in the new coordinate system. Indeed, $\hat{\mathbf{z}} \to -\hat{\mathbf{z}}$ implies $\hat{\theta} \to -\hat{\theta}$, $\theta \to \pi - \theta$, and $P_j^k(\cos\theta) \to (-1)^{j+k} P_j^k(\cos\theta)$ (Fig. 7.2). To retain the original configuration of the gravity field, the quantity $C^*_{j,k} P_j^k(\cos\theta)$ must be unaltered and hence we also change the field coefficients $C^*_{j,k} \to (-1)^{j+k} C^*_{j,k}$. With these transformations, we succeed in describing the identical problem from two different coordinate systems; and, as before, the results of a perturbation cannot depend on the choice of reference directions. For PGRs, f is proportional to one of the $C^*_{j,k}$ and if Q is invariant under the change of coordinate systems, then the constraint on Eq. (7.7) takes the form:

$$C^*_{j,k} \cos(\Psi) = (-1)^{j+k} C^*_{j,k} \cos(\Psi + \pi A_\Omega) \tag{7.18}$$

or $(-1)^{A_\Omega} = (-1)^{j+k}$. The above discussion applies equally well to LRs, but an additional minus sign is introduced when the final cross product in the Lorentz force is calculated in the left-handed coordinate system (Eq. 5.29). Summarizing our results for the three resonances, we have:

SGRs : Sum of node coefficients is always even;

Figure 7.2 An orbit seen from two coordinate systems. In the xyz system, AN marks the position of the ascending node since the polar angle θ decreases as the satellite moves away from this point (*i.e.,* the orbit ascends above the xy reference plane). As seen from the $xy(-z)$ system, however, the polar angle is $\pi - \theta$ and AN is the **descending** node since the polar angle **increases** in the direction of orbital motion.

$$\text{PGRs : j + k + A}_\Omega \text{ is always even;}$$
$$\text{LRs : j + k + A}_\Omega \text{ is always odd.} \tag{7.19}$$

Notice that the results for PGRs and LRs do not exclude first-order inclination resonances. In fact, first-order inclination resonances occur for PGRs when $j+k$ is odd and for LRs when $j+k$ is even (*cf.* $C^*_{3,2}$ term in Table 7.1 and $g_{2,2}$ term in Table 7.2).

The symmetries presented in Eqs. (7.9), (7.12), (7.13) and (7.19) are widely applicable. Besides constraining the form of the expansions of SGRs, PGRs, and LRs presented here, they apply directly to any form of an arbitrary perturbation (*e.g.,* each of the orbit-averaged perturbations given in Chapter 5). Moreover, the symmetries hold for all physical quantities that are written in terms of orbital elements which can be especially useful for spot-checking complicated expressions. For instance, the expansions for $\sin\nu$ and $\sin E$ in Danby (1988, p. 437) do not manifest the symmetry implied by Eq. (7.13) and hence cannot be correct; valid expressions can be found in Smart (1953).

7.3.2 The Hamiltonian

The properties discussed above are shared by all perturbations simply because of the nature of orbital elements. The resulting rules explain many of the patterns that are apparent in Tables 7.1 and 7.2. Additional similarities are present because in each of the problems there is a unique rotating frame in which the perturbation is constant in time; for SGRs the frame rotates at the angular rate of the perturbing satellite, while for PGRs and LRs, it rotates at the planetary spin rate. When expressed in this rotating frame, $\mathbf{F} = m\mathbf{a}$ contains both a centrifugal term and a Coriolis term. Nevertheless, a conserved quantity of the motion (energy) can be found by taking the dot product of the equation of motion with $\mathbf{v}_{rel}$ ($\mathbf{v}_{rel}$ is the velocity relative to the rotating field) and integrating over time; for SGRs, this procedure yields the classical Jacobi constant (Danby 1988, p. 253). To zeroth-order in the perturbing force, the conserved quantity H is given by:

$$H = -\frac{GM}{r} - \frac{1}{2}\Omega_p^2(x^2+y^2) + \frac{1}{2}v_{rel}^2, \tag{7.20}$$

where, for SGRs, Ω_p here and below is understood to be the mean motion of the perturbing satellite. The first term is the gravitational potential energy, the second is the potential corresponding to the centrifugal force, and the final term represents the particle's kinetic energy. Because of its perpendicularity to $\mathbf{v}_{rel}$, the Coriolis force does not contribute to Eq. (7.20). In applying Eq. (7.20) to SGRs, we neglect the small contribution of the perturbing satellite which is a good approximation when one is not too close to the satellite (*cf.* Roy 1978, p. 129). For PGRs, we neglect the higher-order gravitational coefficients which is a reasonable approximation. Finally, the Lorentz perturbation, like the Coriolis acceleration, is perpendicular to $\mathbf{v}_{rel}$ (see the discussion immediately prior to Eq. 5.29) and so its term disappears when dotted with the velocity, leaving the energy integral unaltered. This is true even if the particle's charge varies with time (Horanyi and Burns 1991). Thus Eq. (7.20) is the exact integral of the motion for the Lorentz perturbation. We now convert this constant of the motion into orbital elements to see how it constrains the form of our expansions. This conversion was first accomplished by Tisserand (Roy 1978, Eq. 5.50). We find

$$\frac{R_{syn}}{a} + 2\left(\frac{a}{R_{syn}}\right)^{1/2}(1-e^2)^{1/2}\cos i = C, \tag{7.21}$$

where R_{syn} is the radial position of synchronous orbit and C is a constant. We use

$$\Omega_p = \left(\frac{GM_p}{R_{syn}^3}\right)^{1/2} \tag{7.22}$$

and Eq. (7.4) to replace the distances in Eq. (7.21) with frequencies. Since we are interested in expressing the constraint in terms of our derived time rates of change, we differentiate and obtain

$$\frac{1}{n}\left(\frac{n}{\Omega_p}-(1-e^2)^{1/2}\cos i\right)\frac{dn}{dt}-\left(\frac{3e\cos i}{(1-e^2)^{1/2}}\right)\frac{de}{dt}-3(1-e^2)^{1/2}\sin i\,\frac{di}{dt}=0 \tag{7.23}$$

which, to lowest-order in e and i, reduces to

$$3ne\frac{de}{dt} + 3ni\frac{di}{dt} + \left(1-\frac{n}{\Omega_p}\right)\frac{dn}{dt} = 0. \tag{7.24}$$

Equations (7.23) and (7.24) provide a link between variations in a, e, and i which can be used in a number of applications. For example, Burns and Schaffer (1989) and Horanyi and Burns (1991) have used planar versions of Eq. (7.23) in electromagnetic problems to obtain de/dt when da/dt (or dn/dt) is known, while Schaffer and Burns (1992) were the first to apply a variant of Eq. (7.24) to elucidate properties of Lorentz resonances. The expressions can also be used to check derivations; the orbit-averaged electromagnetic expressions (Eqs. 5.41–5.43), for instance, obey Eq. (7.23) as they must. Indeed, it is not difficult to see what the eccentricity counterpart to Eq. 5.48 must be. We now discuss how Eq. (7.23) constrains the form of our expansions given in Tables 7.1 and 7.2.

For any orbit, the changes in the orbital elements imposed by the full perturbation must satisfy Eq. (7.23). In general, many terms add together to produce these changes, but at resonant locations the effects of a single term dominate all others. At these locations, the resonant term itself must obey Eq. (7.23), but elsewhere it need not. The expansion of PGRs (Table 7.1) illustrates this property nicely; only at resonance, where $n/\Omega_p \approx |A_{\lambda'}/A_\lambda|$, do single resonant terms satisfy Eq. (7.24). The situation for Lorentz resonances is even simpler. As can be seen in Table 7.2, each term satisfies Eq. (7.24), regardless of the value of $\xi(= n/\Omega_p)$, and thus the cumulative perturbation automatically does too.

7.3.3 Additional Patterns

In the previous few sections we have discussed simple physical ideas that put strong constraints on the form of **all** resonances; here we investigate rules of a more limited scope. Some of these apply to just one type of resonance while others follow from mathematical properties of the expansions rather than from simple physical arguments.

Several additional physical rules further constrain the form of Lorentz resonances. First, the Lorentz force must vanish for a circular uninclined orbit at the synchronous distance, since there the velocity relative to the magnetic field is zero (Eq. 5.29). This fact is reflected in the expansion of Table 7.2; all dn/dt, de/dt, di/dt and $d\epsilon/dt$ terms disappear in the limit $n \to \Omega_p, e \to 0, i \to 0$ (see Section 5.4.2). The $d\Omega/dt$ and $d\varpi/dt$ terms need not vanish in this limit as these orbital elements are undefined for planar and circular orbits, respectively (Section 5.4.2). Furthermore, consideration of Eqs. (5.29) shows that the Lorentz expansion splits into two pieces, one arising from the $\mathbf{v} \times \mathbf{B}$ component of the force (ξ terms in Table 7.2), and one due to $(\mathbf{\Omega}_p \times \mathbf{r}) \times \mathbf{B}$ (constant terms in Table 7.2). Since the $\mathbf{v} \times \mathbf{B}$ force can do no work in the non-rotating frame, the orbital energy, and hence dn/dt, is unaltered. Thus there are no ξ terms in Table 7.2's dn/dt entries.

Some patterns can best be explained mathematically. One such regularity seen in both Tables 7.1 and 7.2 is that the powers of the eccentricity and inclination in the dn/dt equation equal or exceed the arguments of the corresponding angular quantities in Ψ. This property can be shown to be true by carefully following through the expansion of the perturbing forces; it stems from the fact that each appearance of a ν or u is accompanied by an e or i respectively. Furthermore, the structure of the perturbation equations (Danby 1988, Eq. 11.5.13) also insures that the power of e in the de/dt and $d\varpi/dt$ equations are, at most, one and two lower than A_ϖ while the power of i in the di/dt and $d\Omega/dt$ terms follow the same pattern with respect to A_Ω. Finally, the fact that the numerical coefficient in the de/dt and $d\varpi/dt$ (di/dt and $d\Omega/dt$) terms are usually identical, to first order, also follows from the structure of the perturbation equations.

For typical resonant arguments, the equality in the patterns discussed in the above paragraph holds exactly. The only exceptions are resonances at synchronous orbit which have arguments of the form $A\lambda - A\lambda'$. Additionally, these strange resonances are the only ones that influence the $d\epsilon/dt$ equation, although the effect is weak since $\xi \approx 1$. Examining the $2\lambda - 2\lambda'$ resonant argument (see the $C^*_{2,2}$ entries of Table 7.1 and in the $g_{3,2}$ entries of Table 7.2), we see that the gravitational version of this resonance has more influence on the orbital elements than the Lorentz version does. This manifests the fact that the Lorentz force weakens in the vicinity of synchronous orbit.

The resonant arguments of Tables 7.1 and 7.2 all have $|A_{\lambda'}| = k$, which follows directly from the fact that the gravitational and magnetic fields for the appropriate coefficients have k-fold longitudinal symmetry. This constraint, taken together with Eqs. (7.11) and (7.19) and the above discussion, allows us to predict which resonant arguments will appear for a given field coefficient. In comparing Table 7.1's $C^*_{3,2}$ and Table 7.2's $g_{2,2}$ entries, for example, we see that all possible first- and second-order resonant arguments (those for which $|A_\Omega| + |A_\varpi| \leq 2$) are present. The $g_{3,2}$ entries also contain all possible arguments of order two, but a few are missing from the $C^*_{2,2}$ entries. The missing arguments are best explained by looking at the mathematical expansion of the planetary gravity resonances (Kaula 1966). Properties of the series expansions for PGRs show that all arguments with $A_\lambda = 0$ and $A_\varpi = \pm 2$ as well as those that satisfy $j - k + A_\Omega < 0$ cannot appear in the expansion. The missing term ($C^*_{2,2} : \Psi = -2\lambda' + 2\varpi$) is an example of the former constraint while ($C^*_{2,2} : \Psi = 4\lambda - 2\lambda' - 2\Omega$)'s absence illustrates the latter.

7.3.4 Global Structure; Considerations of esonance Strength

Although the ideas discussed above significantly constrain the structure of individual resonances, they put few restrictions on the global properties of the entire expansions. Accordingly, in this section we address the distribution and relative strengths of resonances in each of the three cases.

To a first approximation, the distributions of SGRs, PGRs, and LRs relative to synchronous orbit are almost identical because the nodal and apsidal frequencies are slow compared to the mean motions and, consequently, can be ignored when calculating rough resonance positions. For all three problems, Nth-order resonances ($N = |A_\Omega| + |A_\varpi|$) are located inside synchronous orbit when $|A_\lambda| < |A_{\lambda'}|$ and outside that position when $|A_\lambda| > |A_{\lambda'}|$. The radial location of resonance, a, is determined by

$$\frac{a}{R_{syn}} = \frac{\Omega_p}{n}^{2/3} = \left|\frac{A_\lambda}{A_{\lambda'}}\right|^{2/3}. \qquad (7.25)$$

As in Section 7.3.2, for SGRs R_{syn} and Ω_p are understood to be the perturbing satellite's distance and mean motion, respectively. We use Eq. (7.25) to plot the positions of several first-order resonances ($N = 1$) and two second-order ones ($N = 2$) in Fig. 7.3. These resonances cluster together most tightly in the vicinity of synchronous orbit – adjacent resonances become arbitrarily close for large A_λ. Higher-order resonances behave similarly although Eq. (7.25) shows that they extend further from synchronous orbit than their first-order cousins.

Figure 7.3 Location of the several strong first-order (solid lines) and two representative second-order (dashed lines) Lorentz resonances around Jupiter. For Jupiter, $R_{syn} = 2.24$ planetary radii. The figure applies equally well to planetary gravity resonances and, if the perturbing satellite is at R_{syn}, to satellite resonances. In Section 7.4, we find that dust grains spiraling **toward** synchronous orbit can become trapped at resonant locations while those dragged **away from** synchronous orbit experience resonant jumps in either the inclination or eccentricity. Both of the displayed second-order resonances arise from the $g_{4,3}$ component of the magnetic field. Since the second-order 1:3 resonance is found far beyond the 1:2 resonance (Eq. 7.25), we see that higher-order resonances cover a broader radial range than first-order ones do.

Although resonances lie in similar positions for each perturbation, their strengths relative to one another vary depending on the details of the perturbing force. For example, each field coefficient (*e.g.,* $g_{2,2}$) produces two first-order resonances, one inside R_{syn} ($\Psi = \lambda - 2\lambda' + \Omega$) and one outside ($\Psi = 3\lambda - 2\lambda' - \Omega$). For LRs, the strengths of these two resonances are related since, to a sign, they have identical entries (Table 7.2); for PGRs, though, the entries differ (Table 7.1). More important, however, is the morphology of resonances in the vicinity of synchronous orbit. For SGRs, synchronous orbit is occupied by the perturbing satellite and so resonant strengths rise as this location is approached. Since resonances both increase in strength and decrease in separation as synchronous orbit is neared, it is inevitable that resonance overlap eventually occurs. At this point, single-resonance models of orbital motion are inappropriate and chaotic motions predominate; Wisdom (1980) has shown that resonance overlap occurs at a distance proportional to $\mu^{2/7}$, where μ is the satellite-to-planetary mass ratio. Unlike SGRs, PGRs and LRs tend to weaken as synchronous orbit is approached since these resonances depend on successively larger powers of R_p/a (Fig. 7.3, and Tables 7.1 and 7.2). Thus the spacing and strength effects compete, and it is not immediately obvious which dominates; Schaffer and Burns (1987), however, argue that this variety of resonance overlap does **not** occur for Lorentz resonances.

Instead, a different type of resonant overlap happens for PGRs and LRs. Just as the main energy levels of the hydrogen atom resolve into a multiplet of closely-spaced levels, so a detailed examination of resonant locations reveals a similar fine structure. Each individual resonant location (*e.g.,* 3:2 in Fig. 7.3) resolves into a cluster of resonances with a fixed ratio $A_\lambda/A_{\lambda'}$ and different nodal and apsidal coefficients. These resonances lie at slightly different locations due to the non-zero secular precession rates $d\Omega/dt$ and $d\varpi/dt$ which arise from the axisymmetric components of SGRs, PGRs, and LRs (*e.g.,* the $g_{j,0}$ terms of Table 7.2). For SGRs and PGRs, inclination resonances lie further from synchronous orbit than eccentricity resonances; this is due to the fact that secular gravitational perturbations cause orbital nodes to regress and orbital pericenters to precess. For LRs, the situations is more complicated because both gravitational and electromagnetic perturbations influence the precession rates. In some cases, the Lorentz force can cause the opposite behavior, *i.e.,* **nodal precession** and **apsidal regression** (see the $g_{1,0}$ and $g_{3,0}$ components of Table 7.2). Thus inclination resonances may be closer to synchronous orbit than eccentricity resonances. Finally, since the electromagnetic precession rate depends on L, and hence on the charge-to-mass ratio of a dust grain, an ensemble of particles of different sizes will experience resonances in a range of slightly different locations. For some charge-to-mass ratios, the strong first-order inclination and eccentricity resonances are close enough to interfere with one another, leading to resonant overlap and chaos (see Schaffer and Burns 1992's Fig. 5).

In the expansions of PGRs and LRs presented in Tables 7.1 and 7.2, we have assumed that the gravitational and magnetic field coefficients are time-independent and thus the fields rotate as rigid objects (*i.e.,* at a single frequency). In reality, however, these coefficients probably change slowly [*cf.* Levy (1989) for LRs at Jupiter] and, in some cases, even rapidly [*cf.* Marley (1991), Marley and Porco (1993) for PGRs at Saturn]. Unfortunately, the physics driving these changes, especially those of the magnetic field, are poorly understood which precludes a quantitative discussion. Nevertheless, we can determine the qualitative effects of gradual changes in the fields by analogy with satellite resonances. In SGRs, the perturbing satellite has three distinct orbital frequencies: its rapid mean motion and slower nodal and apsidal precession rates. If the precession rates are suppressed, all corotation resonances (whose arguments depend on quantities of the perturber that are gradually changing) disappear from the disturbing function. In an entirely similar manner, the inclusion of slow drift frequencies to both the PGR and LR problems introduces corotation resonances that are slightly separated from the nominal resonant locations (Fig. 7.3).

Because corotation resonances affect only the perturbee's mean motion, they are often of minor importance. When a satellite is the perturber, however, the paired interactions of a corotation resonance and a nearby eccentricity resonance are capable of longitudinally confining ring arcs (Goldreich *et al.* 1986, Porco 1991). Thus the existence of corotation resonances in the other two cases may not be entirely academic. In particular, we suggest that similar trapping mechanisms may operate in some faint rings that are influenced by Lorentz forces.

7.4 oupling wit rag orces

7.4.1 esonant Equations

Acting alone, mean-motion resonances are capable of inducing moderately-large, periodic changes in the orbital elements of nearby particles. Nonetheless, because the majority of possible orbits are far from resonant locations, resonant effects might seem to be unimportant. Not so! When coupled with a drag force, which causes secular evolution of an orbit's mean motion, the importance of resonances is greatly enhanced since drag forces will inevitably transport distant particles into resonant locations where they can be strongly perturbed. Furthermore, drag forces allow resonant perturbations to secularly change orbital eccentricities and inclinations as we will demonstrate below. Depending on the direction of the drift, drag forces acting at resonance can cause jumps in the value of e and/or i as well as resonant trapping with an associated sustained growth in those elements.

The importance of the coupling between drag forces and resonances was first recognized by Goldreich (1965) who argued that tidal drags cause satellites to evolve into, and subsequently become stably trapped in, satellite mean-motion resonances. Since then, the capture process has been reexamined (Greenberg

1973a), individual examples have been analyzed (*e.g.,* Sinclair 1975, Greenberg 1973b), and Hamiltonian methods have been applied to the process (Peale 1976, Henrard 1982, Borderies and Goldreich 1984, Dermott *et al.* 1988, Malhotra 1991). In these next few sections we argue that particles drifting into PGRs and LRs display similar dynamic behavior to that seen at SGRs. We also illustrate how our LR expansion can be applied to the study of particular resonances.

Small particles that make up diffuse ring systems are not significantly influenced by tidal forces; instead several additional drag forces operate on these particles. Plasma and atmospheric drags arise from motion through swarms of charged and neutral molecules that corotate with the planet; accordingly, these drags slow particles inside of R_{syn} and speed up those outside of this position. Orbital evolution, therefore, is away from the synchronous location. Poynting-Robertson drag arises from the asymmetric scattering and re-radiation of photons (Burns *et al.* 1979) and always causes orbits to lose energy and evolve inward. Finally, resonant charge variations arise from the lag in the response of a grain's charge as its orbital motion takes it into regions with different charging currents. Depending on the plasma parameters, resonant charge variations can cause the semimajor axis to either increase or decrease (Burns and Schaffer 1989, Northrop *et al.* 1989). Although these drag forces only operate on small particles they, like tidal evolution, can bring material to resonances and influence the subsequent dynamics. The analogous process for interplanetary dust – evolution under Poynting-Robertson drag into resonances with the planets – was first recognized by Gold (1975) and later numerically studied by Gonczi *et al.* (1982). Several recent papers revisit and extend the early results (*e.g.,* Jackson and Zook 1989, 1992, Weidenschilling and Jackson 1993, Roques *et al.* 1993, Lazzaro *et al.* 1993).

After the discussion of Section 7.3, it should not be surprising that LRs and PGRs behave almost identically to SGRs when coupled with a drag force. The main difference is due to the existence of strong first-order inclination-type PGRs and LRs. In fact for LRs, inclination resonances are usually stronger than the corresponding eccentricity ones (Table 7.2). Thus, while a distribution of dust evolving through a set of SGRs might be expected to remain roughly planar due to the dominance of eccentricity-type resonances, this will not be the case for PGRs and especially LRs as the jovian halo so elegantly demonstrates (Burns *et al.* 1985). To emphasize this point, we treat a first-order inclination-type resonance in this section although the structure of the equations, and hence the resonant dynamics, is identical for an eccentricity resonance (see Table 7.2 and Hamilton and Burns 1993b).

In writing a set of equations valid for the passage of a grain through an isolated resonance, we include the drag force as well as the perturbation's resonant and secular terms. We specialize the equations to the 3:2 first-order Lorentz inclination resonance which is thought to cause the transition from the main jovian ring to its interior halo (Burns *et al.* 1985). Since a first-order inclination resonance does not strongly affect e, ϖ, and ϵ (see Table 7.2), we ignore changes in these elements. The governing equations come from the $(g_{3,3} : \Psi = 2\lambda - 3\lambda' + \Omega)$ entry of Table 7.2. Taking $\xi = n/\Omega_p \approx 3/2$, the appropriate expressions are

$$\frac{dn}{dt} = -3in^2\beta\cos(2\lambda - 3\lambda' + \Omega) + \dot{n}_{drag} \tag{7.26}$$

$$\frac{di}{dt} = -\frac{n\beta}{2}\cos(2\lambda - 3\lambda' + \Omega) + \dot{I}_{drag} \tag{7.27}$$

$$\frac{d\Omega}{dt} = \frac{n\beta}{2i}\sin(2\lambda - 3\lambda' + \Omega) + \dot{\Omega}_{sec}, \tag{7.28}$$

where $\dot{\Omega}_{sec}$ is the nearly constant secular precession rate arising from electromagnetic and gravitational forces; its presence slightly alters the physical location of resonance. Drag terms influence each of Eqs. (7.26–7.28), but contributions to $d\Omega/dt$ are neglected as they are dominated by $\dot{\Omega}_{sec}$. Finally, the limited radial extent of the resonance zone justifies treating $\dot{n}_{drag}$ as a constant. We define the resonance strength to be

$$\beta \approx \frac{\sqrt{10}L}{2} \quad \frac{(g_{3,3}^2 + h_{3,3}^2)^{1/2}}{g_{1,0}} \quad \frac{R_p}{a}^{2} \approx (0.05)L, \tag{7.29}$$

or one-third of the dn/dt coefficient taken from the $(g_{3,3} : \Psi = 2\lambda - 3\lambda' + \Omega)$ entry of Table 7.2. In the final approximation, we have used parameters appropriate for the jovian 3:2 resonance, namely $g_{1,0} \approx 4.218$ G, $g_{3,3} \approx -0.231$ G, $h_{3,3} \approx -0.294$ G (Acuña *et al.* 1983a), and $a/R_p \approx 1.7$. At Jupiter, a micron-sized grain charged to a potential of +5V, has $L \approx 0.028$ and hence $\beta \approx 0.0014$, a value orders of magnitude greater than typical SGR strengths. Furthermore, since drag forces act on small particles much faster than tidal forces influence large ones, evolution of dust particles in Lorentz resonances proceeds correspondingly more rapidly.

To improve our calculation of β, we would need to include additional contributions from the $g_{j,3}$ and $h_{j,3}$ $(j = 5, 7, 9, ...)$ field coefficients, but unfortunately, the values of these coefficients are unknown for all non-terrestrial magnetic fields. Nevertheless, we can get a rough upper bound on the error in β by assuming that the higher-order field coefficients are roughly equal in magnitude to the octupole coefficients [for the terrestrial magnetic field, the coefficients decrease in magnitude with increasing order – (Stern 1976)]. In this case, the higher-order terms contribute $\lesssim 0.5\beta$ to the resonance strength. There are also terms in Eqs. (7.26–7.28) that depend on larger powers of e and i, but these contributions amount to $\lesssim 0.1\beta$ for conditions present in the Jovian ring.

Finally, we note that the structure of the equations (7.26–7.28) is appropriate for all first-order inclination resonances; only the constant coefficients in each equation differ from one resonance to the next (Table 7.2). Second-order (Nth-order) resonances differ only in that the power of i in each of the dn/dt, di/dt, and $d\Omega/dt$ equations is $(N-1)$ larger. The $(g_{4,3} : \Psi = 5\lambda - 3\lambda' - 2\Omega)$ and $(g_{4,3} : \Psi = \lambda - 3\lambda' + 2\Omega)$ entries of Table 7.2 are each second-order inclination resonances; their positions relative to Jupiter are given in Fig. 7.3. Eccentricity resonances of all orders are identical in form to inclination resonances if all i's are replaced by e's. Because all of these different types of resonances have a similar structure, we expect the same type of dynamic behavior at each of them.

7.4.2 esonance Trapping

What happens to particles that drift into resonance? The question is most exactly treated by transforming Eqs. (7.26–7.28) into canonical variables from which a pendulum-like Hamiltonian can be defined (*cf.* Peale 1976). Such an analysis shows that for an isolated resonance there are two possibilities depending on the direction from which the resonance is approached: resonance trapping and resonant jumps. A trapping probability, which depends on the relative strengths of the resonance and the drag force, is associated with the former. Unfortunately, the Hamiltonian results are awkward to interpret in terms of the orbital elements, the variables that have geometric meaning. Accordingly, the purpose of this and the following section is to give simple descriptions and approximate formulae in terms of orbital elements without resorting to a Hamiltonian analysis. In so doing, we further emphasize the similarities between SGRs, PGRs, and LRs.

When a particle enters the resonance zone and subsequently is stable against perturbations that attempt to dislodge it, the particle is said to have been trapped into resonance. For the particle to remain trapped, its orbital period must stay nearly commensurate with the forcing period, and hence the average value of dn/dt must be zero. This can only occur when the first term in Eq. (7.26) balances the second. Thus very large drag rates preclude trapping or, put another way, for a given drag rate many resonances, especially higher-order ones, are too weak to trap passing particles. In Fig. 7.4, we show what happens to a grain that encounters the 3:2 inclination resonance while slowly drifting toward synchronous orbit. Resonant perturbations stop the evolution of the mean motion and simultaneously cause the inclination to grow. The latter growth can be easily explained with the energy constraint, Eq. (7.24).

Although drag forces need not produce changes in the orbital elements that satisfy Eq. (7.24) (resonant charge variations are an exception and will be discussed separately below), the resonant portion of the perturbation must. Since the cumulative perturbations for n, e, and i are written as sums of resonant and drag terms (Eqs. 7.26 and 7.27), we solve for the resonant terms and substitute these into Eq. (7.24). The energy constraint takes the form

$$e\frac{de}{dt} + i\frac{di}{dt} = \frac{\dot{n}_{drag}}{3n} \; 1 - \frac{n}{\Omega_p} \; + e\dot{e}_{drag} + i\dot{I}_{drag}. \qquad (7.30)$$

Figure 7.4 Resonance Trapping. A plot of the orbital evolution numerically determined by Eqs. (7.26–7.28) for jovian parameters $\beta = 1.4 \times 10^{-3}$ and $\dot{n}_{drag} = -10^{-5}\Omega_p^2$. Plotted against N_p, the number of jovian rotations, are the mean motion ratio n/Ω_p, the inclination i, and the resonant angle Ψ. Initial conditions are $n = 1.6\Omega_p, i = 0.01$, and $\Psi = 0$. The resonant angle Ψ librates with small amplitude around a value slightly less than 270° as could have been anticipated by setting equation (7.26) to zero and solving for Ψ. The dashed line comes from Eq. (7.32) and, for these parameters, is $i \approx 0.0037 N_p^{1/2}$. It has been offset slightly to the left for clarity. Integrations of the full equations of motion, both for SGRs (*cf.* Dermott *et al.* 1988's Fig. 11) and LRs (Hamilton, unpublished), show behavior qualitatively similar to this.

As it stands, Eq. (7.30) is directly applicable to mixed resonances (all of the second-order resonances with $g_{j,k}$ coefficients satisfying $j+k$=odd), which influence both e and i. For nearly circular orbits at inclination resonances, however, eccentricities are only weakly perturbed and can usually be ignored. Furthermore, drag forces typically do not strongly affect orbital inclinations so the $\dot{I}_{drag}$ term can be dropped. Taking these approximations yields

$$i\frac{di}{dt} = \frac{\dot{n}_{drag}}{3n} \quad 1 - \frac{n}{\Omega_p} \quad , \tag{7.31}$$

which can be directly integrated to

$$i = \sqrt{i_0^2 + \frac{2\dot{n}_{drag}t}{3n} \quad 1 - \frac{n}{\Omega_p}} \quad , \tag{7.32}$$

where i_0 is the initial inclination and $t = N_p(2\pi/\Omega_p)$ is time, with N_p the number of jovian rotations (*cf.* Hamilton and Burns 1993b). The prediction of Eq. (7.32) agrees well with the numerical integration of Eqs. (7.26–7.28) presented in Fig. 7.4. We note that Eqs. (7.31) and (7.32) are applicable to inclination resonances of all orders and that similar expressions apply to nearly planar orbits at eccentricity resonances. Incidentally, Eq. (7.31) can also be obtained directly for the 3:2 inclination resonance by setting $dn/dt = 0$ in Eqs. (7.26–7.28) and solving for $i\ di/dt$.

As an interesting aside, consider the case where resonant charge variations cause evolution through a Lorentz resonance. Because the drag force is entirely electromagnetic, the full perturbation satisfies Eq. (7.24). If a particle becomes trapped in a resonance, then dn/dt is zero and hence $e\ de/dt + i\ di/dt = 0$. Thus there can be no secular increase in one element without a corresponding decrease in another.

Equation (7.30) shows that particles trapped in resonances systematically change their inclinations and/or eccentricities. Evolution toward synchronous orbit makes i increase while evolution in the opposite sense causes it to decrease (Eq. 7.32). Because Eq. (7.32) gives nonsensical results for shrinking inclinations (the quantity inside the square root becomes negative), particles drifting away from synchronous orbit cannot stay in resonance forever. In fact, by linearizing Eqs. (7.26–7.28) around the equilibrium inclination, it can be shown that solutions in which i decreases are unstable and so particles do not become trapped at all. Conversely, when drifts are toward synchronous orbit, i increases and the linearization yields stable solutions.

Thus we find that trapping into pure inclination-type and eccentricity-type SGRs, PGRs, or LRs occurs only when drifts are toward the synchronous location (Fig. 7.3); in such cases, the energy integral Eq. (7.24) requires that there be an associated "square root" growth in e or i (Eq. 7.32 and Fig. 7.4).

7.4.3 Jumps at esonance

When drifts are away from synchronous orbit, or when the drag rate is too high for resonant trapping to occur, discrete jumps in the inclination (or eccentricity) happen instead. In this section we discuss the mechanism that leads to resonant jumps and derive a simple expression to approximate the jump amplitude in the limiting case of slow drag (*cf.* Hamilton and Burns 1993b).

Figure 7.5 shows the orbital history of a dust grain drifting away from synchronous orbit and through the jovian 3:2 inclination resonance. Far from resonance, the angle Ψ is seen to circulate rapidly and the resonance has little influence on the motion of an orbiting dust particle. As drags bring the particle closer to resonance, however, Ψ starts librating about a value near 90°; because of their $\cos\Psi$ dependence, however, dn/dt and di/dt are still not strongly perturbed (Eqs. 7.26 and 7.27). Eventually, the equilibrium point about which libration occurs becomes unstable (one can solve for the point at which this occurs from Eqs. 7.26–7.28). The resonance variable Ψ drifts away from 90°, and resonant perturbations to dn/dt overwhelm the drag force, quickly pushing orbits across the resonance zone. At this point Ψ starts circulating rapidly in the opposite sense, resonant perturbations dwindle in strength, and drag forces dominate orbital evolution once again.

It is clear from Fig. 7.5 that both n and i experience jumps during passage through resonance. Since the jumps are caused by resonant forces, their magnitudes are necessarily related by Eq. (7.24). In particular, for inclination resonances, eccentricities are unaffected and so

$$i\,di = -\ \frac{dn}{3n} \quad 1 - \frac{n}{\Omega_p} \quad , \tag{7.33}$$

where dn and di are the jump amplitudes; the former can be approximated simply from the width of the region over which resonant perturbations are significant, which we estimate to be roughly the resonance's libration width. We obtain the libration width by setting $d\Psi/dt = 0$ and using Eqs. (7.26–7.28) to solve separately for the largest possible mean motion (n_{max}) and the smallest (n_{min}); the libration width is then simply $|dn| \approx n_{max} - n_{min}$. For grains drifting away from synchronous orbit through a first-order inclination resonance, we find the mean motion jump,

$$dn \approx -\frac{2n\beta}{i}\left|1 - \frac{n}{\Omega_p}\right| \quad 1 - \frac{n}{\Omega_p} \quad , \tag{7.34}$$

which, combined with Eq. (7.33), yields the inclination jump,

$$di \approx \frac{2\beta}{3i^2}\left|1 - \frac{n}{\Omega_p}\right|^3 . \tag{7.35}$$

Figure 7.5 Jumps at resonance. A plot of the orbital evolution determined by Eqs. (7.26–7.28) with parameters appropriate for a 1-micron grain: $\beta = 1.4 \times 10^{-3}; \dot{n}_{drag} = 10^{-5}\Omega_p^2$. Initial conditions are $n = 1.4\Omega_p, i = 0.01$, and $\Psi = 0$. Notice that the jumps in mean motion (semimajor axis) and inclination occur simultaneously near $n \sim 3\Omega_p/2$ as required by Eq. (7.33). The resonant argument Ψ librates around a value near 90° until passage through the resonance occurs, after which it circulates. Integrations of the full equations of motion, both for SGRs (*cf.* Dermott *et al.* 1988's Fig. 5) and LRs (Hamilton, unpublished), show behavior qualitatively similar to this.

As they stand, these expressions are ambiguous since it is unclear what value i has. For nearly circular orbits that drift into strong first-order resonances, however, $di \approx i_f$, where i_f is the inclination immediately after the jump. We approximate the inclination during resonant passage with $i \approx i_f/2 \approx di/2$, which allows us to express each jump amplitude purely as a function of the resonance's location and strength:

$$di \approx 2 \;\; \frac{\beta}{3} \;\; ^{1/3} \left| 1 - \frac{n}{\Omega_p} \right|, \tag{7.36}$$

$$dn \approx -2n(3\beta^2)^{1/3} \;\; 1 - \frac{n}{\Omega_p} \;\; . \tag{7.37}$$

As usual, the above discussion applies equally well to all first-order eccentricity resonances. Applying Eqs. (7.37) and (7.36) to our jovian example and taking the appropriate parameters from Fig. 7.5's caption, we estimate $dn = 0.03\Omega_p$ and $di = 0.08$, values lower than, but in reasonable agreement with, the numerically determined jumps observed in Fig. 7.5. We have also verified the functional dependence of di on β and $1 - n/\Omega_p$ in additional numerical experiments. The numerically-determined final inclination in Fig. 7.5 is $\approx 5.5^{\circ}$ which corresponds to particles rising $\approx 10,000$ km above the jovian equatorial plane, a value in agreement with the ring's observed half-thickness of $8,000 - 10,000$ km measured by Showalter *et al.* (1987). Thus the vertical thickness of the jovian halo is consistent with micron-sized grains drifting through the 3:2 Lorentz first-order inclination resonance.

Here, and in the preceding section, we have demonstrated that when drag forces bring particles to mean-motion resonances, either trapping or resonant jumps can occur. Because the results of a particular encounter depend so strongly on the direction of drag-induced orbital evolution, however, certain resonance-drag combinations manifest only a single type of behavior. For instance, tidal forces typically drive inner satellites toward outer ones and so the most common resonant phenomena for SGRs is trapping (*cf.* Goldreich 1965). Conversely, at Lorentz resonances, plasma and atmospheric drags cause orbits to evolve away from the synchronous location which leads to resonant jumps. In Table 7.3, we summarize the typical outcome of couplings between each of the resonances and drag forces discussed above. In all cases, the dynamical outcome of an interaction depends on the direction of drag-induced orbital evolution at a given resonant location, not on the structure of the particular resonance. This serves to re-emphasize the fact that resonances arising from very different perturbations are dynamically similar.

7.5 Su ary

In this chapter, we present the first disturbing-function-style expansion of the Lorentz force (Table 7.2). Our expansion, which is to second order in eccentricities and inclinations, provides simple equations valid for first-order e and i resonances as well as for second-order e^2, i^2, and ei resonances. To lowest-order, our equations for Lorentz resonances have the same form as those derived for gravitational resonances which accounts nicely for the similar dynamical behavior that we have observed in numerical integrations.

We trace many of the similarities between different types of resonances to basic orbital symmetries that constrain the functional form of all quantities – and hence all perturbations – expressed in terms of orbital elements. In particular, these orbital symmetries account for several of the patterns long noticed in expansions of the satellite disturbing function. Additional regularities are due to the fact that the three perturbations considered in this chapter – SGRs, PGRs, and LRs – are all constrained by a nearly identical integral of the motion. This integral exists for an arbitrary orbital perturbation provided that a rotating frame can be found in which the perturbation, or at least the resonant part thereof, is independent of time.

Our results imply that the orbital dynamics displayed at mean-motion resonances are fundamental. The first-order structure of a given resonance is determined primarily by orbital symmetries and by the integral of the motion. The character of the perturbing force is important only in determining absolute resonance strengths.

Table 7.3 Results of resonance-drag interactions

Resonance	Drag Force	Typical Dynamics
SGR	Tidal	Trapping
SGR	Plasma & Atmos. & Poynt.-Rob.	Trapping & Jumps
PGR†	Tidal & Plasma & Atmospheric	Jumps
PGR	Poynting-Robertson	Trapping & Jumps
LR	Tidal	(Incompatible)
LR	Plasma & Atmospheric	Jumps
LR	Poynting-Robertson	Trapping & Jumps
All	Resonant Charge Variations	???

† Here we have assumed a static gravity field for PGRs which is a good approximation for the terrestrial planets. Since gravitational modes of the giant planets can rotate rapidly (*cf.* Marley 1991), resonant locations are significantly altered and, hence, both types of dynamical behavior can occur.

r 8

r ir i s

In the preceding chapters, we have developed analytical and numerical tools that are useful for treating the orbital motions of dust particles circling asteroids, comets, and planets that themselves move on orbits around the Sun. We have applied our methods to several particular objects, demonstrating the importance of non-gravitational forces for hypothetical centimeter-sized satellites of an asteroid, for particles that make up Saturn's wedge-shaped E ring, and for those that orbit within the main jovian ring. These examples, however, are just a few of the solar system's many dusty features and we hope to apply the intuition gained from the problems considered within these chapters to these additional faint ring structures. In this final chapter, we briefly tour the various planetary systems, discussing the dusty environments of each in turn and highlighting problems in which dust plays an important role. Because planets differ in their sizes, oblatenesses, satellite retinues, magnetic environments, and distances from the Sun, particular dynamical effects vary greatly in importance. Indeed, we maintain that it is the variations in the interplay of dynamical forces that cause the great diversity found in faint ring structures throughout the solar system.

8.1 e Inner Solar Syste

The circumplanetary environments of Mercury and Venus are likely to be among the most pristine in the solar system since each planet is devoid of satellites (Burns 1973) and, hence, of major sources capable of supporting populations of circumplanetary dust. Although impacts with the planetary surface can theoretically loft material into bound orbits, such impacts are necessarily large, and relatively rare. Furthermore, the process itself is very inefficient since bound material typically reimpacts the surface swiftly (see Chapters 2–4 and Burns and Hamilton 1991). Even this meager production mechanism is unavailable at Venus, whose dense atmosphere prevents most – if not all – impact ejecta from escaping into space.

Several decades ago, the terrestrial environment, with its single large and distant Moon, was almost as pure. Scaling the results of Chapter 4 to lunar ejecta, we find that particles with radii $\lesssim 1\,\mu$m are flung from the Earth-Moon system by radiation pressure, while objects with radii up to $\approx 10\,\mu$m are rapidly forced onto highly-eccentric orbits which penetrate Earth's atmosphere (*cf.* Peale 1966, Allan and Cook 1967). More massive ejecta is swept up by the Moon in low velocity collisions that occur with characteristic timescales of thousands of years. Since large-impact events are necessary to raise significant amounts of debris off the lunar surface and such impacts are rare, the inferred ring of lunar debris is very sparsely populated.

Probably the dominant sources of debris near Earth today, however, are the myriad artificial satellites circling our world and the fuel-spraying booster rockets that put them there. The most crowded regions are in low-Earth orbit, where most manned missions have flown, and geosynchronous orbit, which is becoming increasingly crowded with communication satellites (*cf.* Kessler and Cour-Palais 1978, Kessler 1985, and Hechler 1985). These orbiting objects act as sources for small particles as paint chips flake off, and additional material is more forcefully removed by high velocity impacts of orbital debris and interplanetary micrometeoroids. Debris in the $\sim 1-10\,\mu$m range is highly perturbed by radiation pressure and electromagnetic effects; much of this debris is forced to enter Earth's atmosphere within a few years (Horanyi *et al.* 1988). This dynamical effect has interesting ramifications for the current population of orbital debris in the near Earth environment, some of which was sampled by the Long Duration Exposure Facility, a satellite that was recovered in early 1990 after spending nearly six years in low-Earth orbit (*cf.* McDonnell *et al.* 1992).

Most distant of the terrestrial planets, Mars is attended by two small moons, Phobos and Deimos, that orbit at several planetary radii. Such moonlets are ideal sources for circumplanetary dust since velocities needed for debris to escape their surfaces are slight (Soter 1971). Dubinin *et al.* (1990) reported evidence for a putative ring of debris around Mars, and several papers have subsequently addressed the issue theoretically (Horanyi *et al.* 1990, 1991, Juhász *et al.* 1993). These efforts explored the dynamics displayed by orbiting grains, and predicted the size distribution and number density of particles in the martian dust halo. We can improve our understanding of the martian environment by adding several important effects neglected by previous works.

First, the earlier papers ignore the precession induced by Mars' oblateness which is particularly important for grains launched from Phobos. There, the apsidal precession rate (2.78 rad/yr.) is similar to Mars' orbital motion (3.34 rad/yr.) resulting in a close cancellation of the motion of pericenter relative to the Sun. Just as in Saturn's E ring (*cf.* Chapter 6), this near cancellation allows radiation pressure to substantially increase orbital eccentricities. Without the oblateness term, Juhász *et al.* (1993) calculate that grains with radii in the

$1-7\,\mu$m range launched from Phobos will crash into Mars (*cf.* Chapter 6); with oblateness, this range increases to $1-40\,\mu$m. Additionally, the expressions used for the rates at which moonlets sweep up particles are inaccurate for grains highly perturbed by radiation pressure; this makes the calculated dust densities further suspect. Finally, the possibility that a significant population of dust is raised by collisions of ring particles with the tiny martian moons should be considered (*cf.* Chapter 6). Since orbits are highly-perturbed, collisions at speeds of 1-3 km/s are common; and because escape velocities from the moonlets are of order 5 m/s, such impacts can liberate significant amounts of debris.

8.2 e uter Solar Syste

Because their extensive retinues of tiny satellites serve as excellent sources, the giant planets rule significantly dustier environments than their terrestrial counterparts. Satellites vary tremendously in their dust production rates; large objects, such as the Galilean satellites of Jupiter, Saturn's Titan, and Neptune's Triton, are poor sources because they retain nearly all impact ejecta. Rather than discussing each satellite source individually, however, in this section we will instead focus on the expected dynamical behavior in specific regions that are common to all planets, noting possible applications when appropriate. In this manner, we hope to highlight points of particular interest without becoming unnecessarily tedious.

Approaching any planet from the edge of its Hill sphere, the first regime encountered is one in which micron- and larger-sized dust particles are dominated by three forces: planetary gravity, solar gravity, and radiation pressure. The results of Chapters 2–4 are thus directly applicable to grains launched from distant satellites, many of which orbit at significant fractions of a Hill Sphere: Jupiter's retrograde cluster at $\sim 0.4 r_H$, its prograde group and Saturn's Phoebe at $\sim 0.2 r_H$, and Saturn's Iapetus and Neptune's Nereid at $\sim 0.05 r_H$. Grains launched from these objects achieve large eccentricities, but can also attain very high inclinations relative to the planet's equatorial plane since they are not influenced by forces that cause precession about this plane. This class of orbits seems capable of explaining some of the micron-sized debris detected by the Voyager spacecraft's Planetary Radio Astronomy (Warwick *et al.* 1982, 1986, 1989) and Plasma Wave Science (Gurnett *et al.* 1983, 1987, 1991) instruments in the saturnian, uranian, and neptunian systems. Although most impacts occurred near the respective ring plane crossings, some circumplanetary material was also found at large inclinations at Saturn (Gurnett *et al.* 1983) and Neptune (D. Tsintikidis 1992, private communication), and perhaps at Uranus. It seems likely that this material originates from exterior satellites and is distributed along highly-eccentric and inclined orbits.

Such a distribution of very eccentric orbits can also transfer material radially;

in the saturnian system, the observable consequences of this are striking. Dust blasted off Saturn's retrograde Phoebe, for instance, is transported inward to Iapetus, preferentially hitting that satellite's leading hemisphere (Soter 1974, Mignard *et al.* 1994). If we accept an external origin for Iapetus' color, it is not surprising that the satellite's leading side is very dark, much like Phoebe. The trailing side of Iapetus, however, is icy in composition and exceedingly bright. Thus it seems likely that the gradual contamination of icy Iapetus by tiny grains originating from Phoebe has, over billions of years, produced the greatest albedo variation present on any satellite in the solar system.

At distances of typically $\sim 5 - 30$ planetary radii, the effects of planetary oblateness and the electromagnetic force become important. Radiation pressure is still reasonably influential, orbital velocities are larger, and material recollides with source satellites on rapid timescales. Lorentz resonances are relatively unimportant since only very weak, high-order ones are present this far from the planet. These dynamical forces are similar to those dominant in Saturn's E ring (*cf.* Chapter 6), and hence the possibility for E-ring-like objects elsewhere in the solar system should be investigated. At Jupiter, the equivalent region lies amid the Galilean satellites which are too large to be effective sources of dust. Moreover, recollision timescales are rapid and relative velocities are too small to support a self-sustaining ring. For similar reasons, the classical satellites of Uranus, and Neptune's giant moon Triton also prove to be inadequate sources. Ironically, the closest analog to the saturnian E ring may encircle a terrestrial planet – Mars. Maximum orbital eccentricities in the two rings are comparable, and as noted above, the martian ring may be self-generated through energetic grain-moon collisions. Additional types of self-sustaining rings may form preferentially in regimes closer to the central planet.

Approaching to within a few radii of the giant planets, oblateness and electromagnetic forces strengthen, the importance of radiation pressure wanes, and orbital velocities increase as do recollision frequencies. This is the domain dominated by Lorentz resonances, the most powerful bullies of the planetary neighborhood. Charged dust particles drifting into areas controlled by these strong resonances, are forcefully ejected from the resonance zone with a strong kick in their orbital eccentricity or inclination. The most dramatic example of a Lorentz resonance is, of course, the transition between the main jovian ring and its vertically extended inner halo, but Lorentz resonances – both the eccentricity and inclination varieties – are almost certainly important elsewhere in the jovian system, and at Uranus and Neptune (Fig. 8.1). Several opaque rings in the latter two systems are located near strong Lorentz resonances; resonantly perturbed eccentricities may augment relative velocities, collisional yields, and perhaps even cause the dusty component of some rings to be self-sustaining like Saturn's E ring. Great swaths of dust enshroud Uranus and Neptune, and several dusty ring features are found near Lorentz resonances too. Uranus' λ ring nominally lies a few thousand

Figure 8.1 Radial locations of rings and Lorentz resonances at Uranus and Neptune. Rings are drawn as solid arcs and are labeled near the figure's edges while Lorentz resonances are signified by dashed arcs that are labeled in the center of the figure. The stippled regions in Neptune's system represent dust sheets; dust is also found in complex structures throughout the region occupied by the main uranian rings. Solid circles represent satellites: Cordelia at Uranus and (heading inward) Galatea, Thalassa, Despoina, and Naiad at Neptune. Several ring structures are located suggestively near Lorentz resonances: the Adams and Leverrier rings, and the inner and outer boundaries of both the 1989N4R and the uranian ring system. Many of these associations may be coincidental, especially since rings and resonances each inhabit the region of space near the central planet. Nevertheless, Lorentz resonances almost certainly influence the orbital motions of the dusty components at both planets.

kilometers from the strong 2:1 Lorentz resonance, and the ring's five-fold density fluctuation advertises that interesting dynamics are involved (Fig. 8.1). The neptunian ring arcs are in a 42:43 gravitational resonance with the small satellite Galitea, but they are also near the 3:2 Lorentz resonance. The latter may cause dust to leak out of the arc sites into the much fainter Adams ring that completely encircles the planet. A dust sheet begins near the 5:3 Lorentz resonance, only to terminate just short of the 2:1 resonance in an opaque condensation of material that comprises the Leverrier ring (Fig. 8.1).

Back at Jupiter, a more complete analysis of the main ring and the halo is warranted, especially considering the forthcoming observations of the Galileo orbiter. The structure of the vertically extended halo contains information on the properties of the particles passing through the 3:2 resonance – primarily their size distribution and electric potential, Since jump amplitudes depend on particle sizes, halo grains should be non-uniformly layered, with larger ones tending to be found closer to the equatorial plane. Given the optically determined particle size distribution of Showalter *et al.* (1987) and the details of the resonance interaction discussed in Chapter 7, the vertical structure of the halo should be calculable. Such a derivation would provide an independent dynamical check of Showalter (1987)'s optically-derived size distribution, and could also constrain the grain potential.

Finally, particles somewhat smaller than those considered in this thesis are strongly influenced by electromagnetic effects and can exhibit non-intuitive behavior. Neither the perturbations schemes employed in Chapters 5–7, nor the adiabatic theory of charged particles are appropriate for the motions of these submicron grains since the forces of gravity and electromagnetism are comparable in strength. At Jupiter, these forces combine to cause rapid radial ejection of positively charged dust grains. Those interior to synchronous orbit are sent to fiery deaths in the jovian atmosphere, while those exterior are accelerated to high velocities that take them rapidly away from Jupiter into interplanetary, and then interstellar, space. Such high-velocity dust streams have recently been detected by the Ulysses spacecraft, and models assuming a gossamer ring (an outward extension of the main jovian ring) source (Hamilton and Burns 1993c) and an Io source (Horanyi *et al.* 1993a,b) have recently been developed. The motions of neutral atoms and molecules in circumplanetary orbits are dominated by gravity and the resonant scattering of sunlight (Smyth and Marconi 1993); consequently the dynamics displayed by such atoms is akin to the dynamics that we have investigated here. Additional surprises certainly await investigations of the dynamics of atoms and submicron-sized dust.

In these past few pages, we have taken a rapid tour of the solar system, seeking out particular areas where dust is common and where the methods of this thesis might successfully be applied. The results of any such exercise depends strongly on perspectives which are continually changing as progress is made;

thus an area emphasized here may later prove less interesting than another that was overlooked. A thorough study of one area leads to new insights into entirely different problems, and it is the anticipation of these insights that fuels continued investigations of dynamical phenomena.

ix

li r i l x si s

Computer algebra systems are increasingly useful tools in scientific research as advances in both hardware and software design continue to vastly improve the performance of these symbolic packages. A striking example of this trend is Murray and Harper (1993)'s recent expansion of the planetary disturbing function to eighth-order in eccentricities and inclinations; the information contained in each of this volume's 436 pages is generated entirely symbolically. With the aid of computer algebra, the authors improved on the presentation of previous works (Peirce 1849, Le Verrier 1855, Brouwer and Clemence 1961), ferreted out insidious errors made in these earlier expansions, and extended their analysis beyond the limits of human endurance.

In a less dramatic way, the symbolic program used to generate the entries listed in Table 7.2 saved this author weeks of calculations and tedious error checking. It is the purpose of this appendix to encourage and assist those interested in applying symbolic methods to similar problems by presenting and explaining the relatively simple MACSYMA code used in Chapter 7.

As entire books have been written about symbolic algebra (Harper *et al.* 1991 give a wonderful comparison of the main computer algebra packages; for an introduction to MACSYMA, consult Rand 1984), our purpose is not to teach the language, but rather to demonstrate the power of symbolic methods and to clarify the process employed in our orbital expansions. Nevertheless, a certain amount of explanation is necessary. In the sample MACSYMA session that follows, lines beginning with C constitute MACSYMA input (the program) while those starting with D are MACSYMA output. The input lines can be typed interactively, or submitted all at once in a batch mode. Within a single line, ":=" is used to define a function (*e.g.,* line C2) while a single colon sets up a rule for later substitution (*e.g.,* line C22 – here line D22 merely echoes the input).

The particular example under consideration expands the $g_{1,0}$ aligned dipolar portion of the magnetic field, but the extension to the other field components is straightforward. The program's flow roughly follows the explanation given in section 7.2.2. Here, we first obtain equations that define the magnetic field, which we then substitute into expressions for the force components. These, in turn, are plugged into expressions for the time rates of change of the orbital elements. Finally, the rates of change are Taylor-expanded out to the desired order. We now give specific line-by-line comments, following which we present the program itself.

Lines C2 and C3 define the Legendre polynomials and lines C4 and C5 select the symmetric dipole term (TH$\equiv$ θ). The program then performs additional substitution steps and prints the appropriate Legendre polynomial, "P," and its derivative with respect to θ, "dP," in lines D9 and D13. Lines 14–21 are unimportant here, since these pertain to the associated Legendre polynomials necessary for asymmetric magnetic fields (MLON$\equiv$ λ', PHI$\equiv$ ϕ). Lines D22, D23, D32, and D33 simply translate the spherical coordinates into orbital elements [*cf.* Eqs. (5.9), (5.10), and (5.11), BW$\equiv$ Ω] and lines D24–D26 give the r, θ, and ϕ components of an aligned dipolar magnetic field [*cf.* Eqs. (5.37), (5.38), and (5.39), RP$\equiv$ R_p]. Various expressions for elliptic orbital motion appear in lines D27–D33; [*cf.* Eqs. (5.12), (5.35), (5.36), and (5.40), WP$\equiv$ Ω_p]. In lines D34 through D36, the magnetic field components are written in terms of orbital elements.

Lines C37–C42 define the acceleration components in the orbital coordinate system [see Eqs. (5.32), (5.33), and (5.34)], and lines C43–C52 define the rates of change of the orbital elements (Danby 1988, p. 327). The next three lines replace the variables M, u, and ν with the orbital elements that appear in Table 7.2; here PLON$\equiv$ λ and CW$\equiv$ ϖ. The equation of center, expanded out to fourth-order in eccentricity is used for ν; for higher-order expansions this equation would have to be modified. Line C56 determines the order of the expansion; typical runs take from several minutes to several hours on a Sun SPARC 1 workstation, depending on this quantity and on the particular field component being expanded. Each time derivative is then Taylor-expanded to order "EPOW=2" in both of the small quantities E and I. We avoid E^2I^2 terms while retaining E^2 and I^2 terms by employing a trick: we substitute E*IOE in for I and do the Taylor expansion to second-order in E; since the IOE's flag the places where I's belong, the reverse substitution (I replacing E*IOE) is employed after the expansion to correct the expression. Finally, we call trigreduce, which uses identities to eliminate powers of trigonometric functions in favor of trigonometric functions of sums of angles. Lines D63, D70, D77, D84, D99, and D130 give the final output for dn/dt, de/dt, di/dt, $d\Omega/dt$, $d\varpi/dt$, and $d\epsilon/dt$, respectively. These lines contain all of the data found in the $g_{1,0}$ subtable of Table 7.2.

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