ASTR 630
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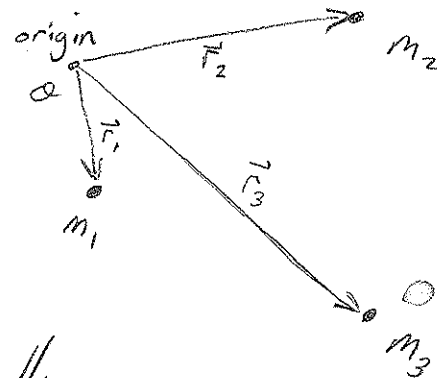
The Three-body Problem



After the 2-body problem, the next logical step is to attempt to solve the 3-body problem. Unfortunately, the 3-body problem is not analytically solvable except in special cases. One can, however, find constraints on allowed motions that prove useful in determining the nature of particular solutions.

terminology for the 3-body problem:

term	assumptions*	example
general	1	3 stars
restricted	1, 2	Sun, Jupiter, an asteroid
elliptic restricted	1, 2	(same as above)
circular restricted	1, 2, 6	same if Jupiter's orbit were circular
mass heirarchy	2, 3	Sun, Jupiter, asteroid
Hill's problem	1, 3, 5	Sun, Earth, Moon
coplanar	4	Sun, Jupiter, distant satellite in the orbital plane

These are all special cases of the three-body problem.

* definitions on next page

assumptions

1) Point masses. All objects are treated as point masses interacting via an attracting inverse square force law.

2) Two large masses and one small mass ($m_1, m_2 >> m_3$): The small mass is assumed to not influence the motions of either of the large masses; thus the large masses (primaries) obey Keplarian two-body motion.

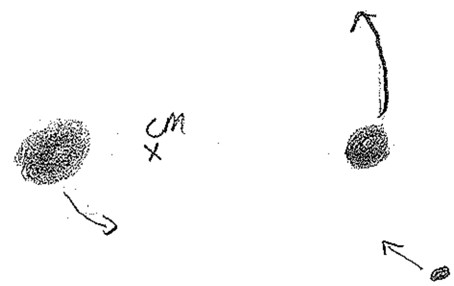
3) One large mass and two small ones ($m_1 >> m_2, m_3$). The small masses do not influence the larger.

4) The position and velocity vectors are all in a common plane. Since there are then no out-of-plane forces, the objects remain coplanar.

5) Two objects are located close together and the third is distant, e.g. the Earth-Moon system is far from the Sun.

6) Circular orbit. An unperturbed circular orbit is a partial solution to the problem

So, the "coplanar-circular-restricted problem" consists of two large objects circling each other an a small (technically infinitesimal) particle moving in the plane of the circular orbits.



Similarly, the "circular Hill's problem" consists of two small bodies in close proximity whose center of mass circles the large object.

Why is the three-body problem unsolvable?

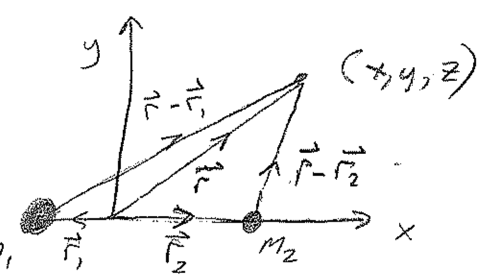
# bodies	2	3	n	3 (coplanar)	n (coplanar)
# degrees of freedom	12	18	6n	12	4n
Center of mass integrals	−6	−6	−6	−4	−4
Conservation of angular momentum	−3	−3	−3	−1	−1
Conservation of energy	−1	−1	−1	−1	−1
"elimination of the nodes"	−1	−1	−1	−1	−1
"use one variable as independent"	−1	−1	−1	−1	−1
remaining degrees of freedom	0	6	6n−12	4	4n−8

⇒ NOT ENOUGH INTEGRALS OF THE MOTION

Circular - Restricted Three-body Problem

In this much studied problem, the primaries are on known circular orbits, so there are only the 6 degrees of freedom of the infinitesimal particle. One more degree of freedom can be removed using Jacobi's integral, as discussed below.

Consider a frame centered on the center of mass of the primaries and rotating at their constant angular velocity.



In such a frame, the bodies will remain on the x-axis if originally placed there. In this frame, the potential field due to the two massive bodies is constant, therefore there is an energy-like quantity that is conserved.

The kinectic "energy" per unit mass of the small body is:

$$KE = \tfrac{1}{2} V^2 = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \qquad (1)$$

let $\rho_1 = |\vec{r} - \vec{r}_1|$, $\rho_2 = |\vec{r} - \vec{r}_2|$

The potential "energy" per unit mass is:

$$-\frac{Gm_1}{|\vec{r}-\vec{r}_1|} - \frac{Gm_2}{|\vec{r}-\vec{r}_2|} = -\frac{Gm_1}{\rho_1} - \frac{Gm_2}{\rho_2} \qquad (2)$$

And the centrifugal potential required in the rotating frame is:

$$-\tfrac{1}{2}\, n^2 (x^2+y^2) \qquad (3)$$

where n is the mean motion of the primaries:

$$n = \left[\frac{G(m_1+m_2)}{|\vec{r}_2-\vec{r}_1|^3}\right]^{1/2} \qquad (4)$$

Summing (1), (2) and (3) and using (4):

$$\tfrac{1}{2} v^2 - \frac{Gm_1}{\rho_1} - \frac{Gm_2}{\rho_2} - \frac{1}{2}\,\frac{G(m_1+m_2)}{|\vec{r}_2-\vec{r}_1|^3}(x^2+y^2) = -\frac{1}{2} C$$

where C is called the "Jacobi constant" and $-\tfrac{1}{2}C$ is the "energy" in the rotating frame.

$$C = \frac{G(m_1+m_2)}{|\vec{r}_2-\vec{r}_1|^3}(x^2+y^2) + \frac{2Gm_1}{\rho_1} + \frac{2Gm_2}{\rho_2} - v^2 \qquad (5)$$

One can simplify the appearance of (5) by introducing

$$\mu = \frac{m_2}{m_1+m_2} \qquad (6)$$

the mass ratio. Here m_2 is the <u>smaller</u> mass so $0 \le \mu \le 1/2$. In addition we non-dimensionalize (5) by choosing units of length, mass, and time.

<u>Unit of length</u>: "distance between primaries is one unit"
$\Rightarrow \; |\vec{r}_2 - \vec{r}_1| \to 1 \qquad (7)$

<u>unit of mass</u>: "total mass is one unit"
$\Rightarrow \; m_1 + m_2 \to 1 \qquad (8)$

<u>unit of time</u>: "time for primaries to move 1 radian is one unit" $\Rightarrow \; n \to 1 \qquad (9)$

(7), (8), and (9) with (4) imply $G \to 1$ in these units. Additionally, $m_2 = \mu(m_1+m_2) \to \mu$
$m_1 = (1-\mu)(m_1+m_2) \to 1-\mu$

so (5) becomes:

$$C = x^2+y^2 + \frac{2(1-\mu)}{\rho_1} + \frac{2\mu}{\rho_2} - v^2 \qquad (10)$$

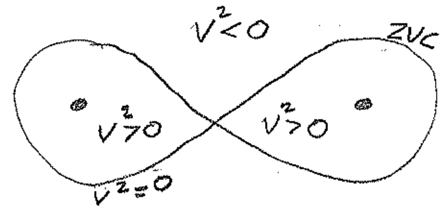
which is a constraint on the motion of the small body, i.e. the body must remain on a curve of constant C.

Notice that only μ appears as a parameter in (10). This shows that two three-body problems will have different families of solutions if the μ's are different. One can show the that the converse (same μ, then same family of solutions) from the full equations of motion.

Now if we set $v=0$ in (10) we obtain the appropriately named zero-velocity curves (ZUCs). These curves are closed one-dimensional curves in the orbital plane, and closed two-dimensional surfaces in three dimensional space.

$$C = x^2 + y^2 + \frac{2(1-\mu)}{\rho_1} + \frac{2\mu}{\rho_2} \qquad (11)$$

Considering (10) as a function of v^2, we see that $v^2=0$ on the curve, $v^2<0$ to one side and $v^2>0$ on the other



(true except for special points where the ZUCs cross as the "X" in the figure) $v^2<0$ implies an imaginary velocity, so clearly motion is confined to those regions where $v^2 \geq 0$.

To apply this constraint to a particle with given initial conditions, simply

1. solve for C from (10)
2. use this value of C, obtain the locus of points that satisfy (11)
3. the particle is prohibited from following a path that crosses the solution found in 2.

The 5 panel figure shows the zero-velocity curves for succesively smaller Jacobi constants (larger "energies") for $\mu \sim 1/4$. Hatched areas are off limits to a particle starting in a white area. As "energy" is increased, more portions of the space become available to the particle.

The next two figures show numerically computed orbits for a Sun, asteroid, test particle system with $\mu = 5 \times 10^{-12} \approx 0$. Note that the orbit never crosses the ZVCs. The asteroid is marked by an 'x', ☆'s are equilibrium points where all forces balance, a ▲ marks the particles initial position, and a ■ marks the end of the integration

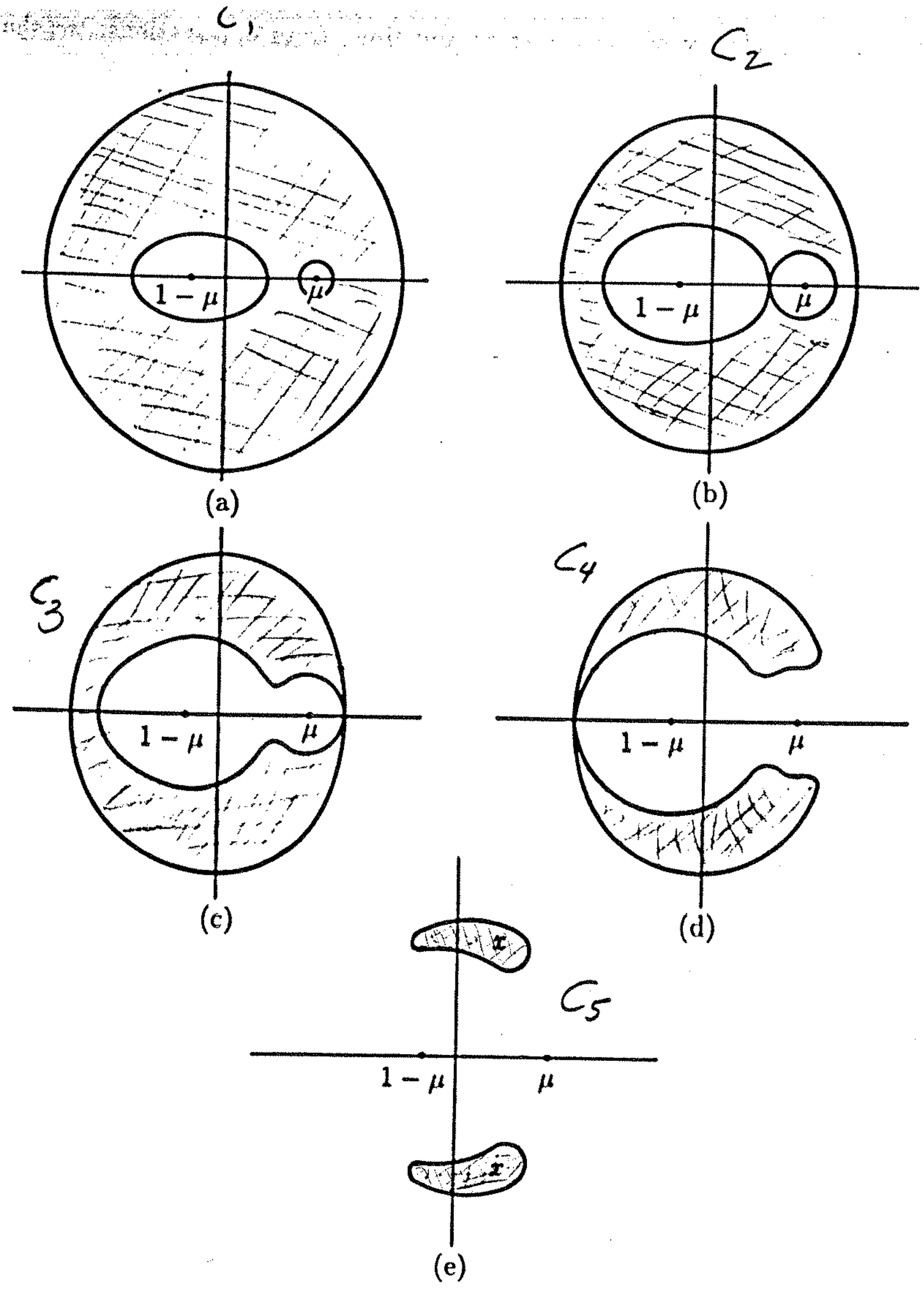
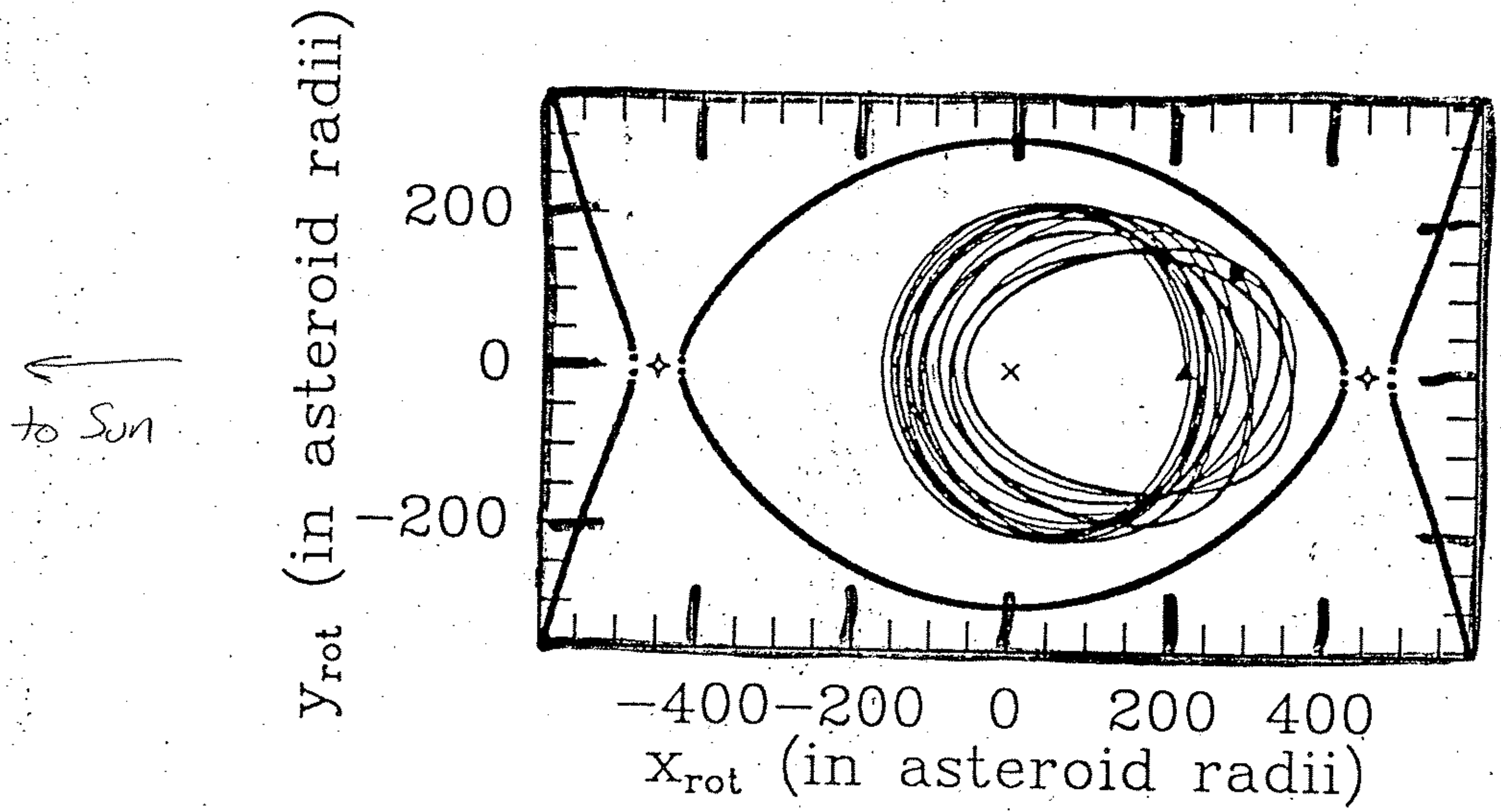


Figure 8.2 from Danby

Shaded areas are off limits

$C_1 > C_2 > C_3 > C_4 > C_5$

Prograde Orbit



Prograde escape orbit